Thursday, March 13, 2014 – last class

What did we do in this course?

We started with free groups – defn (univ property) and existence (by construction). Then ping-pong – i.e. how to find free groups.

We briefly reviewed π_1 and covering spaces – TA did more – showed subgps of free gps are free.

Next was the geometry of the hyperbolic plane – geodesics, isometries, metric. Calculated length of line segments, area of circles. Showed triangles are thin, geodesics diverge exponentially and geodesics are contracting.

– Cayley graphs, Examples (F_n, $B(1,2)$, $\mathbb{Z}^n$, S_3,

– Hyperbolic metric space, hyperbolic group

- Quasi-isometry

  (G,S) q.i. to (G,S').

Towards $\longrightarrow$ Thm: Hyperbolicity is a qi-invariant

- Divergence of geodesics in hyperbolic X
- quasi-geodesics
- quasi-geodesics are bounded distance from geodesics
- Proof of theorem $X \sim X'$, X' hyperbolic $\Rightarrow$ X hyperbolic

Thm Hyperbolic groups are finitely presentable

Thm: Hyp gps have only finitely many conjugacy classes of finite elements.

<u>Švarc-Milnor lemma</u>: G acts properly & cocompactly on geodesic $X \Rightarrow G$ qi to X.

(and consequences: All F_n are qi, $\pi_1 S_g$ is hyperbolic, M^3, finite-index subgps of hyp gps are hyp, quotients by finite normal subgps

$\longrightarrow \partial$ of a hyperbolic space

3 definitions: rays, quasi-rays, sequences (Gromov product)

~~Defs~~ Defs are equivalent

$X \cup \partial X$ is compact

X qi $Y \to \partial X \approx \partial Y$

bdd cancellation for free group autos.

Fixed pt subgps fin gen.

∞ order elts of a hyperbolic group are quasi-geodesics, determine pts of ∂

There is a geodesic "joining" $g_{-\infty}$ to g_{∞}

g acts with North-South dynamics on $\partial(G,S)$

Ping-pong on $\partial(G,S)$ to find free subgroups

Quasi-convex subgroups

$\cap$ of q-c subgps is q-c

Howson's Thm for free groups

Hanna Neumann conjecture / (now Thm)

Some Key ideas: Thin Δ's $\Rightarrow$ 

$\Rightarrow (x,y)_w \geq \min((x,z)_w, (z,y)_w) - 2\delta$

<u>Δ inequality</u>   Švarc-Milnor

<u>IVT</u>   Quasi-geodesics close to geodesics

Building geodesics from ∞ sequences

Some things we <u>didn't</u> do

Gromov originally suggested that geometric techniques could be applied to study the classical combinatorial group theory problems about finitely-generated groups, namely the

word, conjugacy and isomorphism

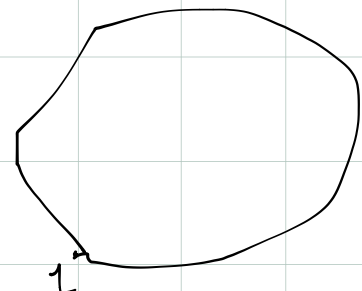
problems and variations (eg membership, equivalence under an automorphism, ...)

For free groups, we solved those problems.

<u>Max Dehn</u> solved them for surface groups

For hyperbolic groups:

<u>Word problem</u> solved by <u>Dehn's algorithm</u>



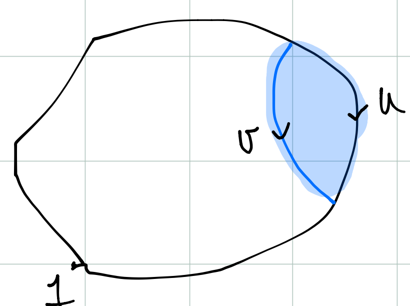
given $w \in F(S)$,

if it represents 1 in G,

Then the associated path in $\mathcal{C}(G,S)$ is a loop.

If it is long, $(\geq 8\delta)$ we can find a shortcut (see Lecture 14, Feb 10)

If we use as <u>relaters</u> all words of length $< 16\delta$, then



the blue patch (uv^{-1}) is a relater.

So, to solve the word problem. (Is $\overline{w} \underset{G}{=} 1$?)

① Write down all relaters of length $< 16\delta$

② Look for a subword u of w which is more than half of one of the relaters.

If you find such a u, replace it with the rest of the relator (a shorter word v) and start again.

If you can't find such a u, then $\bar{w}$ is not trivial.

This technique implies that in a hyperbolic group in order to "fill" a loop with relators you only need to use $\frac{|w|}{16\delta}$ of them, ie the number is a linear function of the length of the loop. $w = \prod_{i=1}^{k} u_i r_i u_i^{-1}$, $k \leq |w|/16\delta$.

The number of relators you need to fill a loop in this sense is called the <u>Dehn function</u> of the group, so hyperbolic groups have linear Dehn functions. In fact, Gromov proved that if a Dehn function is less than quadratic, then the group is hyperbolic (so the Dehn function is linear). There are groups with Dehn function $A \cdot \ell^{p/q}$ for all $p/q \geq 2$, however!

The conjugacy problem (is there an algorithm to decide whether $\overline{w}, \overline{w}'$ are conjugate in G?) is harder, but we do have the tools to do it, and could with another lecture or two.

The isomorphism problem (is there an algorithm to decide whether or not two hyperbolic groups are isomorphic) is <u>much</u> harder, solved in 1995 for torsion-free hyperbolic groups by Sela and in general by Dahmani and Guirardel in 2010

We haven't seen many concrete examples of hyperbolic groups, but there is a sense in which if you write down a "random" presentation, the

group you get will be hyperbolic
(Gromov, Champetier, ...)

Given a finite presentation, it is undecidable whether the corresponding group is hyperbolic.
However, it *may* be decidable whether *this* presentation is a Dehn presentation, i.e. whether $\overline{w} =_G 1 \Rightarrow w$ contains a subword which is more than half of a relator.

Note that the usual presentation for $\mathbb{Z}^2$ is *not* a Dehn presentation, and in fact the Dehn function is quadratic.

Some things no one knows about hyperbolic groups:

Do they all have finite-index subgroups with no torsion?

This turns out to be equivalent to:

Is every hyperbolic group residually finite?

We know surface groups are hyperbolic.

Does every 1-ended hyperbolic group contain a surface subgroup?

We know Baumslag-Solitar groups $B(1,2)$ $= \langle a, b \mid b^{-1}ab = b^{2} \rangle$ are <u>not</u> hyperbolic

In general $B(m,n) = \langle a, b \mid b^{-1}a^{n}b = a^{m} \rangle$ are not hyperbolic.

Q: If G is finitely-presented and doesnt contain any $B(m,n)$, is G hyperbolic

A: No (Brady)

Q': What if G has a finite $K(G,1)$?

(A $K(G,1)$ is a cell complex with $\pi_1 = G$ whose universal cover is contractible)

Hyperbolicity is an attempt to capture essential features of <u>negative</u> curvature.

Although hyperbolic groups are "generic" in some sense, in another sense they are very rare: <u>most</u> of the infinite discrete groups encountered in other areas of mathematics are <u>not</u> hyperbolic (eg $GL(n,\mathbb{Z})$)

Turns out related geometric ideas can also help there –

— Relatively hyperbolic groups:

"cone off" some subgroups (and all their cosets) in the Cayley graph — it may become hyperbolic

- CAT(0) geometry

A notion of non-positive curvature, also combinatorial in nature