CAT(0) groups

Even though "most groups are hyperbolic" there are lots and lots of naturally occuring groups that are NOT hyperbolic

eg G, H infinite and hyperbolic

$\Rightarrow G, H$ have ∞-order elements $g \in G$, $h \in H$

so $G \times H$ contains $\langle g \rangle \times \langle h \rangle = \mathbb{Z}^2$

(so is not hyperbolic.)

Many of the most interesting groups contain $\mathbb{Z}^2$, eg

- $SL_n(\mathbb{Z})$, $n \geq 3$ $\quad \begin{pmatrix} 1 & * & * & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
- surface mapping class groups
- Automorphism groups of free groups

But there are also groups that don't contain $\mathbb{Z}^2$ that are not hyperbolic an example is the Baumslag-Solitar group

$$BS(m,n) = \langle a,b \mid b a^m b^{-1} = a^n \rangle$$

If you draw the Cayley graph for $BS(1,2)$ this presentation, you can find arbitrarily fat geodesic triangles

($G = BS(m,n)$ is interesting for another reason: there is a surjective homomorphism $G \longrightarrow G$ that is not an isomorphism.)

We would still like to study more groups using the Svarc-Milnor Lemma and curvature constraints.

A notion of curvature for a general metric space was studied by Gromov, based on work by Caratheodory, Alexandrov and Toponogrov,

so he called it CAT(K), where $K \in \mathbb{R}$ is the "curvature".

In particular, CAT(0) is a notion of <u>non-positive</u> curvature

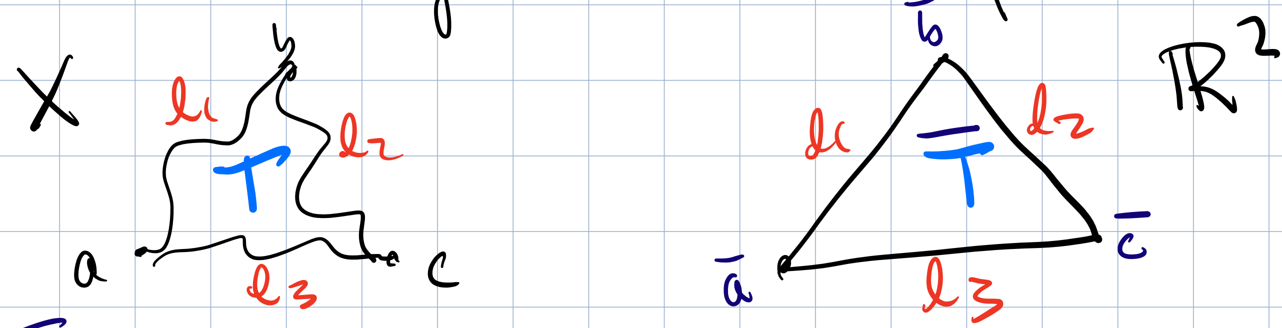
It is also defined in terms of triangles, but is "less fuzzy" than the notion of δ-thin triangles used in the definition of hyperbolicity.

We will also prove it is a <u>local</u> condition, (you only need to check it on small triangles), unlike hyperbolicity.

So it is more like the notion of curvature in a Riemannian manifold.

It's time to <u>define</u> CAT(0).

Let X be a geodesic metric space



T = geodesic triangle

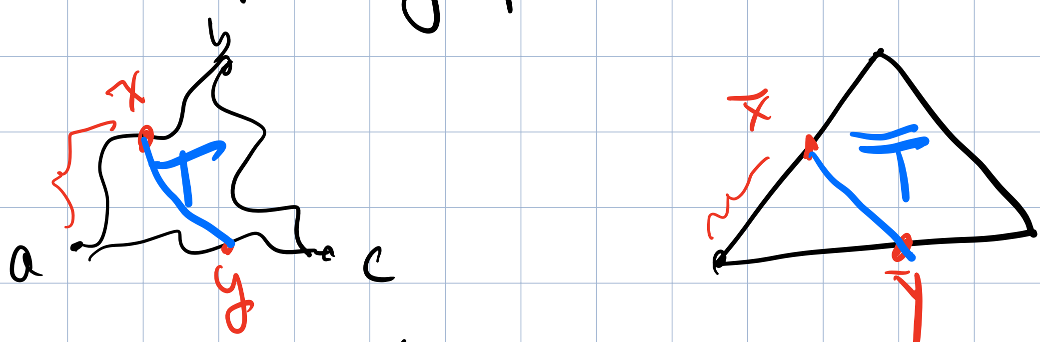
l_1, l_2, l_3 satisfy triangle inequality

$\bar{T}$ = triangle in $\mathbb{R}^2$ with same side lengths.

"comparison triangle"

<u>**Def**</u> X = proper geodesic metric space is <u>CAT(0)</u> if every geodesic triangle T is thinner than $\bar{T}$, ie

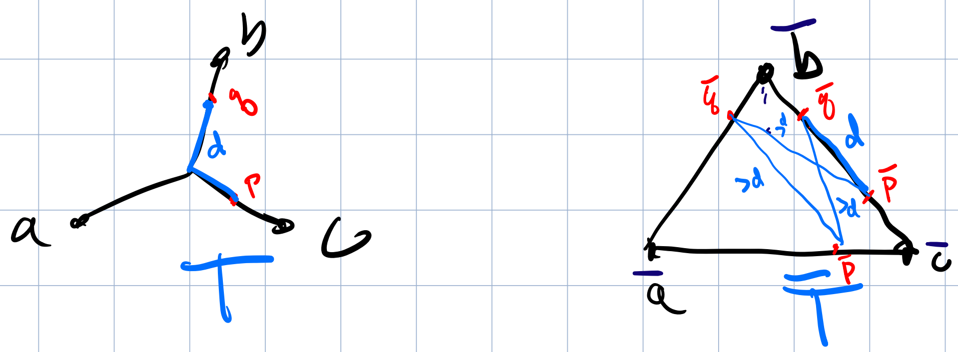
for $x, y \in T$, let $\bar{x}$, $\bar{y}$ be the corresponding points in $\bar{T}$:



then $d_X(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$

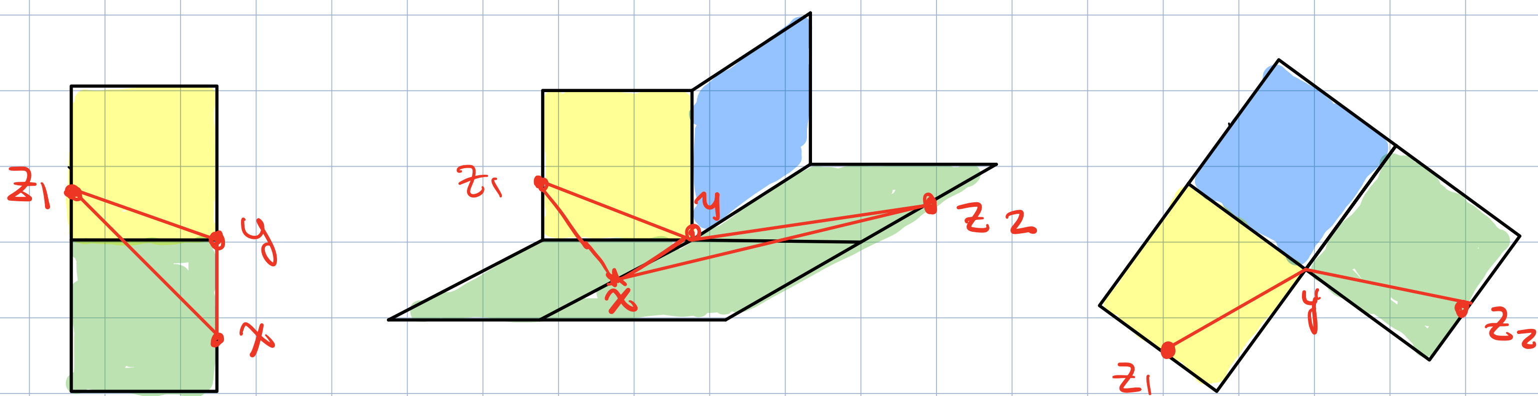
Example: $\mathbb{H}^2$ is CAT(0)

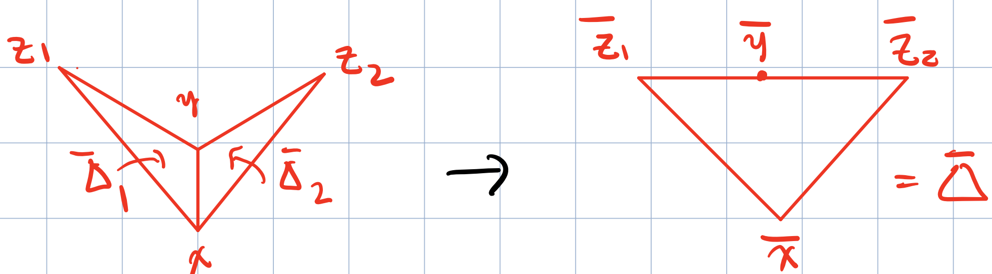
Example A tree is CAT(0)



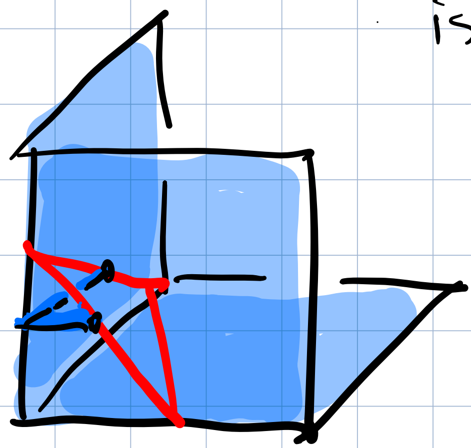
Example $\mathbb{R}^n$ is CAT(0) for all n

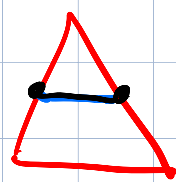
Example The corner of a hallway is CAT(0)





Example: The boundary of a cube is **not** CAT(0)





Exercise: If X, Y are CAT(0) then $X \times Y$ is, where

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$$

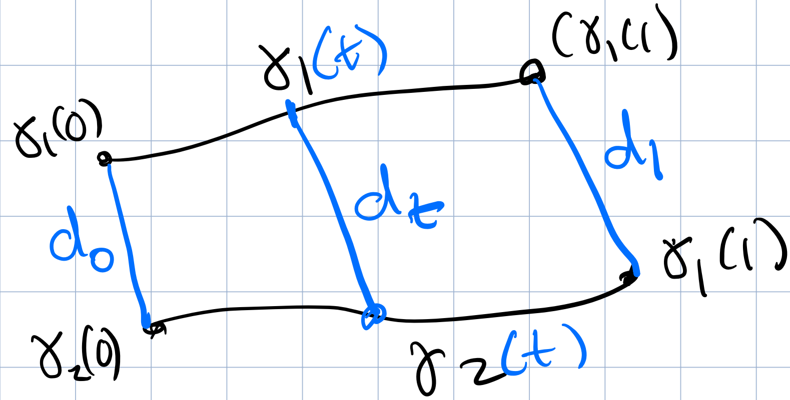
In particular, products $T_1 \times T_2$ of trees are CAT(0)

Proposition The metric on a CAT(0) space X is convex, ie if $\gamma_1, \gamma_2 : [0,1] \to X$ are geodesics then

$$d(\gamma_1(t), \gamma_2(t)) \le (1-t)\, d(\gamma_1(0), \gamma_2(0)) + t\, d(\gamma_1(1), \gamma_2(1))$$

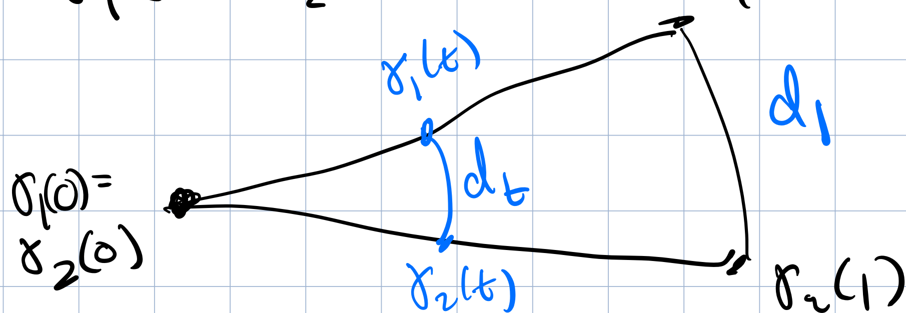
Proof:

Picture:

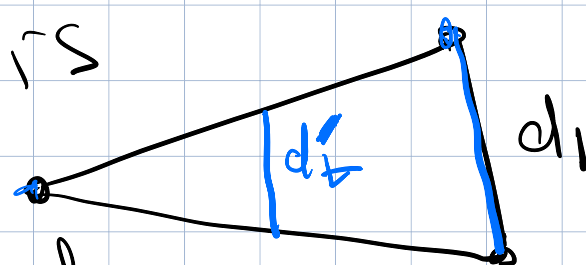


We want: $d_t \le (1-t) d_0 + t \cdot d_1$

If $\gamma_1(0) = \gamma_2(0)$

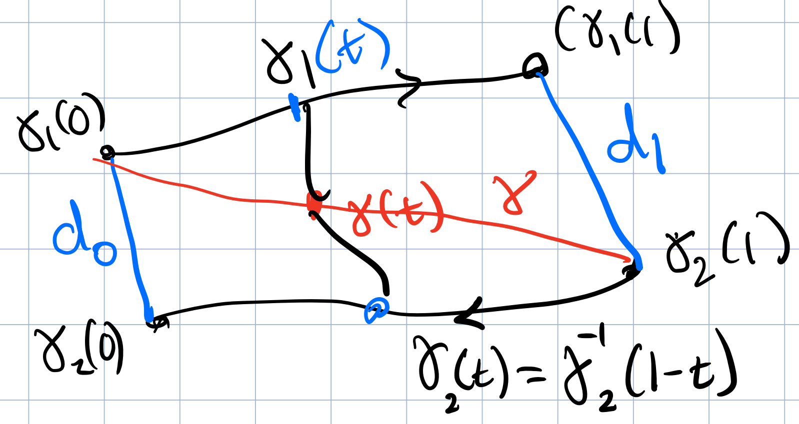


The comparison triangle is



So $d_t \le d_t' = t \cdot d_1$

In general

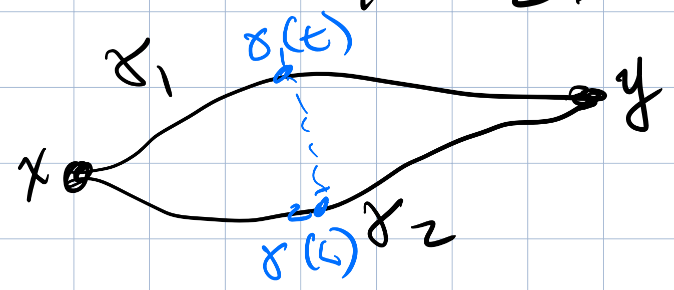


$$d_t = d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma_1(t), \gamma(t)) + d(\gamma(t), \gamma_2(t))$$
$$\leq t d_1 + (1-t) d_0 \quad ✓$$

<u>Corollary</u>: X CAT(0) $\Rightarrow$ there is a <u>unique</u> geodesic between any two points.

