THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: MAY 2016

ALGEBRAIC TOPOLOGY – MA3H60

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) Suppose that the map $f : S^n \to X$ extends to a map $F : D^{n+1} \to X$. Show that $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(X)$ is the zero map. **[3]**

b) Let X be a torus with the interiors of two small disjoint discs removed, and let ∂X denote the union of the two circular boundaries of the discs. What is $H_1(X, \partial X)$? Make a drawing showing a minimal set of generators for this homology group. Do not justify your answer. **[6]**

c) Let $A_\bullet$ and $B_\bullet$ be chain complexes, and let $f, g : A_\bullet \to B_\bullet$ be morphisms of chain complexes. What does it mean to say that f and g are *chain-homotopic*? Show that if f and g are chain homotopic then $f_* : H_k(A_\bullet) \to H_k(B_\bullet)$ and $g_* : H_k(A_\bullet) \to H_k(B_\bullet)$ are equal. **[4]**

d) (i) State the excision property of homology.
(ii) Let X be an n-dimensional manifold and $x \in X$. Use excision (with other techniques) to calculate $H_n(X, X - x)$.
(iii) Let X be the cone $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$. Compute the local

homology group $H_2(X, X - \{(0,0,0)\})$.
(iv) Show that the space X from (iii) X is not a 2-dimensional manifold. **[8]**

e) Suppose that f and g are loops in X based at x_0, and suppose that they are end-point-preserving homotopic. Show that, considered as members of $C_1(X)$, they are homologous (they differ by a boundary). **[6]**

f) It was shown in lectures that $\mathbb{RP}^n$ has a CW structure consisting of one k-cell for each value of k between 0 and n, and that in the resulting cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

$d_k = 0$ when k is odd or $k = 0$ and d_k is multiplication by 2 when $k > 0$ is even. Use this to calculate $H_*(\mathbb{RP}^4)$ and $H_*(\mathbb{RP}^5)$. **[4]**

g) Suppose that the diagram of abelian groups and homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} \\ C & \xrightarrow{g} & D \end{array}$$

is commutative, with ϕ and ψ isomorphisms. Show that $\operatorname{coker} f \simeq \operatorname{coker} g$. **[4]**

h) Let X be a path-connected space. Suppose that $\varphi_i : S^{n-1} \to X$, $i = 1, \ldots, k$, are homeomorphisms onto their images in X, which are disjoint from one another. Let Y be the space obtained from X by gluing in k copies of D^n using these maps. If Y is contractible, what can you say about the homology of X? Justify your answer. **[5]**

OPTIONAL QUESTIONS

2. a) What is meant by the *degree* of a map $S^n \to S^n$? State the degree of

(i) the map $r : S^n \to S^n$ defined by reflection in a hyperplane

(ii) the map $f_A : S^n \to S^n$ defined by $f_A(x) = A(x)/||A(x)||$, where $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear isomorphism.

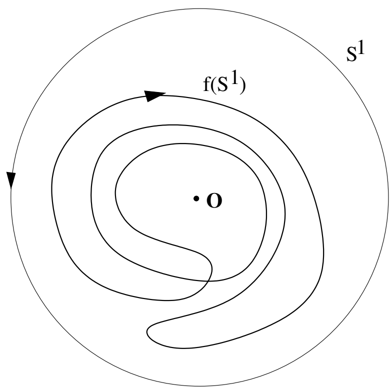
Justify your answers. **[8]**

b) Let $f : S^n \to S^n$ be a map, and suppose that $f^{-1}(y) = \{x_1, \ldots, x_m\}$ with $m < \infty$.

(i) Define the *local degree of* f *at* x_i, denoted by $\deg(f)|_{x_i}$, carefully justifying the steps in your definition.

(ii) State (without proof) the relation between $\deg(f)$ and the local degrees $\deg(f)|_{x_i}$. **[8]**

c) The following diagram shows the image of a map $f : S^1 \to \mathbb{R}^2$, with an arrow indicating the image under $f_\#$ of a generator of $H_1(S^1)$. It also shows S^1 with another arrow indicating a generator of $H_1(S^1)$.



