MA4H4 - GEOMETRIC GROUP THEORY

Contents of the Lectures

1. Week 1

Introduction, free groups, ping-pong, fundamental group and covering spaces.

Lecture 1 - Jan. 6

(1) Introduction
(2) List of topics: basics, core topics, possible topics
(3) Definition of free group
(4) Proof of existence of free groups (part 1)

Lecture 2 - Jan. 7

(1) Proof of existence of free groups (part 2)
(2) Theorem: Free groups are isomorphic if and only if they have the same rank
(3) Action of F_2 on infinite telephone pole

Lecture 3 - Jan. 9

(1) Ping-pong lemma, statement and proof
(2) Application of ping-pong lemma to a matrix group
(3) Recall of fundamental groups and covering spaces (TA did this more thoroughly in support class).

2. Week 2

The geometry of the hyperbolic plane: geodesics, isometries and the metric.

Lecture 4 - Jan. 13

(1) Theorem: Subgroups of free groups are free.
(2) Introduction to the hyperbolic plane $\mathbb{H}^2$
(3) Definition of geodesics in $\mathbb{H}^2$
(4) Proposition: Any two points are joined by a unique geodesic
(5) Inversions in circles
(6) Proposition: Inversions preserve angles
(7) (Exercise: Inversions preserve circles)
(8) Definition of hyperbolic isometry

Lecture 5 - Jan. 14

(1) Proposition: $\text{Isom}(\mathbb{H}^2)$ acts transitively on $\mathbb{H}^2$. The stabilizer of a point is isomorphic to O_2.
(2) Proposition: Isometries preserve $\frac{ds}{1-r^2}$.

3. Week 3

More geometry of $\mathbb{H}^2$ and introduction to the Cayley graph.

Lecture 6 - Jan. 21

(1) Calculations in $\mathbb{H}^2$
  (a) Length of a ray
  (b) Circumference of a circle
  (c) Area of a circle
  (d) Radius of largest circle inscribed in a geodesic triangle
(2) Definition of projection onto a geodesic
(3) Proposition: Geodesic triangles in $\mathbb{H}^2$ are $\ln(3)$-thin.

Lecture 7 - Jan. 23

(1) Proposition: Geodesics in $\mathbb{H}^2$ are contracting
(2) Proposition: Geodesics in $\mathbb{H}^2$ diverge exponentially fast
(3) Definition of Cayley graph
(4) Examples of Cayley graphs for $\mathbb{Z}$, S_3.

4. Week 4

Examples of Cayley graphs, definitions of hyperbolic metric space, hyperbolic group, quasi-isometry and quasi-geodesic.

Lecture 8 - Jan. 27

(1) Examples of Cayley graphs for $S_3, F_2, \mathbb{Z}^2$.
(2) Definition of group presentation
(3) Statement of word, conjugacy and isomorphism problems.
(4) Example of an interesting presentation of the trivial group
(5) The Baumslag-Solitar group $B(1,2)$ and its Cayley graph

Lecture 9 - Jan. 28

(1) Definition of geodesic metric space
(2) Definition of (Gromov) hyperbolic metric space
(3) Definition of hyperbolic group
(4) Examples: finite groups, $\mathbb{Z}$, F_2 are hyperbolic
(5) Examples: $\mathbb{Z}^2$ is not hyperbolic, (Exercises: $B(1,2)$ is not hyperbolic)
(6) Fundamental domain for surface groups action on $\mathbb{H}^2$.

Lecture 10 - Jan. 30

(1) Surface groups (continued)
(2) Definition of quasi-isometretic map, quasi-isometry, quasi-inverse
(3) Example: $\mathbb{Z} \sim \mathbb{R}$.
(4) Lemma: f is a quasi-isometry if and only if it has a quasi-inverse. (Part done in class, part exercise)

(5) Example: Cayley graphs for same group with different generating sets are quasi-isometric
(6) Outline of proof that a space quasi-isometric to a hyperbolic space is itself hyperbolic.

5. Week 5

Proof that a space quasi-isometric to a hyperbolic space is hyperbolic. This went via theorem that in a hyperbolic space quasi-geodesics stay a bounded distance from geodesics, and to prove that we first proved that in a hyperbolic metric space the divergence function is exponential.

Lecture 11 - Feb. 3

(1) Definition of divergence function
(2) Theorem: In a hyperbolic metric space the divergence function is exponential
(3) Definition of (λ, C)-quasi-geodesic
(4) Theorem: A (λ, C)-quasi-geodesic joining two points in a δ-hyperbolic metric space is within bounded Hausdorff distance of a geodesic joining the same points. The bound depends only on λ, C and δ.
(5) Second outline of proof that a space quasi-isometric to a hyperbolic space is itself hyperbolic.

Lecture 12 - Feb. 4

(1) Proof that quasi-geodesics stay within bounded distance of geodesics. continued. Case of continuous quasi-geodesics first
(2) Statement of lemma: A quasi-geodesic stays within bounded distance of a continuous quasi-geodesic.