<u>ie</u>

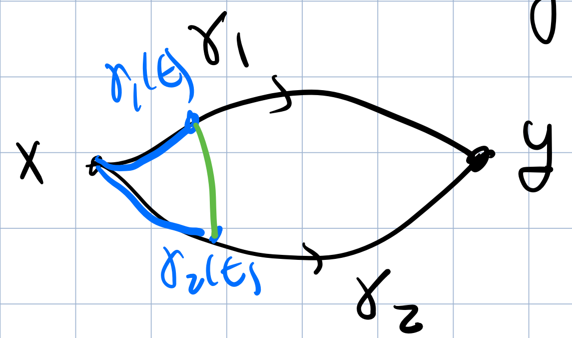
Let γ_1, γ_2 be two geodesics from x to y. Then $\gamma_1 = \gamma_2$.

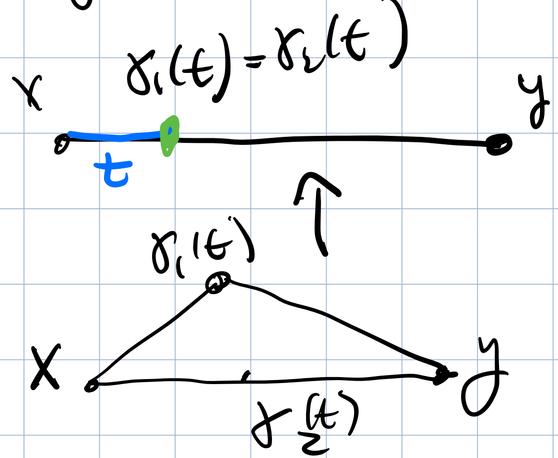


<u>Pf</u>

$$d(\gamma_1(t), \gamma_2(t)) \leq t d_1 + (1-t) d_0$$
$$= t \cdot 0 + (1-t) \cdot 0 = 0. \quad ✓$$

Note: Can also see this using a degenerate triangle:





Look at comparison Δ for

Get $d(\gamma_1(t), \gamma_2(t)) \leq 0$

$\Rightarrow \gamma_1(t) = \gamma_2(t)$. ✓

We can use this to prove the following:

Theorem: A CAT(0) metric space X is contractible

We need

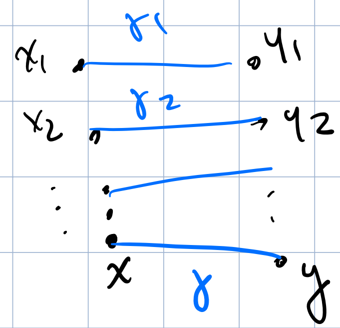
Lemma: Geodesics vary continuously with their endpoints.

ie $x_n \longrightarrow x$ $\quad y_n \longrightarrow y$

Let $\gamma_n : [0,1] \to X$ be the geodesic x_n to y_n

$\gamma : [0,1] \to X$ the geodesic x to y

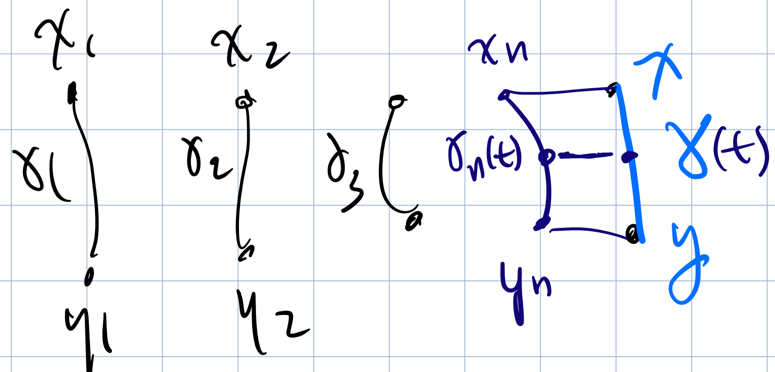
Then $\{\gamma_n\} \longrightarrow \gamma$



Proof Convergence here means uniform convergence, ie

$\forall \varepsilon > 0,\ \exists N$ s.t. $n > N \Rightarrow |\gamma_n(x) - \gamma(x)| < \varepsilon$ for all $x \in [0,1]$

This follows from the previous lemma:



$\{x_n\} \to x$ and $\{y_n\} \to y \Rightarrow \forall \varepsilon,$
$\exists N$ s.t. for $n > N$, $d(x_n, x) < \varepsilon$ and $d(y_n, y) < \varepsilon$
$\therefore\ d(\gamma_n(t), \gamma(t)) < \varepsilon$ for all t

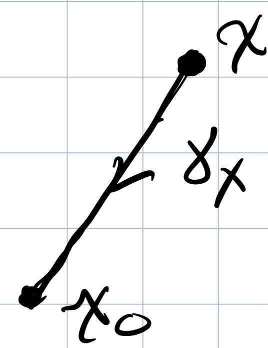
✓

Proof of theorem (contractibility):

Fix $x_0 \in X$

For any $x \in X$,

let γ_x be the unique geodesic x to x_0



Define $F: X \times [0,1] \longrightarrow X$

$$(x, t) \longmapsto \gamma_x(t)$$

This is a continuous map by the Lemma.

$$F(0) = \mathrm{id}$$
$$F(1) \equiv x_0$$

ie F is a homotopy from X to $\{x_0\}$ ✓

In particular, the CAT(0) property is not a quasi-isometry invariant

(eg if there is any non-trivial relation, the Cayley graph has a loop)

So you can't find CAT(0) groups by studying their Cayley graphs.

How *do* you find CAT(0) groups?

For manifolds, the classical Cartan-Hadamard theorem says:

<u>Theorem</u>: If M is a complete Riemannian manifold of non-positive sectional curvature, then local geodesics in $\tilde{M}$ can be continued uniquely to geodesic rays and the exponential map $T_x(\tilde{M}) \to \tilde{M}$ is an isometry, for any $x \in \tilde{M}$.

So if M is compact, its fundamental group acts freely and cocompactly on $\tilde{M} \cong T_x(\tilde{M}) \cong \mathbb{R}^n$, ie $\pi_1 M$ is CAT(0).

Having non-positive sectional curvature is a <u>local</u> condition - it depends only on a neighborhood of each point.

There is a version of the CH theorem for locally CAT(0) spaces, where

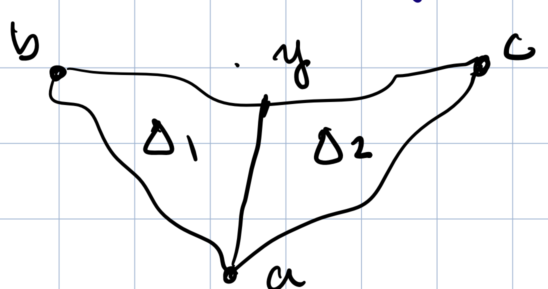
X is <u>locally</u> CAT(0) if every point has a CAT(0) neighborhood

CAT(0) version of Cartan-Hadamard theorem If X is locally CAT(0), then in $\tilde{X}$ the lifted metric is geodesic, convex and complete.

The propositions we proved about CAT(0) spaces are $\therefore$ true for $\tilde{X}$: geodesics exist, are unique and vary continuously with their endpoints, and $\tilde{X}$ is contractible.

To get from this statement to the statement that $\tilde{X}$ is actually CAT(0), we need the following:

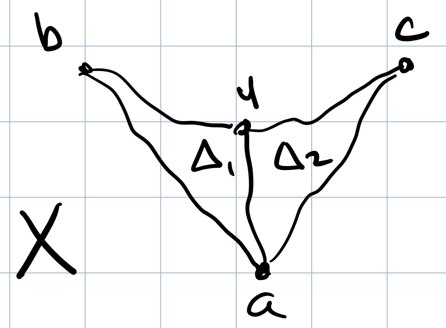
Alexandrov Lemma Let $\Delta(a,b,c)$ be a geodesic triangle with vertices a,b,c and y a point on the geodesic $[b,c]$

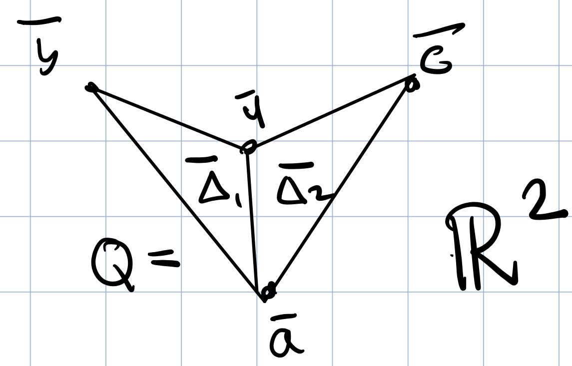


$\Delta_1 = \Delta(x, y, z_1)$, $\Delta_2 = \Delta(x, y, z_2)$

If Δ_1 and Δ_2 satisfy the CAT(0) condition, then so does $\Delta(x, z_1, z_2)$.

Proof: Look at the quadrilateral Q formed by comparison triangles $\bar\Delta_1$ and $\bar\Delta_2$ in $\mathbb{R}^2$ that share the edge $\bar a \bar y$

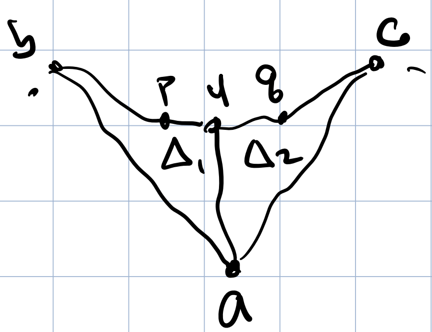


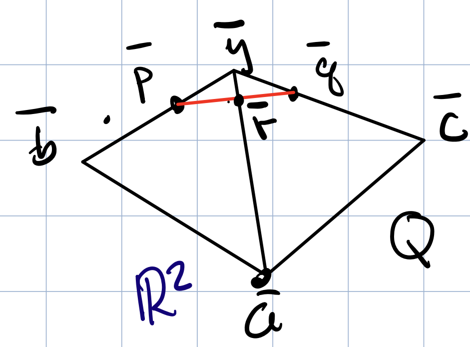


<u>Claim</u> $\angle \bar b \bar y \bar a + \angle \bar a \bar y \bar c \geqslant \pi$.

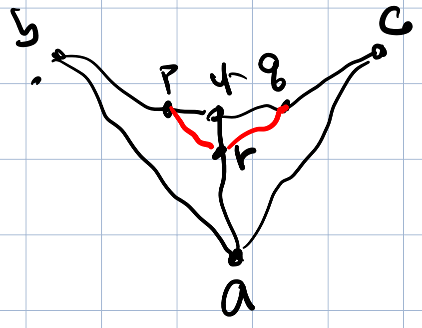
<u>Proof</u> Suppose it is $< \pi$

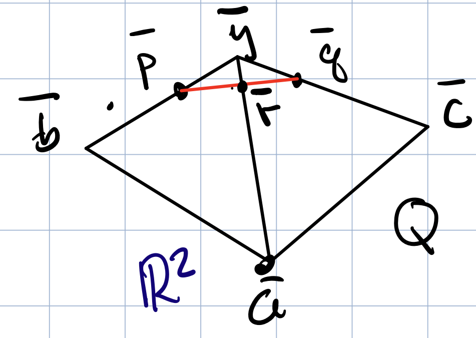
Choose $p \in [b,y]$, $q \in [y,c]$ close to y, with corresponding points $\bar p, \bar q$ in Q:





The straight line segment $\bar p \bar q$ crosses $\bar a \bar y$ at a point $\bar r$. Let r be the corresponding point on ay: and draw geodesics $[pr]$ and $[rq]$:





Then $d(p,q) = d(p,y) + d(y,q)$ since $[b,c]$ is a geodesic

$$= d(\bar{p},\bar{y}) + d(\bar{y},\bar{q})$$

$> d(\bar{p},\bar{q})$ by Δ inequality in $\mathbb{R}^2$

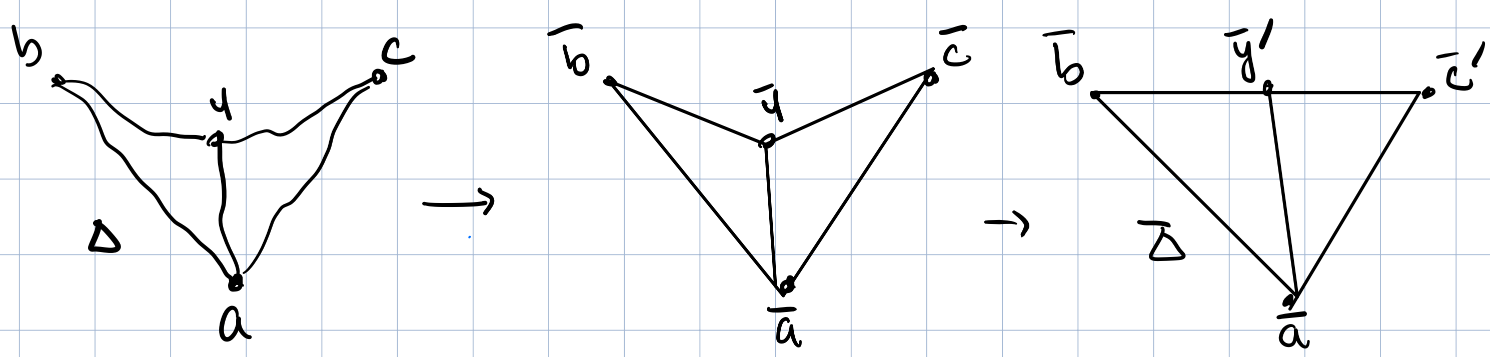
$= d(\bar{p},\bar{r}) + d(\bar{r},\bar{q})$