Let $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ be radial projection, and let $g = r \circ f$. What is the degree of g? Make a drawing and use it to illustrate your answer. **[4]**

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3. a) Write down the long exact sequence of homology resulting from a short exact sequence of complexes $0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$ **[3]**

b) Explain the construction of the connecting homomorphism in this long exact sequence, and prove exactness of the sequence at the target of the connecting homomorphism. **[6]**

c) Suppose that (X, A, B) is a triple. What is the long exact sequence of homology associated with the triple? What short exact sequence of complexes gives rise to it? **[3]**

d) Given a commutative diagram of abelian groups and homomorphisms with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0
\end{array}$$

show that there is an exact sequence

$$0 \longrightarrow \ker f_1 \longrightarrow \ker f_2 \longrightarrow \ker f_3 \longrightarrow \operatorname{coker} f_1 \longrightarrow \operatorname{coker} f_2 \longrightarrow \operatorname{coker} f_3 \longrightarrow 0\ .$$

[4]

e) Given a commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all three columns, and the first two rows, are exact, and the third row is a complex, show that in fact the third row is exact. **[4]**

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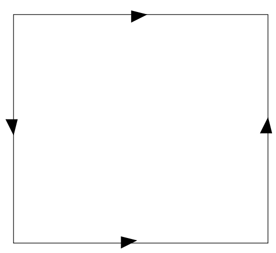
4. a) Describe a CW complex structure on the n-sphere S^n. **[2]**

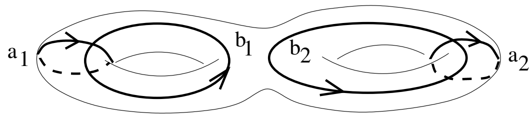
b) Let X be a CW complex. What is the *cellular chain complex* $C_\bullet^{CW}(X)$? Explain what are the groups and what is the differential. **[4]**

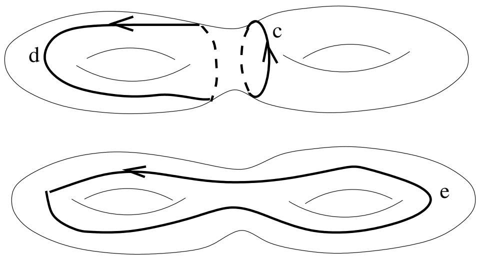
c) The Klein bottle K is the quotient of the square with opposite edges identified as shown.



Find a CW structure on K, and use cellular homology to calculate the homology of K, carefully explaining your calculation. **[6]**

d) Let M_2 be the genus 2 oriented compact surface without boundary. In the following three pictures, the first shows curves a_1, b_1, a_2, b_2 whose homology classes give a basis for $H_1(M_2)$, and the second and third show three curves representing other homology classes.





Express $[c]$, $[d]$ and $[e]$ as linear combinations of $[a_1], [b_1], [a_2], [b_2]$, justifying your answer with the help of suitable drawings. **[8]**

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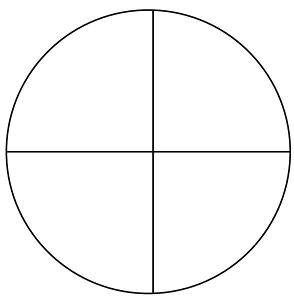
5. a) Let X be the graph shown in the following diagram.



(i) Calculate the Euler characteristic $\chi(X)$. **[2]**

(ii) Calculate $H_1(X)$ by any method you choose, briefly explaining your procedure, and give a basis for $H_1(X)$. **[5]**

(iii) Let $f_1 : X \to X$, $f_2 : X \to X$ be anticlockwise rotation through π about the centre O and reflection in the vertical line through the centre, respectively. Write down the matrices of $f_{1*} : H_1(X) \to H_1(X)$ and $f_{2*} : H_1(X) \to H_1(X)$ with respect to your chosen basis. **[5]**

b) Let Y be the space obtained from S^3 by identifying all pairs of antipodal points on the equator $E := \{(x_1, x_2, x_3, x_4) \in S^3 : x_4 = 0\}$. Calculate $H_*(Y)$. [Suggestion: Let Y_+ and Y_- be the images in Y of the upper and lower hemispheres of S^3. Each is homeomorphic to $\mathbb{RP}^3$.] **[8]**

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Model Solution No: 1

a), (c),(d)(i)(ii),(e),(f) are bookwork; (b) is unseen but close to a course exercise; (e)(iv) is a course exercise; (g) is material covered in lectures; (h) is unseen.

a) As $f = F \circ i$ so $f_* = F_* \circ i_*$. As $H_n(D^{n+1}) = 0$, $i_* = 0$ and so $f_* = 0$.

b) $H_1(X, \partial X) \simeq \mathbb{Z}^3$. Generators are e.g. generators of $H_1(T^2)$ and a path from one boundary component to the other.

c) f and g are chain homotopic if there exists a collection of linear maps $h_i : B_i \to A_{i+1}$ such that $\partial h + h\partial = f - g$. If f and g are chain homotopic then given $a_n \in Z_n(A_\bullet)$, we have

$$f(a_n) - g(a_n) = \partial h_n(a_n) + h_{n-1}(\partial a_n) = \partial h_n(a_n).$$

That is, $f(a_n)$ and $g(a_n)$ differ by a boundary. Thus $f_*([a_n]) = g_*([a_n])$.

d) (i) Excision: If $\bar{Z} \subset \mathring{A}$ then the inclusion $(X - Z, A - Z) \to (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \to H_n(X, A)$.