Lecture 13 - Feb. 6

(1) More detailed statement of Lemma, and proof:
    (a) A (λ, C)- quasi-geodesic stays within bounded Hausdorff distance of a continuous quasi-geodesic β, and
    (b) The length of the sub-geodesic from $\beta(s)$ to $\beta(t)$ is bounded in terms of the distance from $\beta(s)$ to $\beta(t), \lambda, C$ and δ.
(2) Proof that a space quasi-isometric to a hyperbolic space is itself hyperbolic (third time).
(3) Some properties of hyperbolic groups
(4) Statement of next theorem to prove: Hyperbolic groups are finitely presentable.

6. Week 6

Group-theoretic consequences of hyperbolicity, then the Švarc-Milnor Lemma.

Lecture 14 - Feb. 10

(1) Theorem: Hyperbolic groups are finitely presentable.
(2) Beginning of proof

(3) Definition of k-local geodesic
(4) Lemma: In a δ-hyperbolic metric space, and8δ-local geodesic is with 6δ of a geodesic
(5) End of proof of theorem
(6) Proof of Lemma

Lecture 15 - Feb. 11

(1) Theorem: Hyperbolic groups have only finitely many conjugacy classes of finite elements
(2) Definition of proper action
(3) Theorem: If G acts properly and cocompactly on a geodesic metric space, then G is finitely generated.
(4) Theorem (Švarc-Milnor Lemma): If G acts properly and cocompactly on a geodesic metric space X, then G is quasi-isometric to X.

Lecture 16 - Feb. 13

(1) Consequences of Švarc-Milnor Lemma
  (a) Surface groups are hyperbolic
  (b) Finitely generated free groups of rank at least 2 are all quasi-isometric
  (c) Fundamental groups of n-dimensional hyperbolic manifolds are hyperbolic.
  (d) Finite-index subgroups of hyperbolic groups are hyperbolic
  (e) Quotients of hyperbolic groups by finite normal subgroups are hyperbolic
(2) Definitions of commensurable, virtually isomorphic, rigid
(3) Hints on exercise: The real line is not quasi-isometric to the infinite trivalent tree.

7. Week 7

Gromov product and the boundary of a hyperbolic space.

Lecture 17 - Feb. 17 Three definitions of boundary

(1) $\partial_r X$ = equivalence classes of geodesic rays from x_0
(2) $\partial_q X$ = equivalence classes of quasi-geodesic rays
(3) $\partial_s X$ = equivalence classes of sequences
  (a) Definition of Gromov product
  (b) Definition of hyperbolicity using Gromov product
  (c) Proposition: new definition equivalent to old definition (proof deferred)
  (d) Definition of "$x_i \to \infty$"
  (e) Equivalence relation on sequences $x_i \to \infty$

Lecture 18 - Feb. 18

(1) Proof that thin triangle definition of hyperbolicity implies Gromov product definition.
(2) Theorem: $\partial_r X \to \partial_q X$ is bijective for a locally finite hyperbolic graph
  (a) Lemma: a quasi-geodesic ray is bounded Hausdorff distance from a geodesic ray

   (b) Lemma: The concatenation of a geodesic segment with a quasi-geodesic ray is a quasi-geodesic ray
(3) Remark: Theorem is true for any proper hyperbolic metric space, using Arzela-Ascoli.

Lecture 19 - Feb. 20

(1) Theorem: $\partial_r X \to \partial_s X$ is a bijection.
(2) Extension of Gromov product to ∂X
(3) Topology on $\hat{X} = X \cup \partial X$: neighborhoods of points in ∂X.

8. Week 8

Boundary of a hyperbolic space, continued. Application to fixed subgroups of free group automorphisms via bounded cancellation.

Lecture 20 - Feb. 24

(1) Proposition: $\hat{X}$ is metrizable.
(2) Proposition: $\hat{X}$ is compact.

Lecture 21 - Feb. 25

(1) Prop: X quasi-isometric to Y implies ∂X homeomorphic to ∂Y.
(2) Examples of boundaries of hyperbolic groups: surface groups, hyperbolic 3-manifold groups, free groups.

Lecture 22 - Feb. 27

(1) Bounded cancellation lemma for free groups.
(2) Application to fixed point subgroup of an automorphism of a free group.

9. Week 9

Infinite-order elements of hyperbolic groups.

Lecture 23 - March 3 *Cancelled because room was unavailable.*

Lecture 24 - March 4

(1) Infinite-order elements of hyperbolic groups are quasi-geodesic rays (start)

Lecture 25 - March 6

(1) Infinite-order elements of hyperbolic groups are quasi-geodesic rays (end)
(2) There's a bi-infinite geodesic between g_∞ and $g_{-\infty}$

10. Week 10

Infinite-order subgroups of hyperbolic groups, continued. Quasi-convex subgroups.

Lecture 26 - March 10

(1) North-south dynamics of an infinite-order element
(2) Ping-pong with two infinite-order elements to find a free subgroup
(3) Definition and examples of quasi-convex subgroup
(4) Prop: q-c subgroups of finitely-generated groups are finitely generated

Lecture 27 - March 11

(1) Prop: q-c subgroups of hyperbolic groups are hyperbolic
(2) Intersection of quasi-convex subgroups is quasi-convex
(3) Application to free groups: Howson's theorem
(4) Hanna Neumann conjecture (statement, history)

Lecture 28 - March 13

(1) Review of course
(2) Algorithmic problems about hyperbolic groups
(3) Unsolved problems about hyperbolic groups