$\geq d(p,r) + d(r,q)$ since the small Δ's are CAT(0)

$\geq d(p,q)$ by Δ inequality in X

$\divideontimes$

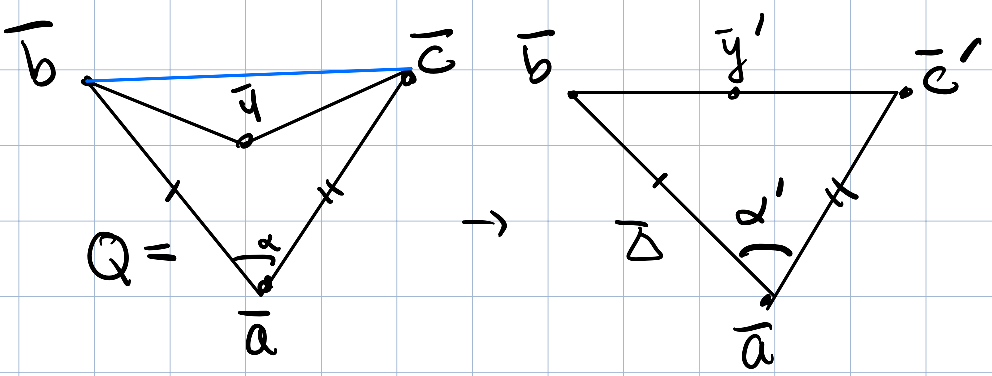
Now we straighten Q to obtain the comparison triangle $\bar{\Delta}$ for Δ



:

<u>Claim</u>: straightening increases the angles at $\bar{a}$, $\bar{b}$ and $\bar{c}$

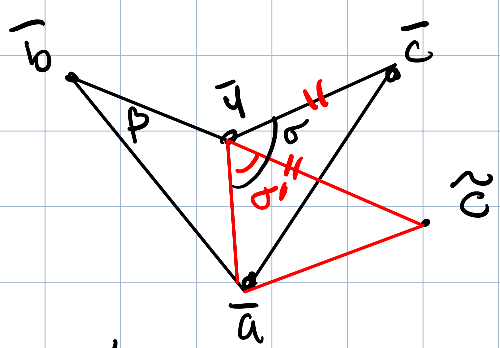
For the angle α at $\bar{a}$:



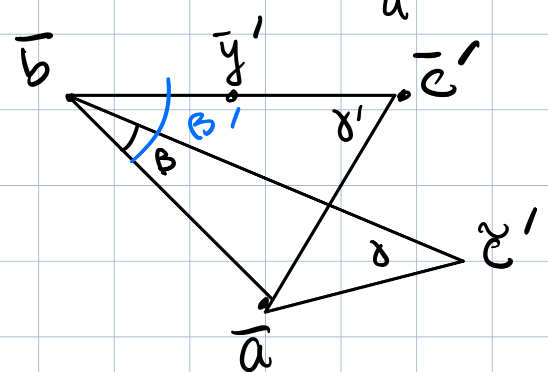
$d(\bar{b},\bar{c}) < d(\bar{b},\bar{y}) + d(\bar{y},c) = d(\bar{b},\bar{c}')$

$\Rightarrow \alpha' > \alpha$ by the law of cosines

For the angle β at b: Continue the edge $\bar{b}\bar{y}$ to a point $\tilde{c}$ with $d(\bar{y},\bar{c}) = d(\bar{y},\tilde{c})$:



$\sigma' < \sigma$, so By the law of cosines, $d(\bar{a},\bar{c}) < d(\bar{a},\tilde{c})$

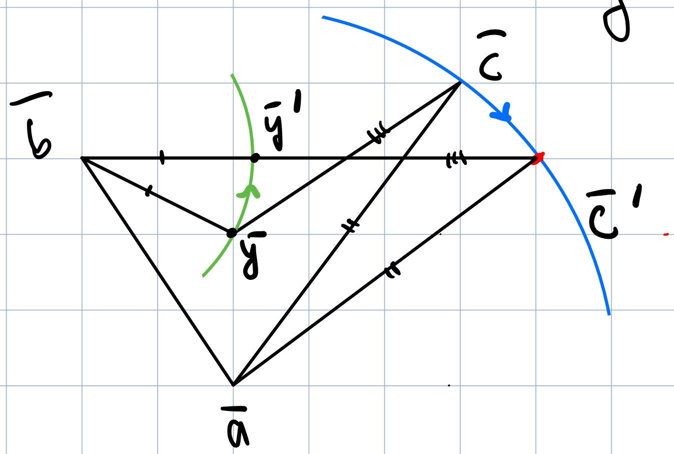


Therefore in the straightened figure $\beta' > \beta$ (law of cosines again)

Since $\beta' > \beta$, $d(\bar{a}, \bar{y}') > d(\bar{a}, \bar{y})$, so $\gamma' > \gamma$ too

<u>Claim</u>: The quadrilateral can be straightened so that the angles at $\bar{a}, \bar{b}, \bar{c}$ increase monotonically

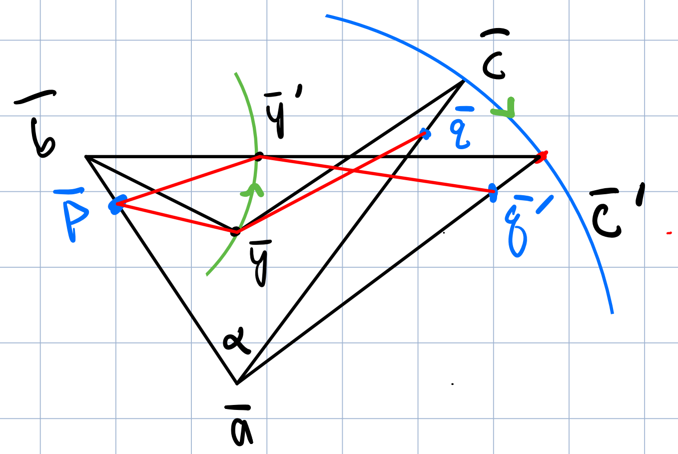
<u>Proof</u> straighten the figure by keeping $\bar{a}\bar{b}$ fixed and rotating $\bar{c}$. Of course $\bar{y}$ will also move:



Then $\bar{y}$ moves along the green arc, $\bar{c}$ along the blue arc in the directions indicated, so α and β grow monotonically

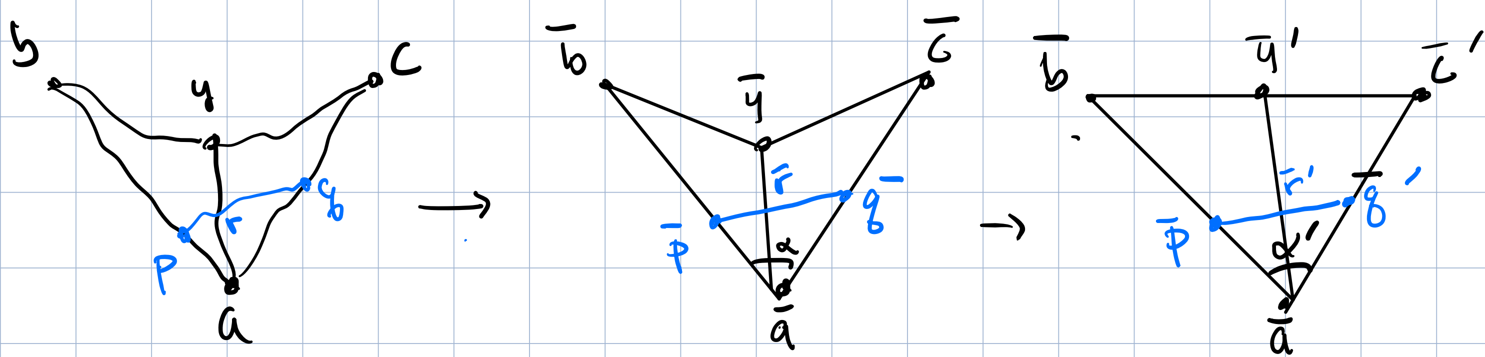
Now for the CAT(0) inequality:

Let p, q be points in $\Delta(a,b,c)$, with $p \in \Delta_1$ and $q \in \Delta_2$. We need to show $d(p,q) \leq d(\bar{p}, \bar{q}')$



The angle $\bar p \bar y \bar q$ is $< \pi$
The angle $\bar p \bar y' \bar q'$ is $> \pi$

If $[\bar p, \bar q]$ stays in Q, this is clear.



Let $\bar r$ be the intersection of $\bar p \bar q$ with $\bar a \bar y$, and r the corresponding point in $[a, y]$

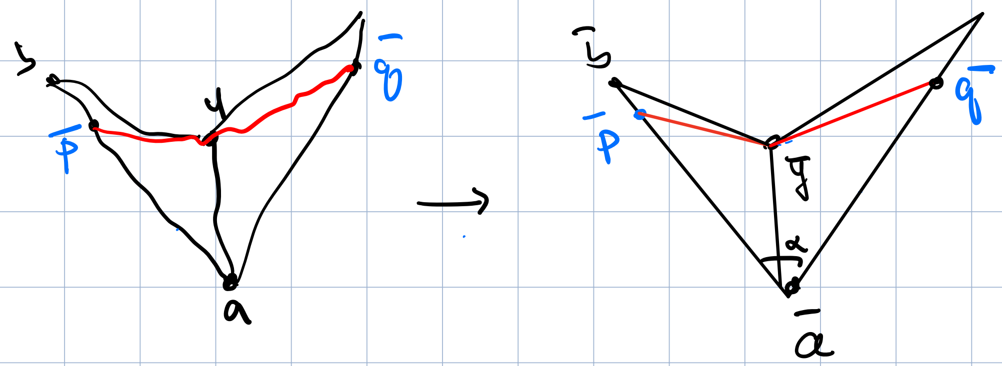
$d(p,q) \le d(p,r) + d(r,q) \le d(\bar p, \bar r) + d(\bar r, \bar q)$

(because Δ_1, Δ_2 are CAT(0))

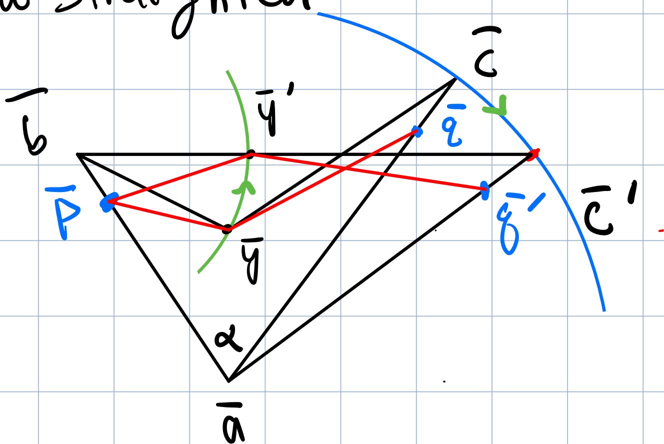
$= d(\bar p, \bar q)$

$\le d(\bar p, \bar q')$ since $\alpha' > \alpha$

If $p \in [a,b]$, $q \in [a,c]$ and $[\bar{p}, \bar{q}]$ leaves Q:



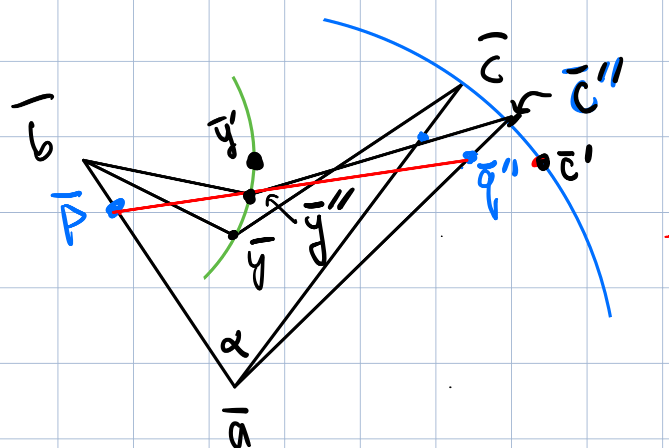
Then when you straighten



The angle $\bar{p}\bar{y}\bar{q}$ is $< \pi$

The angle $\bar{p}\bar{y}'\bar{q}'$ is $> \pi$

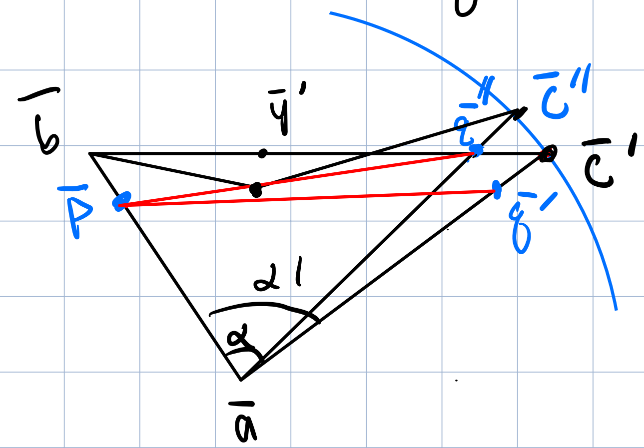
So somewhere in between it is $= \pi$



since the angles at $\bar{b}$ and $\bar{c}$ have grown,

$$d(\bar{p},\bar{y}'') > d(\bar{p},\bar{y})$$
$$\text{and } d(\bar{y}'',\bar{c}'') > d(\bar{y},\bar{c})$$