(ii) Application: x has a neighbourhood U homeomorphic to a ball. The inclusion $(U, U - x) \to (X, X - x)$ induces an isomorphism $H_n(U, U - x) \to H_n(X, X - x)$ by excision – we are excising $X - U$, which is contained in the interior of $X - x$. The l.e.s. of reduced homology of the pair $(U, U - x)$ shows $H_n(U, U - x) \simeq H_{n-1}(U - x)$, as U is contractible. As $U - x$ is homotopy equivalent to S^{n-1}, $H_n(U, U - x) = H_{n-1}(U - x) = \mathbb{Z}$.

(iii) As the cone is contractible, the boundary map in the l.e.s. of the pair $(X, X - x)$ shows $H_2(X, X - x) \simeq H_1(X - x)$. Now $X - x$ consists of two path components, each homotopy equivalent to a circle. So $H_2(X, X - x) \simeq H_1(S^1) \oplus H_1(S^1) \simeq \mathbb{Z}^2$.

(iii) It follows that X is not a 2-manifold, since $H_2(X, X - x) \neq \mathbb{Z}$.

e) Let $F : [0, 1] \times [0, 1] \to X$ be an end-point-preserving homotopy. Define a singular 2-chain c_2 in X by $c_2 = F_\#([A, B, C] - [A, D, C])$. Then

$$\partial c_2 = F_\#[B, C] - F_\#[A, C] + F_\#[A, B] - F_\#[D.C] + F_\#[A, C] - F_\#[A, D]$$

$$= F_\#[B, C] + f - g - F_\#[A, D].$$

Now $F_\#[B, C]$ and $F_\#[A, D]$ are both constant 1-simplices, and therefore boundaries (they lie in the chain complex of a point). Hence $f - g$ is a boundary.

f) For $\mathbb{RP}^3$ the chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so

$$H_3(\mathbb{RP}^3) = \mathbb{Z}, H_2(\mathbb{RP}^3) = 0, H_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}, H_0(\mathbb{RP}^3) = \mathbb{Z}.$$

For $\mathbb{RP}^4$ the chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so

$$H_4(\mathbb{RP}^4) = 0, H_3(\mathbb{RP}^4) = \mathbb{Z}/2\mathbb{Z}, H_2(\mathbb{RP}^4) = 0, H_1(\mathbb{RP}^4) = \mathbb{Z}/2\mathbb{Z}, H_0(\mathbb{RP}^4) = \mathbb{Z}.$$

g) Define $\bar{\phi} : \text{coker}B \to \text{coker}D$ by $\bar{\phi}(b + f(A)) = \phi(b) + g(C)$.

This is well defined because $b \in f(A) \implies \exists a \in A$ s.t. $f(a) = b \implies \phi(b) = \phi(f(a)) = g(\psi(a))$ so that $\phi(b) + g(C) = 0$.

It is injective because $\bar{\phi}(b+f(A)) = 0 \implies \phi(b) \in g(C) \implies \exists c \in C$ s.t. $g(c) = \phi(b) \implies b = f(\psi^{-1}(c))$.

It is surjective because φ is.

It is a homomorphism:

$$\bar{\phi}((b_1 + f(A)) + (b_2 + f(A)) = \bar{\phi}(b_1 + b_2 + f(A) = \phi(b_1) + \phi(b_2) + g(C) =$$

$$= \big(\phi(b_1) + g(C)\big) + \big(\phi(b_2) + g(C)\big) = \bar{\phi}(b_1 + f(A)) + \bar{\phi}(b_2 + f(A))$$

h) Mayer Vietoris for reduced homology: take $A = X$, $B = \coprod_{i=1}^k D^n$, so $A \cup B = Y, A \cap B = \coprod_{i=1}^k S^{n-1}$. As $\tilde{H}_i(Y) = 0$ for all i and $\tilde{H}_i(B) = 0$ for $i > 0$, the connecting homomorphism $\tilde{H}_i(X) \to \tilde{H}_{i-1}(A \cap B)$ in Mayer-Vietoris is an isomorphism for i¿1. It is also an isomorphism for $i = 1$, since moreover $\tilde{H}_0(\coprod_{i=1}^k S^{n-1}) \to \tilde{H}_0(X) \oplus \tilde{H}_0(\coprod_{i=1}^k D^n)$ is injective. And $\tilde{H}_0(X) = 0$ since X is path connected. Thus