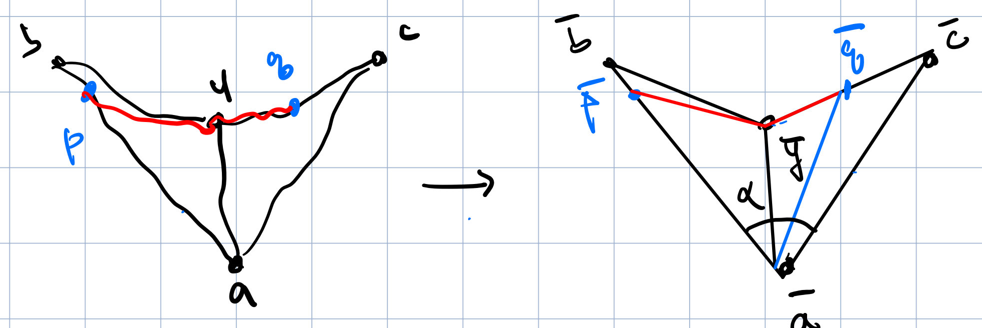
Now continue straightening



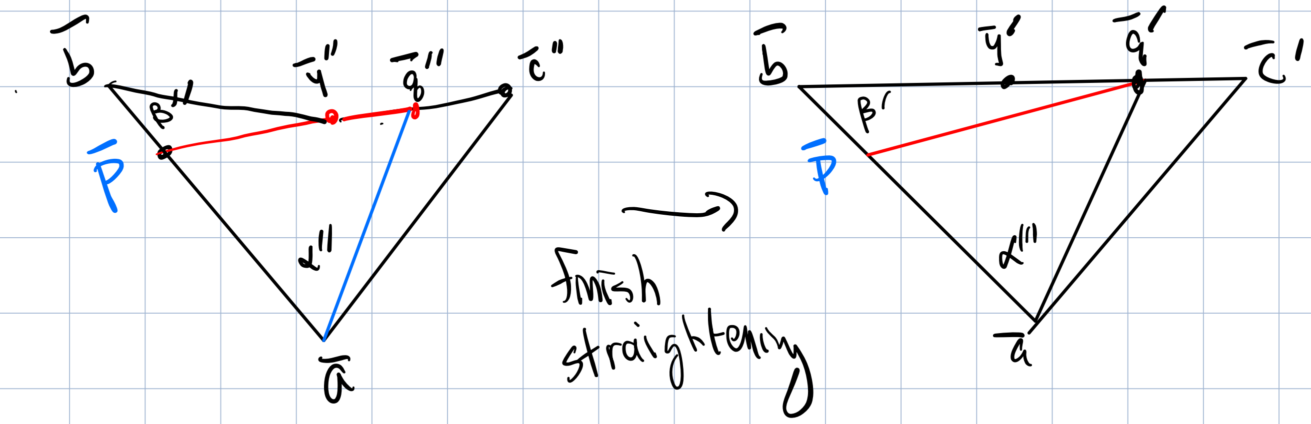
Since $\alpha' > \alpha$, $d(\bar{p},\bar{q}') > d(\bar{p},\bar{q}'')$

So in all
$$\begin{aligned} d(p,q) &= d(p,y) + d(y,q) \\ &\leq d(\bar{p},\bar{y}) + d(\bar{y},\bar{q}) \quad \text{since both small triangles are CAT(0)} \\ &\leq d(\bar{p},\bar{y}'') + d(\bar{y}'',\bar{q}'') = d(\bar{p},\bar{q}'') \\ &\leq d(\bar{p},\bar{q}') \quad \checkmark \end{aligned}$$

If $p \in [a,b]$, $q \in [a,c]$ and $[\bar{p},\bar{q}]$ leaves Q:



partially straighten



Since β increases monotonically as you straighten

$$d(p,q) = d(p,y) + d(y,q) \le d(\bar p,\bar y) + d(\bar y,\bar q)$$
$$\le d(\bar p,\bar y'') + d(\bar y,\bar q'') \quad \text{since } \beta'' > \beta$$
$$< d(\bar p,\bar q') \quad \text{since } \alpha'' < \alpha''' \quad \checkmark$$

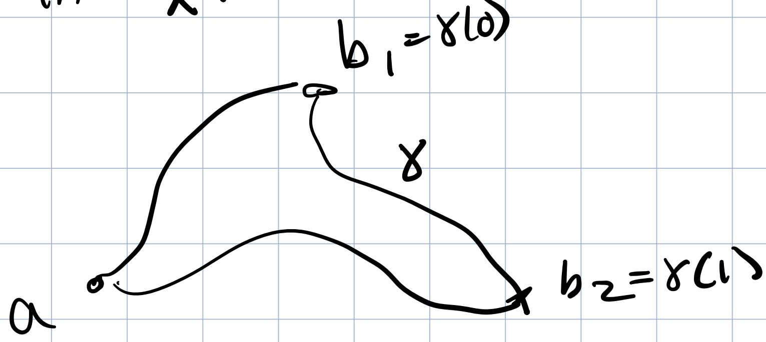
This finishes the proof of Alexandrov's Lemma

The proof of the CAT(0) version of the Cartan-Hadamard theorem is in Bridson-Haefliger. It is not hard but it is long, and we will not do it.

Using Alexandrov's Lemma and the Cartan-Hadamard Theorem, we can now prove

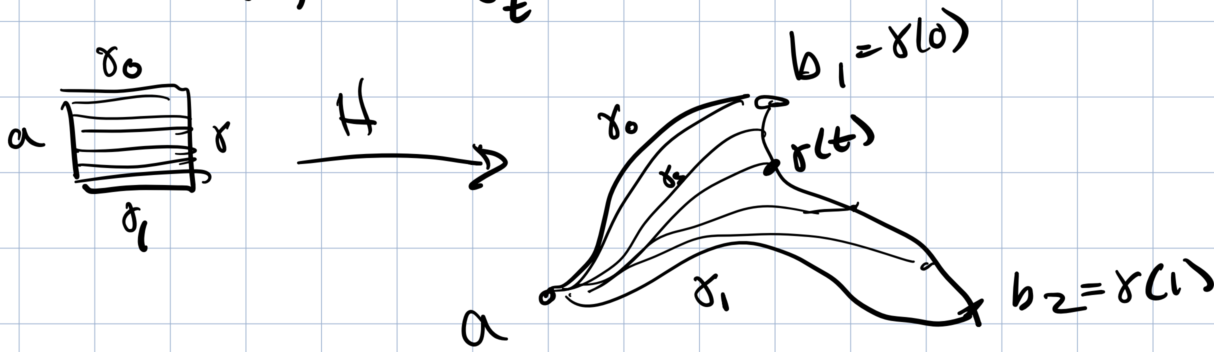
<u>Thm:</u> X locally CAT(0) (= NPC), Then $\tilde X$ is CAT(0).

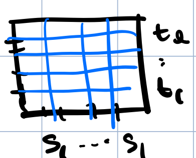
Proof: Let $\Delta(a, b_1, b_2)$ be a geodesic triangle in $\tilde{X}$.



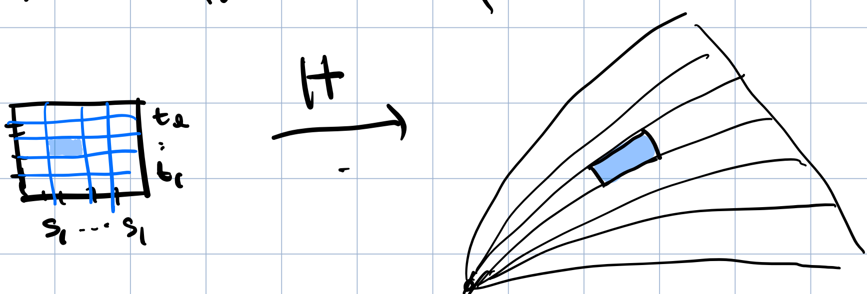
By Cartan-Hadamard, there is a unique geodesic γ_t from a to $\gamma(t)$, that varies continuously with $\gamma(t)$

This gives a map $H: [0,1] \times [0,1] \to X$ with $H(s,t) = \gamma_t(s)$

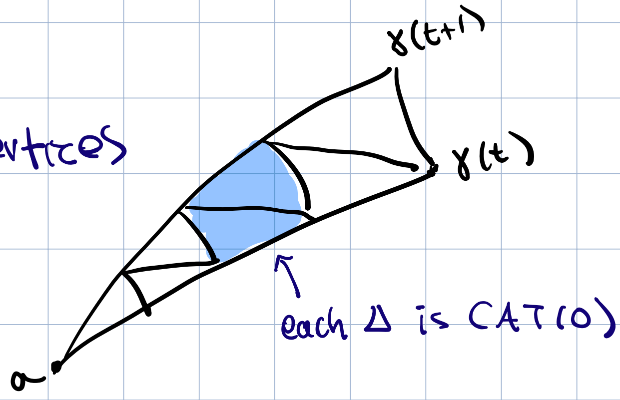


Divide  into rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$

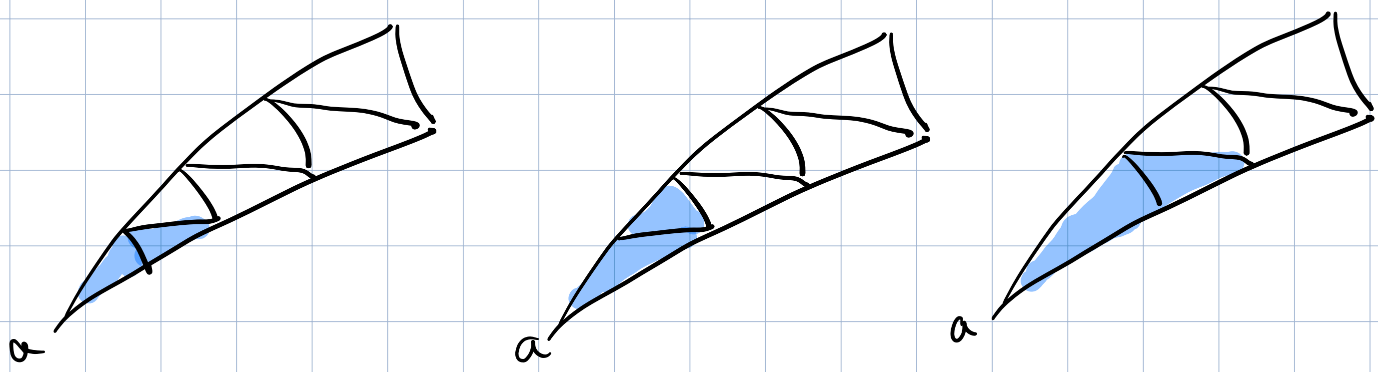
small enough so that each maps into a CAT(0) ball.



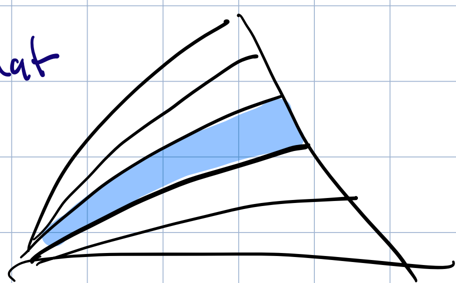
Look at one of the triangles with vertices $a, \gamma(t), \gamma(t+1)$, cut the blue squares into triangles:



Apply Alexandrov's lemma repeatedly



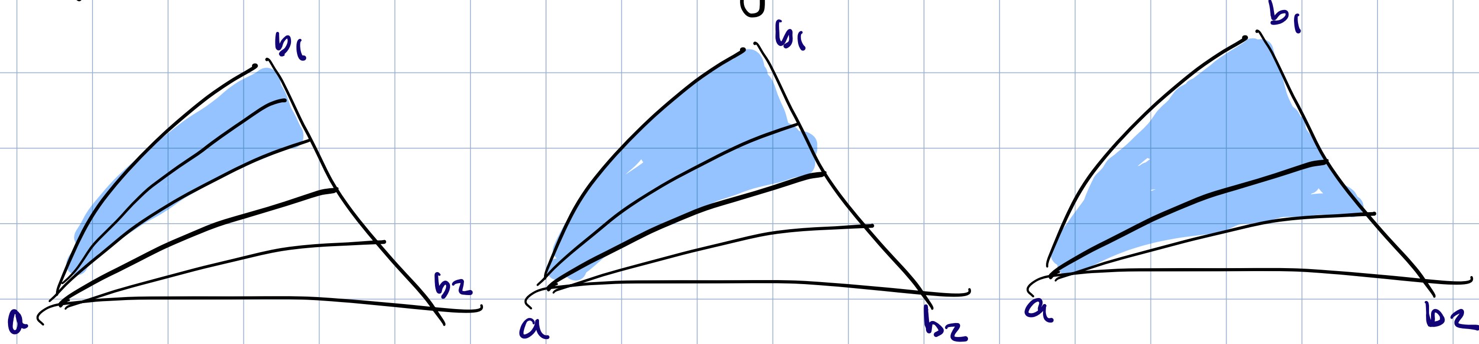
get that



is CAT(0)

Do this for each Δ.

Then use Alexandrov again



To get the whole triangle ab_1b_2 is CAT(0). ✓

We can now look for compact locally CAT(0) spaces; their fundamental groups will be CAT(0) groups

It is still non-trivial to find locally CAT(0) spaces X
but this is now a local problem

CAT(0) groups have many nice properties in common with hyperbolic groups, but the proofs are usually very different.

For example: Let G be a CAT(0) group. Then there are only finitely many conjugacy classes of finite-order elements of G.

For G hyperbolic, you did this in your exercises, using the fact that long ($>16\delta$) loops in a δ-hyperbolic Cayley graph have short ($<8\delta$) segments that are not geodesic

That's not generally true in CAT(0) groups—
eg in $\mathbb{Z}^2$ it's not true.

Next idea: Show any finite set in a CAT(0) space has a unique "center"!

Then look at an orbit of a finite-order element $h \in G$

$$A = x, hx, h^2x \dots h^n x = x$$

This is a finite set, so has a center. c_A

The action of h sends A to itself, so fixes c_A

The action of G on X is proper, so h is in the finite group $\mathrm{stab}(c_A)$.

(Note: this argument actually works for any finite subgroup $H < G$, not just cyclic ones)

Then we can use cocompactness of the action to show there are only finitely many possibilities for $\mathrm{stab}(c_A)$, then that any finite order element is conjugate into one of these.

So how do we find a canonical center for a finite set?

Let A be a finite set. To find a center:

For any $x \in X$, define
$$r(x,A) = \max_{a \in A} d(x,a) \qquad (\text{so } A \subset \overline{B}_r(x))$$

And $$r(A) = \inf_{x \in X} r(x,A)$$

If $r(x,A) = r(A)$, x is called a circumcenter for A.

Proposition Every finite set A has a unique circumcenter.

To prove this, we will prove an inequality relating $r(x,A)$, $r(y,A)$, $d(x,y)$ and $r(A)$:

$$\boxed{d(x,y)^2 \leq 2\left(r(x,A)^2 + r(y,A)^2 - 2r(A)^2\right)}$$

This immediately gives uniqueness:

If $r(x,A) = r(A)$ and $r(y,A) = r(A)$
then $d(x,y) = 0$

To get existence, take a sequence $\{x_n\}$ of points in X such that

$$r(x_n, A) \longrightarrow r(A)$$

Then $d(x_i, x_j)^2 \leq 2\left(r(x_i, A)^2 + r(x_j, A)^2 - 2r(A)\right)$

$$\longrightarrow 0 \text{ as } i, j \to \infty$$

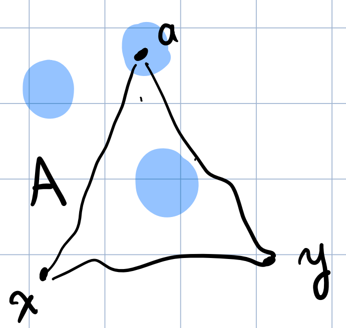
So $\{x_n\}$ is a Cauchy sequence

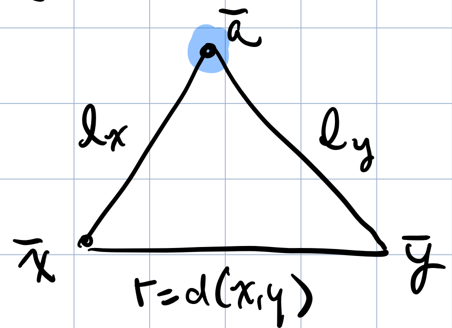
The Cartan-Hadamard theorem implies that a CAT(0) space with a proper cocompact group action must be complete,

$$\text{so } \{x_n\} \longrightarrow x \in X$$

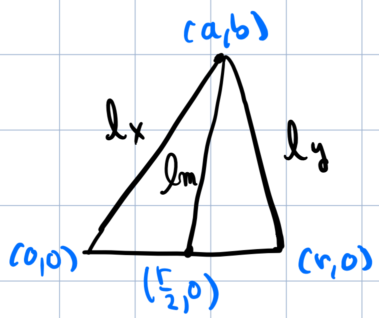
$$\text{and } r(x_n, A) \longrightarrow r(x, A) \checkmark$$

To prove the boxed inequality, form a geodesic triangle with vertices x, y and some $a \in A$ then consider the comparison triangle in $\mathbb{R}^2$.





We may assume $\bar{x} = (0,0)$ and $\bar{y} = (r, 0)$:



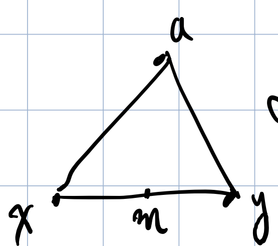
$$l_x = \sqrt{a^2+b^2} \qquad l_y = \sqrt{(a-r)^2+b^2}$$

$$l_m = \sqrt{\left(a-\tfrac{r}{2}\right)^2+b^2}$$

So $\frac{1}{2}(l_x^2+l_y^2) = \frac{1}{2}\left(a^2+b^2+(a^2-2ar+r^2+b^2)\right)$
$$= a^2+b^2-ar+\frac{r^2}{2}$$

and $l_m^2 = a^2-ar+\frac{r^2}{4}+b^2$
$$= \frac{1}{2}(l_x^2+l_y^2)-\frac{r^2}{4} = \frac{1}{2}(l_x^2+l_y^2)-\frac{d(\bar{x},\bar{y})^2}{4}$$

So back in X we have



$$d(m,a)^2 \leq l_m^2 = \frac{1}{2}\left(\overset{l_x}{\overset{\parallel}{d(x,a)}}{}^2+\overset{l_y}{\overset{\parallel}{d(a,y)}}{}^2\right)-\frac{\overset{d(x,y)}{\overset{\parallel}{d(x,y)}}{}^2}{4}$$

or

$$d(x,y)^2 \leq 2\left(d(x,a)^2+d(y,a)^2-2d(m,a)^2\right)$$

For a with $d(m,a)$ maximal, ie $d(m,a)=r(m,A)$

we get $d(x,y)^2 \leq 2\left(d(x,a)^2+d(y,a)^2-2r(m,A)^2\right)$
$$\leq 2\left(r(x,A)^2+r(y,A)^2-2r(m,A)^2\right)$$
Since $r(m,A) \geq r(A)$ this becomes

$$d(x,y)^2 \leq 2\left(r(x,A)^2+r(y,A)^2-2r(A)^2\right) \quad \checkmark$$

Now we can prove the theorem.

<u>Theorem</u> A CAT(0) group G has only finitely many conjugacy classes of finite subgroups.

<u>Proof</u>

Suppose H is a finite subgroup of a CAT(0) group G.

G acts properly and cocompactly by isometries on a CAT(0) space X, so H fixes the circumcenter x of an orbit, ie H is contained in the (finite) stabilizer G_x.

Cover X by translates of a compact set K containing x
The action is proper, so $\Sigma = \{g \mid gK \cap K \neq \emptyset\}$ is finite. Since $G_y \subset \Sigma$ for <u>any</u> $y \in K$, there are only finitely many possibilities for G_y

If H fixes some $y \notin K$ then $y \in gK$ for some g, so $g^{-1}y \in K$ and $G_y = g G_{g^{-1}y} g^{-1}$ is conjugate to (one of the finitely many) stabilizers of points in K. ✓

Remark: It is also true that hyperbolic groups have only finitely many conjugacy classes of finite subgroups – but we only proved it for finite cyclic subgroups

Like hyperbolic groups, CAT(0) groups are finitely presented, but the proof is much different

$G \curvearrowright X$ cocompactly, so Svarz-Milnor $\Rightarrow$ they are finitely generated. (same proof as for hyperbolic groups)

Recall you get the finite generating set S by taking a compact set K whose translates cover X, and letting $S = \{g \in G \mid gK \cap K \neq \emptyset\}$ – this is finite by properness of the action

For hyperbolic groups we then built the Cayley graph, which is δ-hyperbolic for some δ, and let $R = \{$reduced words w in $S \cup S^{-1}$ giving loops in $\mathcal{C}(G,S)$ of length $< 16\delta\}$

For CAT(0) groups this obviously doesn't work; instead take $R = \{s_1 s_2 s_3^{-1} \mid s_i \in S,$ $s_1 s_2 = s_3$ and $K \cap s_1 K \cap s_3 K \neq \emptyset\}$

Theorem $G \cong \langle S \mid R \rangle$, ie $\ker F(S) \longrightarrow G$ is normally generated by R.

Proof: Let $\mathcal{C}(G,S)$ be the Cayley graph for G.

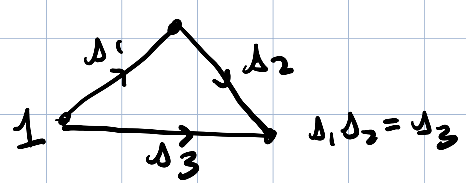
Note there is an edge $g \xrightarrow{e} h$ if and only if $h = gs$ for some $s \in S$

ie $g^{-1}h \in S$

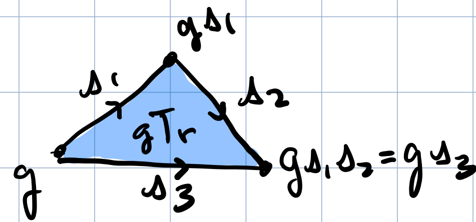
ie $K \cap g^{-1}hK \neq \emptyset$

ie $gK \cap hK \neq \emptyset$

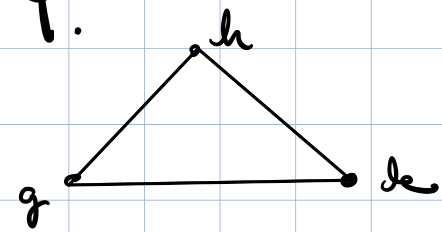
For each relation $s_1 s_2 s_3^{-1}$



Glue triangles gT_r to $\mathcal{C}$:



Call the resulting complex Y.

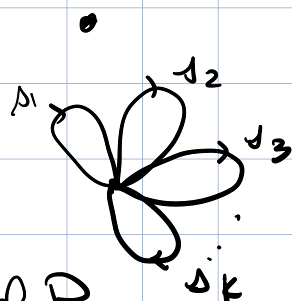


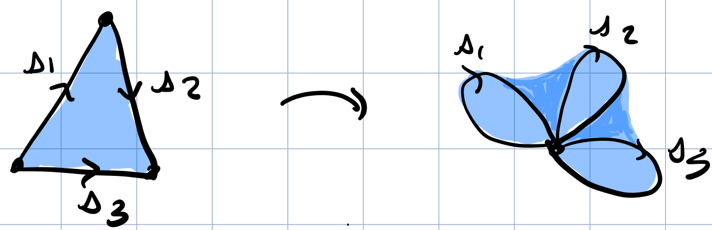
The relation says Y has a triangle whenever

$$gK \cap hK \cap kK \neq \emptyset.$$

G acts freely and properly on Y, so $Y \to G\backslash Y$ is a covering map. The quotient $Q = G\backslash Y$ has

- One vertex
- One loop for each element of S
- One triangle for each element of R





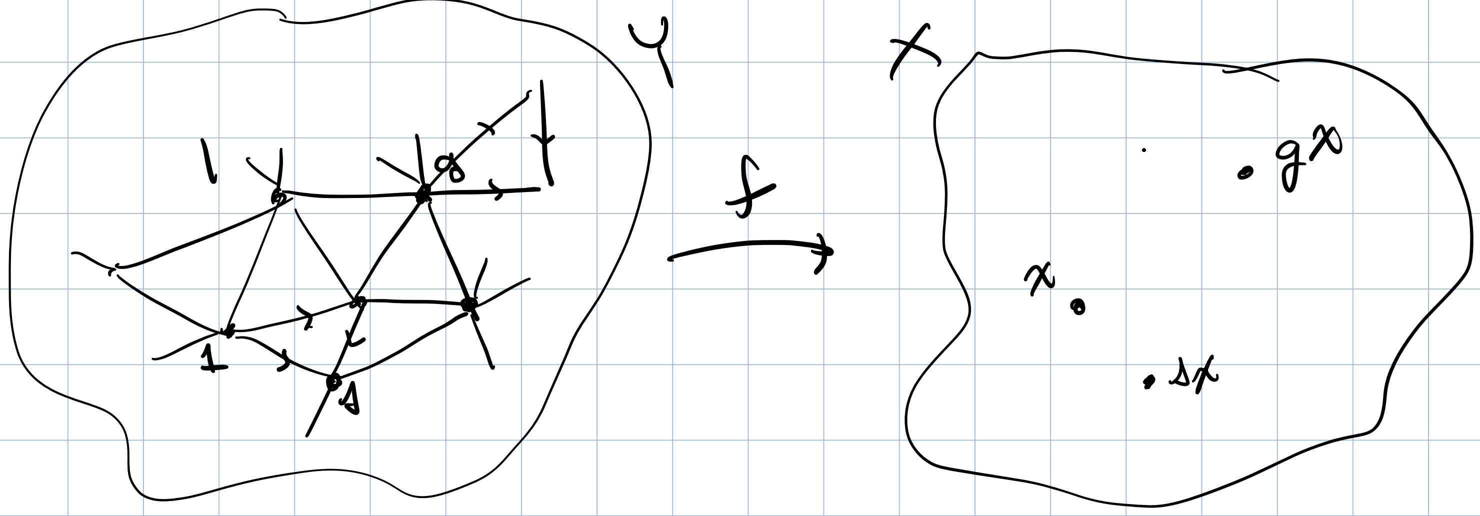
So $\pi_1(Q) = \langle S \mid R \rangle$ by Van Kampen's Theorem.
If we can show Y is simply connected, then Y is the universal cover of Q and G is the group of deck transformations,

so $G \cong \pi_1 Q = \langle S \mid R \rangle$.

We don't know whether Y is simply-connected, but we do know that X is.
The idea is to define a map $f : Y \to X$, and use the fact that loops $\sigma : S^1 \to X$ can be extended to maps $D^2 \to X$ to show the same is true for Y.