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}^k & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

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Model Solution No: 2

(a) and (a)(i) are bookwork. (a)(ii) is course exercise. (b) is bookwork. (c) is unseen but similar to example done in class.

a) A map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Conjugating by an isomorphism $H_n(S^n) \simeq \mathbb{Z}$, f_* corresponds to a homomorphism $\mathbb{Z} \to \mathbb{Z}$, which must be multiplication by an integer. This integer is the degree of f. Because the two possible isomorphisms $H_n(S^n) \simeq \mathbb{Z}$ differ only by a sign, $\deg f$ is independent of the choice of isomorphism.

  (i) S^n is homeomorphic to the union of two standard n-simplices σ_1 and σ_2, glued along their common boundary. The mapping r interchanges them. $H_n(S^n)$ is generated by the class of $\sigma_1 - \sigma_2$. Thus $r_\#(\sigma_1 - \sigma_2) = \sigma_2 - \sigma_1$ so $\deg r = -1$.

  (ii) By the row operations of adding multiples of one row to another, and multiplying a row by a positive scalar, a real invertible matrix A can be reduced to a diagonal matrix B with 1's and -1's along the diagonal. These row operations are homotopic to the identity map, so the resulting maps $f_A : S^n \to S^n$ and $f_B : S^n \to S^n$ are homotopic also, and so have the same degree. Moreover since A is deformed to B through a family of invertible matrices, $\det A$ and $\det B$ have the same sign. The map $f_B : S^n \to S^n$ is the composite of k reflections in hyperplanes, where k is the number of -1's on the diagonal of B. Thus

$$\deg(f_A) = (-1)^k = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} = \begin{cases} 1 & \text{if } \det A > 0 \\ -1 & \text{if } \det A < 0 \end{cases}$$

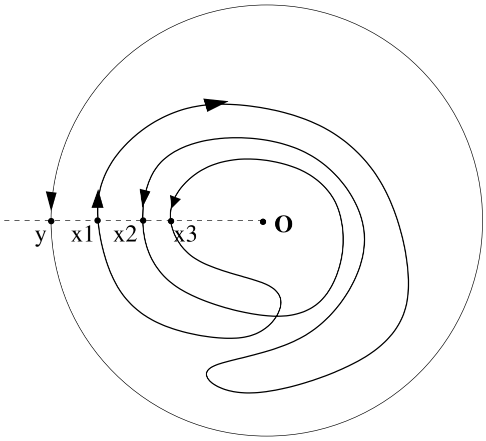
b) (i) The local degree at x is defined as follows. Pick a neighbourhood V of y and neighbourhood U of x such that $f(U) \subset V$ and x is the only point of $f^{-1}(y)$ in U. Then f induces a map of pairs $(U, U - x) \to (V, V - y)$ and therefore a homomorphism $f_* : H_n(U, U - x) \to H_n(V, V - y)$. Each of these two groups is canonically isomorphic to $H_n(S^n)$, from which it follows that f_* is conjugate to multiplication by an integer. This integer is $\deg f|_x$.

  The canonical isomorphisms are as follows:

  - by excision, $(U, U - x) \to (S^n, S^n - x)$ induces an isomorphism $H_n(U, U - x) \to H_n(S^n, S^n - x)$.
  - $S^n - x$ is contractible, so in the long exact sequence of reduced homology of the pair $(S^n, S^n - x)$, the morphism $H_n(S^n) \to H_n(S^n, S^n - x)$ is an isomorphism.
  - Both these isomorphisms are induced by inclusions, so are independent of any choices. Thus $H_n(U, U - x) \simeq H_n(S^n)$ independent of choices. Similarly $H_n(V, V - y) \simeq H_n(S^n)$.

(ii) If y has $m < \infty$ distinct preimage points x_i then $\deg f = \sum_i \deg f|_{x_i}$.

c)



$g^{-1}(y) = \{x_1, x_2, x_3\}$. We have $\deg g|_{x_1} = -1$, $\deg g_{x_2} = \deg g|_{x_3} = 1$ so $\deg g = -1 + 1 + 1 = 1$.