We define f first on vertices, then edges, then triangles:

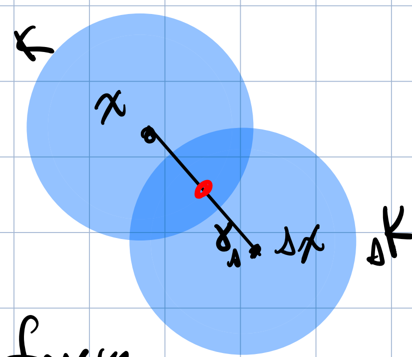
So, choose $x \in X$ and define $f(g) = g \cdot x$



$1 \xrightarrow{e} s$ is an edge $\iff K \cap sK \neq \emptyset$,

so connect x and sx by a path γ_s in $K \cup sK$,

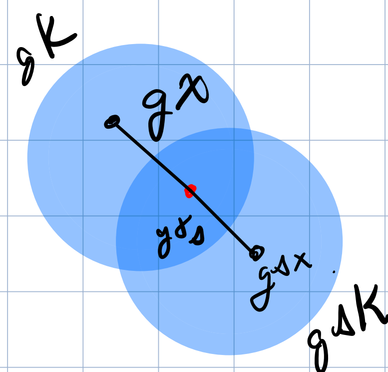
and define $f(e) = \gamma_s$

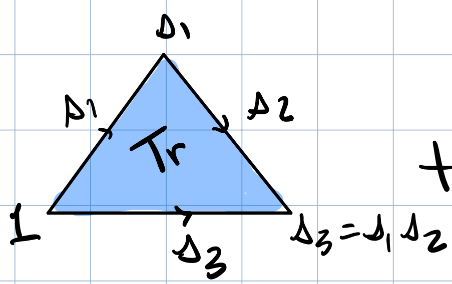


Every edge in Y is of the form

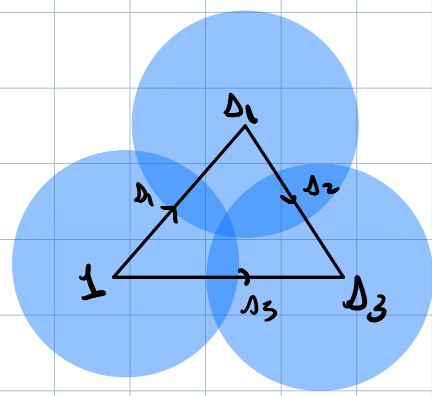
$$g \xrightarrow{ge} gs$$

Define $f(ge) = g\gamma_s$





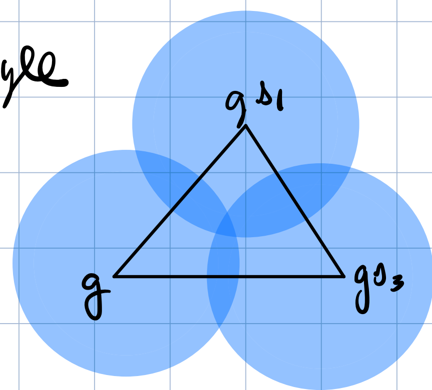
is a triangle in Y $\iff K \cap s_1 K \cap s_3 K \neq \emptyset$



Since X is 1-connected the map we've defined on on ∂T_r extends to a map $T_r \longrightarrow X$.
This defines f on T_r.

Now define f on an arbitrary triangle gT_r by

$$f(gT_r) = g \cdot f(T_r)$$

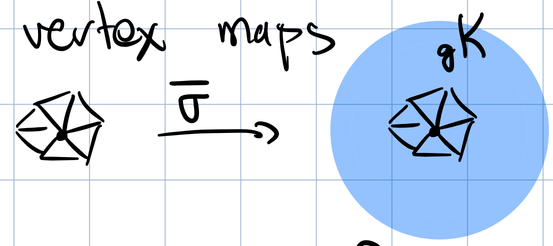


This completes the definition of $f : Y \longrightarrow X$

Now suppose we have a loop $l : S^1 \longrightarrow Y$.

$\sigma = f \circ l : S^1 \longrightarrow X$ can be extended to a map $\bar{\sigma} : D^2 \longrightarrow X$ since X is simply-connected.

Triangulate D^2 and subdivide enough so that the entire link of every vertex maps into gK for some K

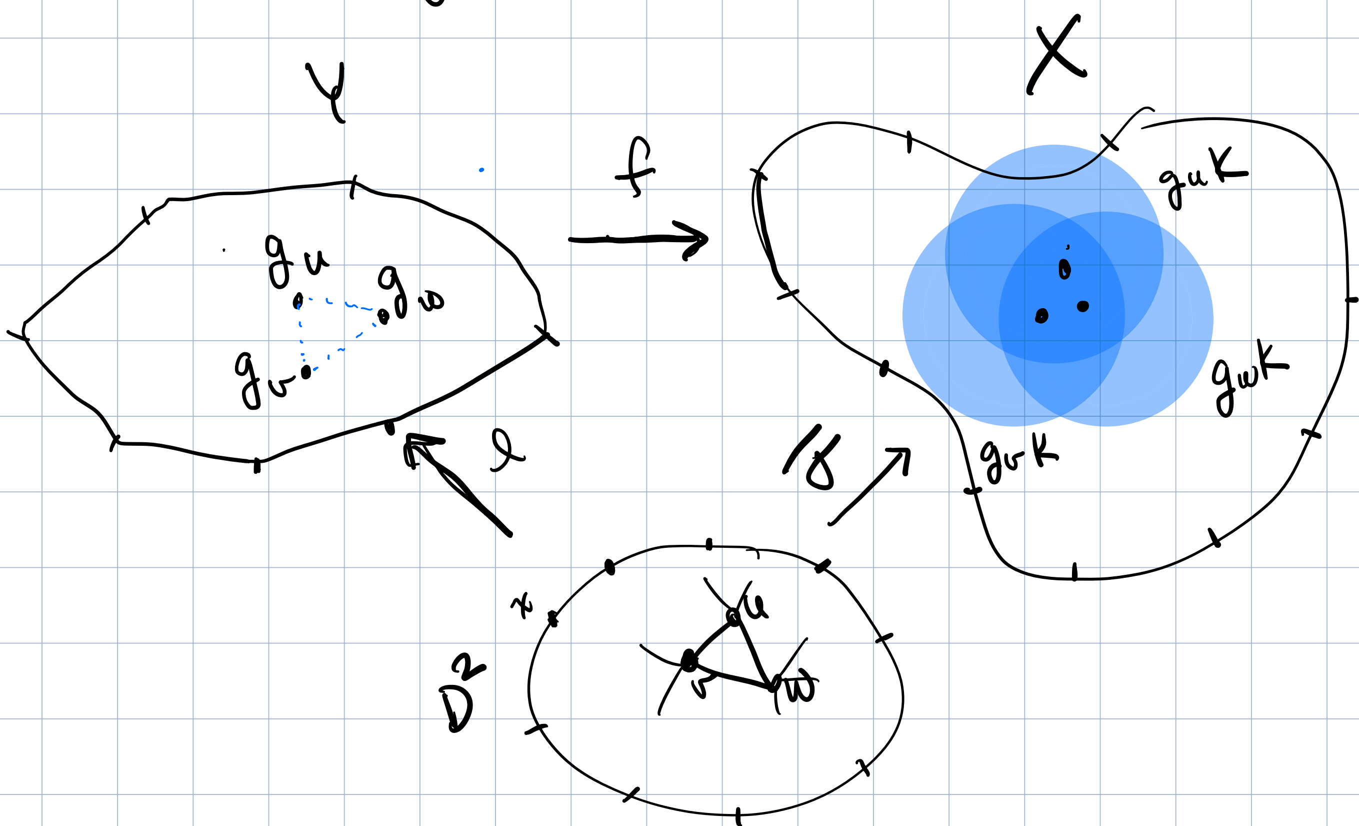


Let $\Delta(u,v,w)$ be one of the triangles in D^2

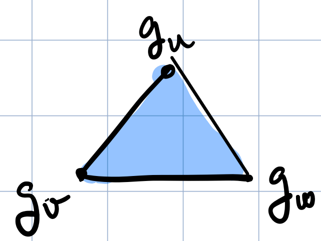
$\bar\sigma(u) \in g_u K$ for some g_u

$\bar\sigma(v) \in g_v K$ for some g_v

$\bar\sigma(w) \in g_w K$ for some g_w



Now g_uK also contains the images of v and w, since they are in $\mathrm{lk}(u)$. Similarly, g_vK and g_wK contain $\bar\sigma(u), \bar\sigma(v)$ and $\bar\sigma(w)$, so all 3 are contained in $g_uK \cap g_vK \cap g_wK$

This means  is a triangle in Y, call it T_{uvw}

so mapping $\Delta(u\,v\,w) \longrightarrow T_{uvw}$ for every triangle in D^2 extends $\ell : S^1 \to Y$ to a map $\overline{\ell} : D^2 \to Y$

This shows Y is simply-connected ✓

Recall we also showed an infinite hyperbolic group has an infinite-order element.

This is also true for CAT(0) groups.
For hyperbolic groups we used the fact that they have only finitely many cone types, but that may not be true for CAT(0) groups (at least our proof doesn't work...).

Other facts that need a new proof for CAT(0) groups:

We showed hyperbolic groups cannot contain $\mathbb{Z}^2$.

CAT(0) groups can, of course, but

* Every abelian subgroup is finitely generated
* Every solvable subgroup is virtually abelian (contains an abelian subgroup of finite index)

Subgroups of hyperbolic groups are not necessarily hyperbolic themselves. But you <u>can</u> say

In hyperbolic groups, subgroups are either virtually cyclic or contain a non-abelian free group.

Similarly, subgroups of CAT(0) groups are not necessarily CAT(0)

In CAT(0) groups, it is <u>conjectured</u> that all subgroups are either virtually abelian or contain a non-abelian free group, but this is still a conjecture (though proved in many cases.)

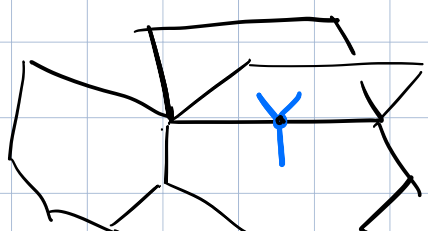
We saw that if X is <u>locally</u> CAT(0) and compact, then $\tilde{X}$ is CAT(0), so $\pi_1(X)$ is a CAT(0) group.

<u>Def</u> A <u>polygonal complex</u> X is a union of Euclidean polygons, glued together by isometries of their edges so that

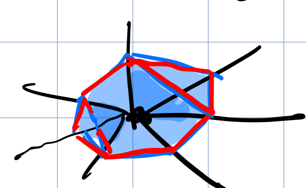
- each polygon injects into X and
- two polygons share at most one edge or vertex

If X is a polygonal complex, then

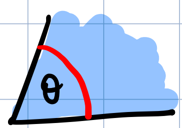
- If $x \in X$ is in the interior of a polygon it has a neighborhood isometric to a ball in $\mathbb{R}^2$
- If x is on an edge, it has a neighborhood that is a tree $T_\varepsilon \times (-\varepsilon, \varepsilon)$ so is CAT(0)



- If x is a vertex, a small neighborhood looks like a cone on a graph L_x, called the <u>link</u> of x



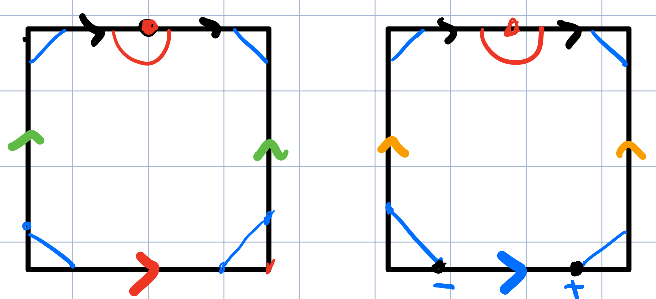
Each polygon has an angle at x –
use that to define a length
for the edge of L_x

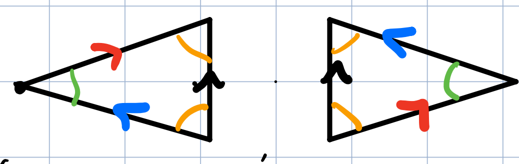


$\theta = \pi/3$

Gromov showed: The neighborhood of x
is CAT(0) if and only if there are no
loops of length $< 2\pi$ in L_x

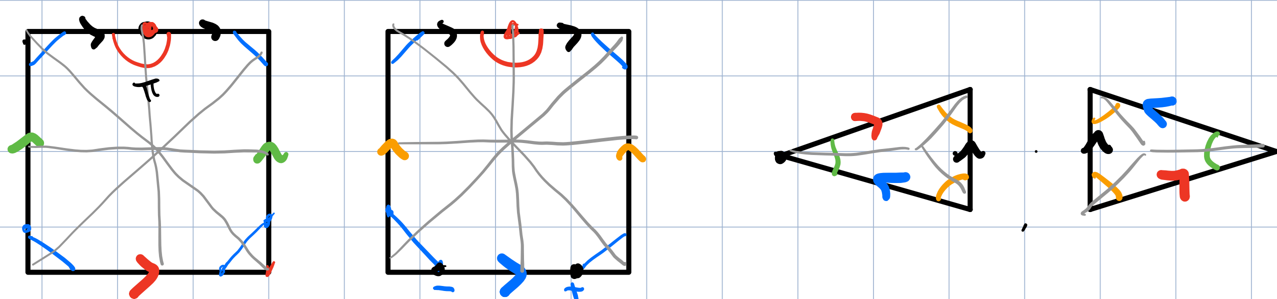
Interesting example





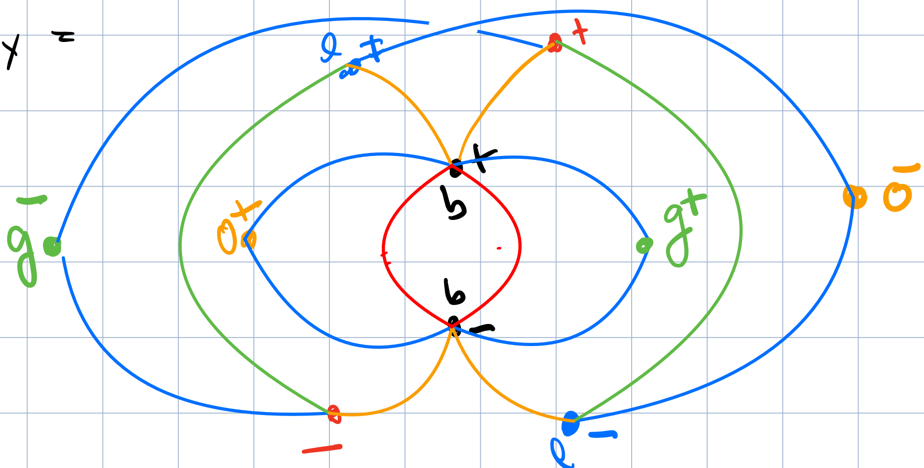
(angles are $\arccos(\frac{1}{4})$ and
$2\arcsin(\frac{1}{4})$)

(You have to subdivide to make this into
a polygonal complex by our definition)



Glue edges with the same colored arrow together
Then – There is only one vertex x, and 5 edges
b(lack), r(ed), o(range), g(reen), (b) blue

$L_X =$



You can check that there are no loops in L_X of length $< 2\pi$, so the resulting space X is CAT(0)

$$\pi_1 X = \langle b, r, g, o, l \mid g b^2 g^{-1} r^{-1},\ o b^2 o^{-1} l^{-1},\ l r b^{-1},\ r l b^{-1} \rangle$$

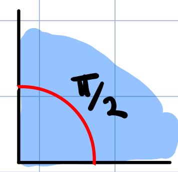
D. Wise found a surjective homomorphism

$$\pi_1 X \twoheadrightarrow \pi_1 X$$

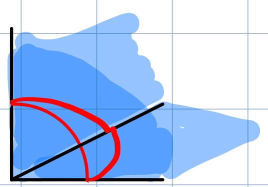
that is not injective.