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Model Solution No: 3

(a), (b) , (c) are bookwork. (d) and (e) are unseen, though (e) is in the textbook.

a) The l.e.s. is

$$\begin{array}{ccccc} & & & & \cdots \\ & & & \swarrow & \\ H_n(A_\bullet) & \xrightarrow{i_*} & H_n(B_\bullet) & \xrightarrow{j_*} & H_n(C_\bullet) \\ & & \overset{\partial}{\swarrow} & & \\ H_{n-1}(A_\bullet) & \xrightarrow{i_*} & H_{n-1}(B_\bullet) & \xrightarrow{j_*} & H_{n-1}(C)\bullet) \\ & \swarrow & & & \\ \cdots & & & & \end{array}$$

b) Given a homology class in $H_n(C_\bullet)$, pick a cycle $c_n \in C_n(C_\bullet)$ representing it. By exactness of the s.e.s., $j_n : B_n \to C_n$ is surjective so there exists $b_n \in B_n$ mapping to c_n. By commutativity, $j_{n-1}\partial b_n = \partial j_n b_n = \partial c_n = 0$ so by exactness, $\partial b_n = i_{n-1}(a_{n-1})$ for some a_{n-1}. Then a_{n-1} is a cycle. Define $\partial[c_n] = [a_{n-1}]$.

We have to show exactness at $H_{n-1}(A_\bullet)$. We have $i_*\partial[c_n] = i_*([a_{n-1}])$ where a_{n-1} is chosen as described above. But by construction, $i(a_{n-1}) = \partial b_n$, so is zero in homology. Conversely, if a_{n-1} is a cycle and $i_*[a_{n-1}] = 0$ in $H_{n-1}(B_\bullet)$, then $i(a_{n-1}) = \partial b_n$ for some $b_n \in B_n$. Then $[a_{n-1}] = \partial[jb_n]$ according to the definition of ∂ above.

c) There is a l.e.s.

$$\begin{array}{ccccc} & & & & \cdots \\ & & & \swarrow & \\ H_n(A,B) & \longrightarrow & H_n(X,B) & \longrightarrow & H_n(X,A) \\ & & \swarrow & & \\ H_{n-1}(A,B) & \longrightarrow & H_{n-1}(X,B) & \longrightarrow & H_{n-1}(X,A) \\ & \swarrow & & & \\ \cdots & & & & \end{array}$$

coming from the s.e.s. of complexes

$$0 \longrightarrow \frac{C_\bullet(A)}{C_\bullet(B)} \longrightarrow \frac{C_\bullet(X)}{C_\bullet(B)} \longrightarrow \frac{C_\bullet(X)}{C_\bullet(A)} \longrightarrow 0$$

d) Expand the diagram to

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1}
$$

Then each column becomes a complex, and the diagram becomes a s.e.s of complexes. Indexing these complexes so that the A_i have index 1 and the B_i have index 0, the homology of the i'th column is $H_1 = \ker f_i, H_0 = \text{coker} f_i$. So the l.e.s. we are asked for is simply the l.e.s. of homology coming from the s.e.s. of complexes (1).

e) The diagram is a s.e.s of complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$. Because the first two rows are exact, the homology of $A_\bullet$ and $B_\bullet$ is 0, so in the l.e.s. of homology resulting from the s.e.s., the only possibly non-zero terms are the $H_i(C_\bullet)$. But each of these is flanked by 0's, so $H_i(C_\bullet) = 0$ also, i.e. the complex $C_\bullet$ is exact.

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Model Solution No: 4

(a)(b) are bookwork, (c) was covered in class, (d) is unseen though close to class exercises.

a) S^n has CW structure with one vertex and one n-cell, glued to the vertex by the constant map on its boundary.

b) The cellular chain complex is the complex

$$\cdots \longrightarrow H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X_{n-2}) \xrightarrow{d_{n-1}} \cdots \longrightarrow H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \longrightarrow 0\ .$$

The differential d_n is the composite of the differential

$$\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$$

in the l.e.s. of homology of the pair (X^n, X^{n-1}) with the morphism

$$H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

in the l.e.s. of homology of the pair (X^{n-1}, X^{n-2}).

c) The identifications indicated in the diagram identify the four edges in two pairs, and identifies all vertices to one. So there is a CW structure with one 0-cell, two 1-cells and one 2-cell. Thus the cellular chain complex

$$0 \longrightarrow H_2(K, K^1) \xrightarrow{d_2} H_1(K^1, K^0) \xrightarrow{d_1} H_0(K^0) \longrightarrow 0$$