That can't happen for linear groups (ie finitely generated subgroups of $GL(n, \mathbb{C})$)

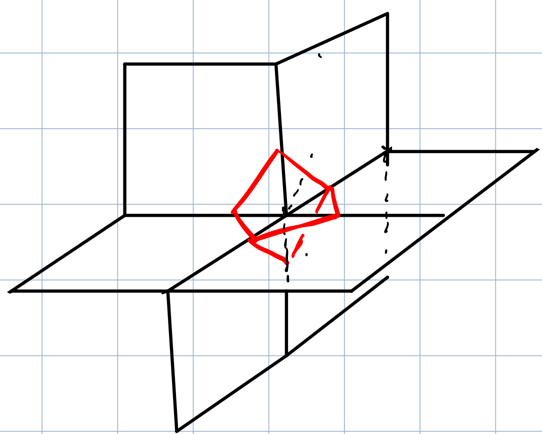
So that proves CAT(0) groups are not necessarily linear. (answering a question of Gromov)

If all the polygons are squares, the angles are 

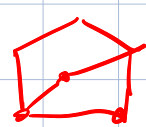
So Gromov's criterion says there can't be any triangles in L_x



This is not CAT(0)
(we already saw this example)



This is CAT(0)

L_x = 

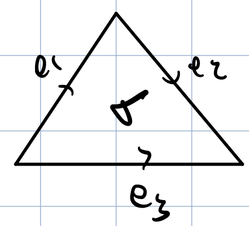
Gromov's link condition, appropriately defined, also applies to higher-dimensional spaces made out of polyhedra glued together. But it is difficult to verify — UNLESS the polyhedra are all cubes (or right-angled parallelepipeds)

Then it is called a cubical complex.

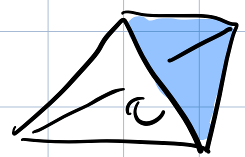
In that case a simple combinatorial condition determines whether or not X is CAT(0).

Def A simplicial complex K is ___flag___ if whenever it contains the boundary of a simplex of dimension ≥ 2, it also contains the simplex.

ie



if all the e_1, e_2, e_3 are in K, then so is σ



if all faces of τ are in K, so is τ.

etc.

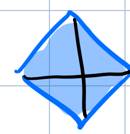
In a cubical complex the link of every vertex is a simplicial complex (by our restrictions on the gluing)

Thm (Gromov) A cubical complex is CAT(0) if and only if the link of every vertex is a flag complex.

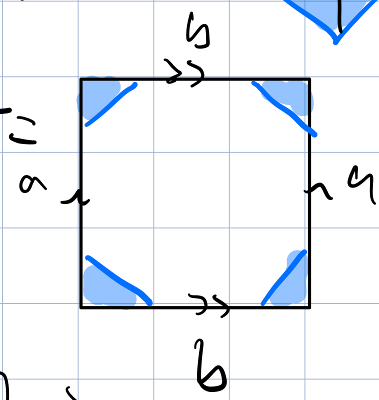
Note that you only have to check the vertices!

Now for some examples:

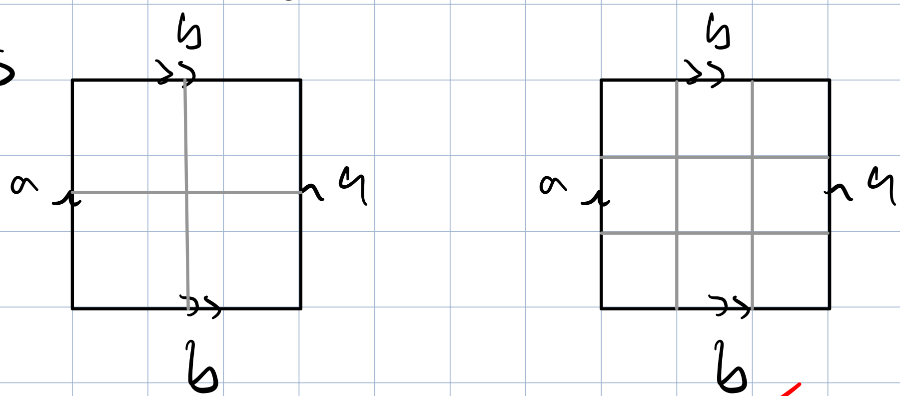
We already know $\mathbb{Z}^2$ is CAT(0)



view $\mathbb{Z}^2$ as $\pi_1(T^2)$, $T^2 =$



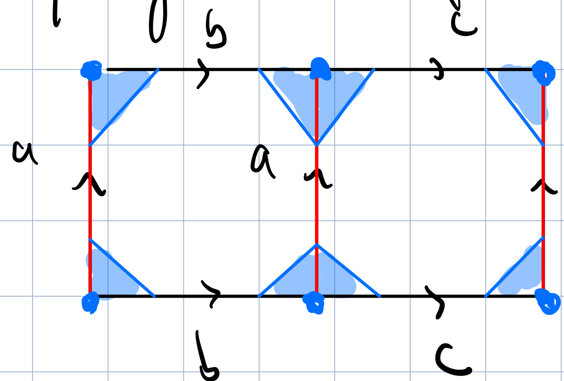
(This is not quite a cubical complex by our definition, but we can subdivide so it is

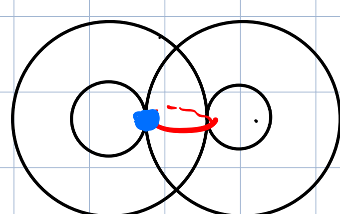


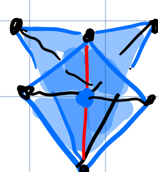
NOT YET ✓

All the vertices we added have Euclidean neighborhoods, so the only one we really have to check is the original vertex)

Slightly more complicated example: $T^2 \cup_{S^1} T^2$







✓

Right-angled Artin groups (RAAGs)

<u>Definition</u> A RAAG is a group given by a finite presentation $\langle S \mid R \rangle$ where all elements of R are of the form $s_i s_j s_i^{-1} s_j^{-1}$ (ie $s_i s_j = s_j s_i$) with $s_i, s_j \in S$.

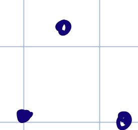
The presentation is conveniently encoded by drawing a graph Γ with

- one vertex for each element of S
- one edge from s_i to s_j for each element $s_i s_j s_i^{-1} s_j^{-1}$ of R

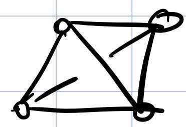
The RAAG is then denoted A_Γ.

<u>Examples</u>

① Γ is discrete (ie has no edges, only vertices)
Then $A_\Gamma = F(S)$, the free group on S.

$\Gamma =$  $\qquad A_\Gamma = F_3$

② Γ is complete (all possible edges - ie Γ is a clique) then $A_\Gamma = \mathbb{Z}^{|S|}$, the free abelian group of rank $|S|$.

$\Gamma =$  $A_\Gamma = \mathbb{Z}^4$

③ If Γ is a join $\Gamma_1 * \Gamma_2$

(every vertex of Γ_1 is connected to every vertex of Γ_2)

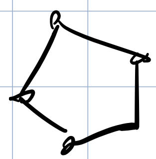
Then $A_\Gamma = A_{\Gamma_1} \times A_{\Gamma_2}$

④ If Γ is a disjoint union $\Gamma_1 \sqcup \Gamma_2$

then $A_\Gamma = A_{\Gamma_1} * A_{\Gamma_2}$ (free product)

(very unfortunate notation)

Most graphs don't fall into one of these categories, and A_Γ can have surprising properties:

eg • Γ =  Then A_Γ has a subgroup isomorphic to $\pi_1 S_g$ $(g>2)$

• if Γ is a <u>tree</u>, then A_Γ is isomorphic to the fundamental group of a closed 3-manifold

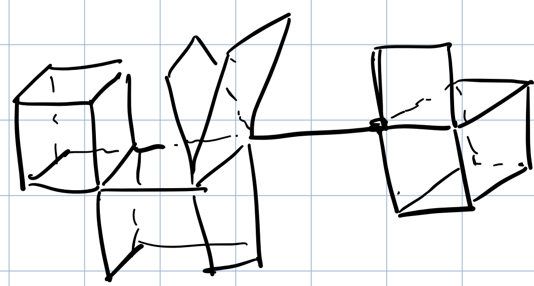
RAAGs have been important in group theory and low-dimensional topology partly because they can have very surprising subgroups, such as surface groups and 3-manifold groups so they are a source of counterexamples in group theory and information about the structure of 3-manifolds in topology.

<u>Claim</u> Every RAAG A_Γ is CAT(0).

We've seen that it suffices to find a NPC space X with $\pi_1 X = A_\Gamma$

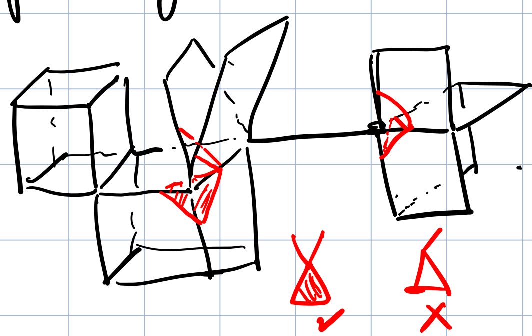
Recall a <u>cubical</u> <u>complex</u> X is a union of Euclidean cubes, glued together by isometries of their faces so that

- each cube injects into X and
- two cubes share at most one face

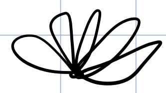
eg $X =$ 

Also recall Gromov's link condition for cube complexes:

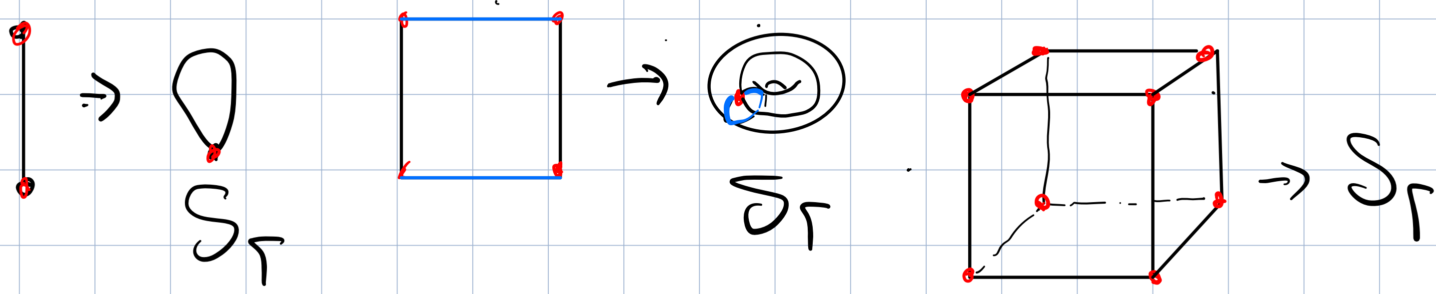
<u>Gromov</u>: X is CAT(0) if and only if the link of every vertex is a <u>flag</u> complex



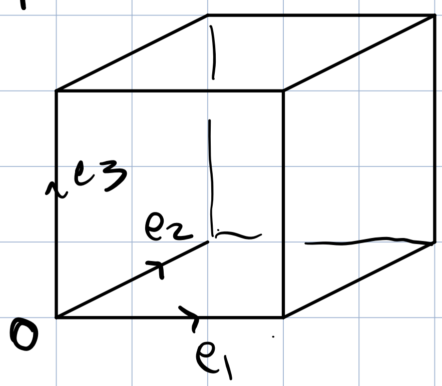
We will find a cube complex S_Γ with $\pi_1(S_\Gamma) \cong A_\Gamma$ that satisfies Gromov's link condition

For Γ discrete, $S_\Gamma =$ 

For Γ a clique, $S_\Gamma = T^n = [0,1]^n /$ opposite faces identified



For arbitrary Γ, S_Γ is a subcomplex of T^n defined as follows:

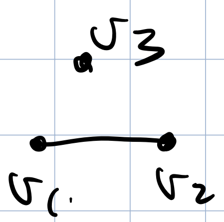
$I^n =$ 

$v_1, \dots, v_n$ = vertices of Γ

$v_i \leftrightarrow e_i$

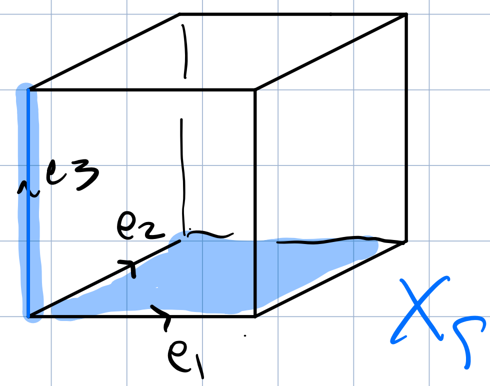
Each subset of $\{e_1, \dots, e_n\}$ defines a face of I^n.

Let $X_\Gamma \subset I^n$ be the subcomplex consisting of the faces spanned by cliques $v_{i_1} \dots v_{i_k}$ in Γ

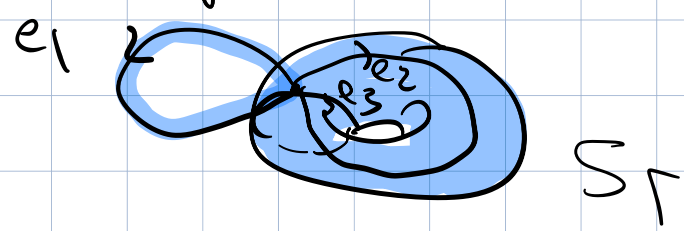
eg $\Gamma =$ 

$v_1v_2 \leftrightarrow \langle e_1, e_2 \rangle$

$v_3 \leftrightarrow \langle e_3 \rangle$



Then S_Γ = image of X_Γ in T^n



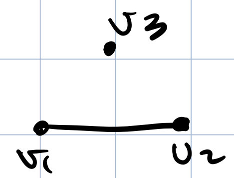
Another way to describe S_Γ:

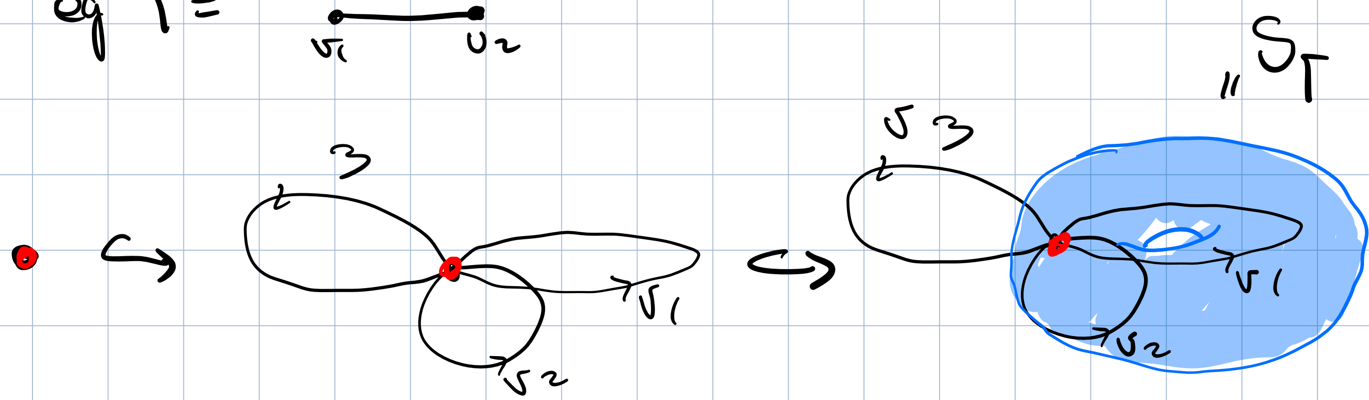
It has
- one vertex

You attach
- an edge for each vertex of Γ (forming a loop)
- a square for each triangle in Γ (forming a 2-torus)

⋮

- a k-dimensional cube for each k-clique in Γ

eg $\Gamma =$ 

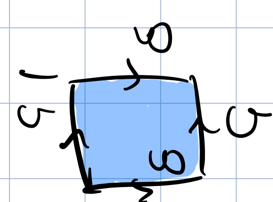


<u>Claim</u>: • $\pi_1(S_\Gamma) \cong A_\Gamma$

• S_Γ is locally CAT(0)

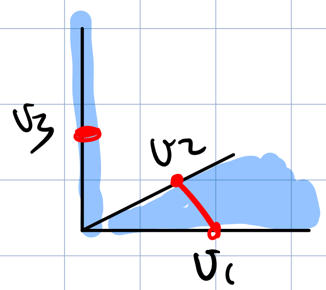
To prove the first claim, we only have to check the 2-skeleton ($\pi_1(X^{(2)}) = \pi_1 X$ for any CW-complex)

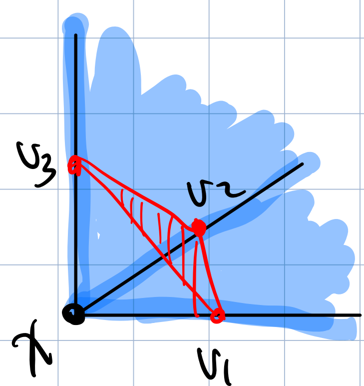
This has • one generator for each vertex of Γ

• A 2-cell  iff v and w are connected by an edge of Γ.

so $\pi_1 S_\Gamma = A_\Gamma$ by van Kampen. ✓

To see that it's CAT(0), just have to check the link at the unique vertex.



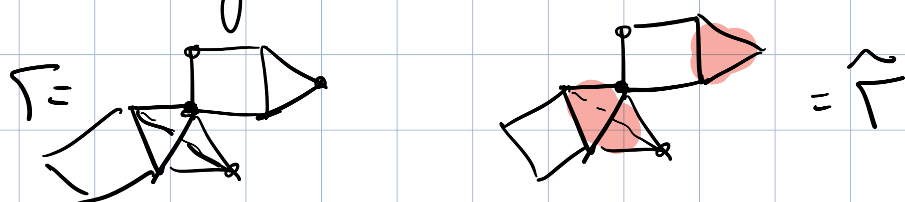


in X_Γ, this has

- one vertex for each vertex of Γ (= generator)
- one edge for each edge of Γ (= relation)
- one triangle for each triangle in Γ
- etc

— ie $Lk_{X_\Gamma}(x)$ is the flag completion of Γ.

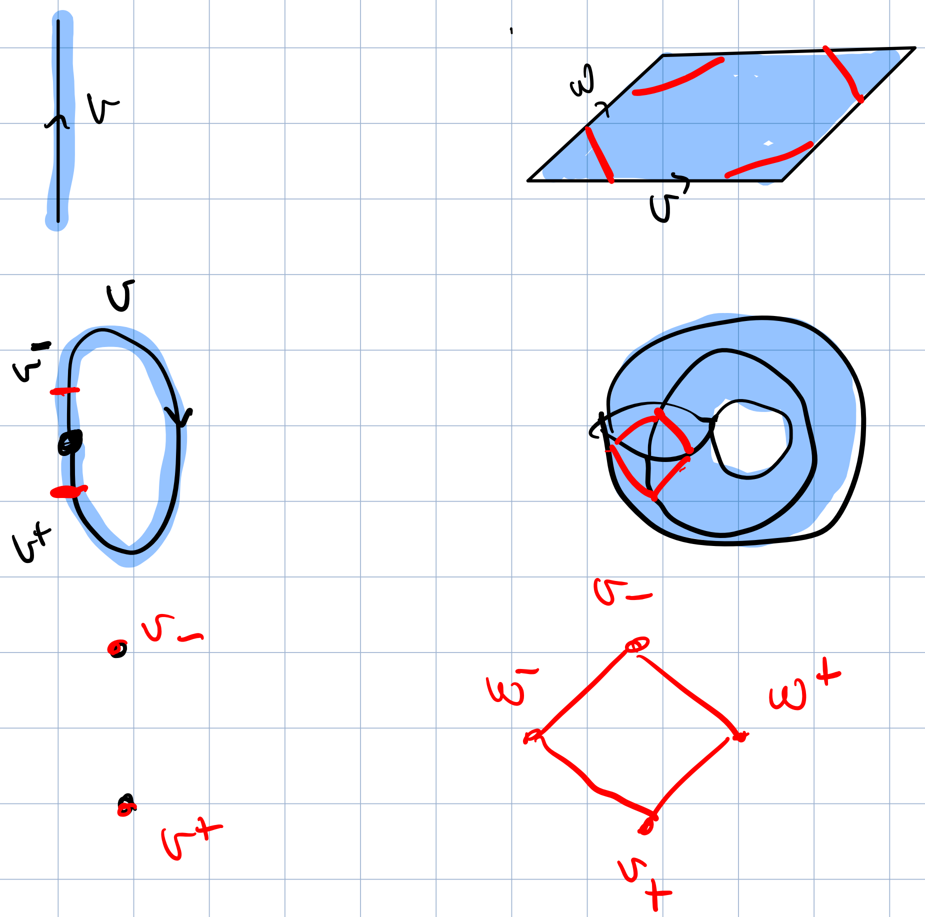
(If Γ is a simplicial graph, the flag completion $\tilde{\Gamma}$ of Γ is the simplicial complex formed by adding a k-simplex to Γ whenever Γ contains the 1-skeleton of a k-simplex



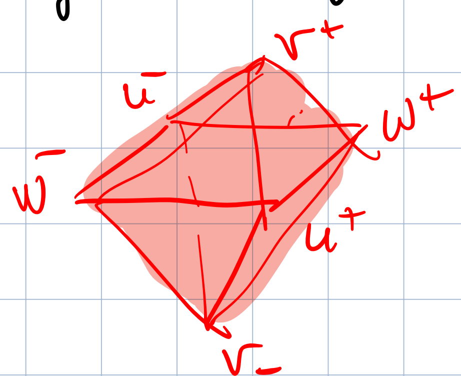
)

But we're interested in $Ile_{S_\Gamma}(x)$.

In S_Γ, each edge becomes a loop, so each generator of A_Γ corresponds to 2 vertices instead of 1, each edge v—w gives 4 edges

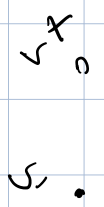


Each triangle in Γ gives 8 triangles in $d\Gamma$

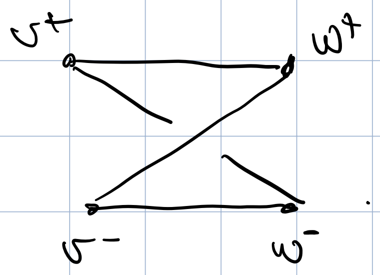


etc.

Def Let Γ be a simplicial graph.
The <u>double</u> $d\Gamma$ is a graph with



- 2 vertices v^+, v^- for each vertex of Γ
- 4 edges $\begin{cases} v^+ - w^+ \\ v^+ - w^- \\ v^- - w^+ \\ v^- - w^- \end{cases}$ for each edge $v - w$ of Γ



Then $Lk_{S_\Gamma}(x)$ is the flag completion of the double $d\Gamma$.

In particular, it is a flag complex, so S_Γ is CAT(0). ✓

RAAGs are special among CAT(0) groups in many ways.
- They are torsion-free.
- Davis and Januszkevich proved they embed in right-angled <u>Coxeter</u> groups

$$C_\Gamma = \langle v_1 \dots v_k \in V(\Gamma) \mid v_i v_j = v_j v_i \text{ if } v_i v_j \in E(\Gamma) \text{ and } v_i^2 = 1 \text{ for all } i \rangle$$

Theorem (Davis-Januszkiewicz): A_Γ is a subgroup of finite index in $C_{\Gamma'}$, where Γ' is all edges in $\Delta_V * \Gamma$ <u>except</u> the edges $v_i^- - v_i^+$.

C_Γ is always linear: Define
$$C_\Gamma \longrightarrow GL(\mathbb{R}^V)$$
$$v_i \longmapsto (\delta_i : \mathbb{R}^V \longrightarrow \mathbb{R}^V)$$
by
$$\delta_i(e_i) = -e_i$$
$$\delta_i(e_j) = e_j \quad \text{if} \quad v_i v_j = v_j v_i$$
$$\delta_i(e_k) = e_k + 2e_i \quad \text{otherwise.}$$
You can check that $\delta_i^2 = 0$ and $[\delta_i, \delta_j] = 0$ if $v_i - v_j$ is an edge.

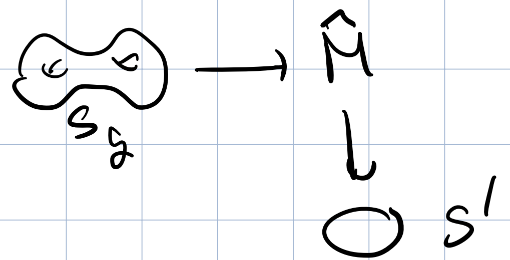
Proposition

$v_i \longmapsto \delta_i$ is a <u>faithful</u> linear representation.

Cor A_Γ is linear.

In particular, any surjective homomorphism $A_\Gamma \to A_\Gamma$ is an isomorphism.

This was a key fact in Agol's celebrated proof that finite-volume hyperbolic 3-manifolds are virtually Haaken (they have finite covers that fiber over a circle)



To prove this he proved $\pi_1 M$ has a finite-index subgroup that embeds into a RAAG. Since subgroups of linear groups and finite extensions of linear groups are linear, this shows $\pi_1 M$ is linear

$\boxed{\text{Recap:}}$ What we did in this course

1 Free groups (universal property, existence)

2. Group presentations $G = \langle S | R \rangle = F(S) / \langle\langle R \rangle\rangle$

3. Group actions: free, proper, cocompact, faithful
ping-pong
$\pi_1 X \curvearrowright \tilde{X}$ $(F_n, \mathbb{Z}^n, \pi_1 S_g \ldots)$
Galois correspondence
$SL_2\mathbb{Z} \curvearrowright \mathbb{H}^2$

4. Quasi-isometry and the Svarz-Milnor lemma

5. Cayley graphs

6. Ends (A group has 0, 1, 2 or ∞ many)

7. δ-Hyperbolic metric spaces and groups.
- hyperbolicity is a qi invariant
- finite presentation
- fin. many conj. classes of finite-order elts
- Centralizers of ∞-order elements ($\Rightarrow$ no $\mathbb{Z}^2$)
- Existence of ∞-order elements in ∞ hyperbolic groups
- <u>sketched</u>: ∞ many ends $\Rightarrow$ contains F_2

8. CAT(0) spaces and groups

- convexity of metric, $\Rightarrow$ contractible
- centers of finite sets $\Rightarrow$ fin. many conj. classes of finite order subgroups
- finitely presented

- Alexandrov's lemma ($\Rightarrow$ CAT(0) is a local condition if X is nice enough) and stated the Cartan-Hadamard theorem, which $\Rightarrow$ that if Y is locally CAT(0), then $X = \tilde{Y}$ is nice enough).

- Gromov's link condition

- RAAGs are CAT(0)