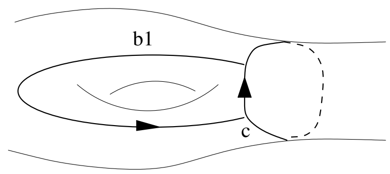
is

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

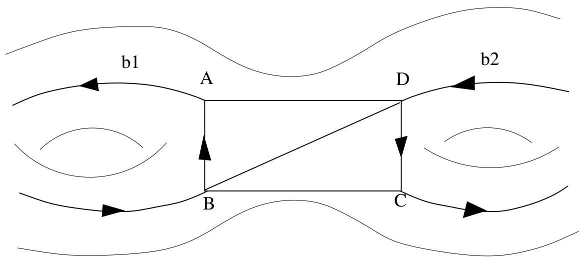
Taking as generators of $H^1(K^1, K^0)$ the two loops a and b, the boundary map $H_2(K^2, K^1)$ maps the generator e^2 to $0a + 2b$, since the two vertical edges in the diagram traverse b in the same direction whereas the two horizontal edges traverse a in opposite directions. Hence the differential d_2 has matrix $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Thus d_2 is injective and $H_2(K) = 0$. The differential d_1 must be 0, since both ends of each edge glue to the unique vertex in K^0. So

$$H_1(K) = H_1(K^1, K^0)/d_2(H_2(K, K^1)) = \mathbb{Z}^2 / \left\langle \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\rangle = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

d) The loop c is the boundary of the right-hand component of its complement in M_2. Thus $[c] = 0$. Then $d = b_1 - c$ so $[d] = [b_1]$.



The loops b_1 and b_2 can be homotoped to contain the segment BA and DC as shown. Then up to homotopy $b_1+b_2+\partial([B,D,A]-[B,D,C])$ is the loop e shown. So $[e]=[b_1]+[b_2]$.



MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS – MAY 2016

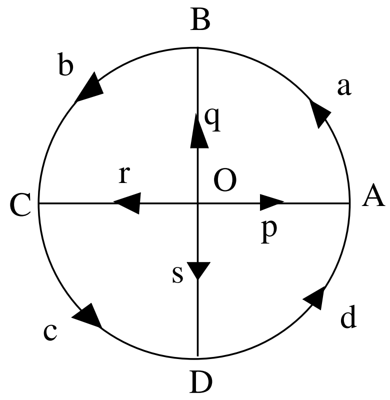
Course Title: ALGEBRAIC TOPOLOGY – MA3H60

Model Solution No: 5

(a) is unseen, (b) is unseen.

a) (i) X is a graph with 5 vertices and 8 edges. So $\chi(X) = 5 - 8 = -3$.

(ii) As X is connected, $H_0(X) = \mathbb{Z}$ and so it follows that $H_1(X)$ has rank 4. Give X a Δ-complex structure with 0-simplices O, A, B, C, D, and 1-simplices a, b, c, d, p, q, r, s, oriented as shown. Then $H_1(X)$ has basis the classes $z_1 = [p + a - q]$, $z_2 = [q + b - r]$, $z_3 = [r + c - s]$, $z_4 = [s + d - p]$.



We have

$$f_{1\#}(z_1) = z_3, \quad f_{1\#}(z_2) = z_4, \quad f_{1\#}(z_3) = z_1, \quad f_{1\#}(z_4) = z_2$$

so the matrix of f_{1*} with respect to the chosen basis is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

For any 1-simplex σ, if we define $r : [0,1] \to [0,1]$ by $r(t) = 1 - t$ then $\sigma \circ r$ is homologous to $-\sigma$. Hence,

$$f_{2*}(z_1) = -z_2, \quad f_{2*}(z_2) = -z_1, \quad f_{2*}(z_3) = -z_4, \quad f_*(z_4) = -z_3$$

and so f_{2*} has matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b) Each of Y_+ and Y_- is homeomorphic to $\mathbb{RP}^3$. Use Mayer Vietoris for reduced homology. We have $Y_1 \cup Y_2 = Y$, $Y_1 \cap Y_2 = \mathbb{RP}^2$, so it gives

$$\begin{array}{ccccc} 0 & \longrightarrow & H_3(\mathbb{RP}^3) \oplus H_3(\mathbb{RP}^3) & \longrightarrow & H_3(Y) \\ & \swarrow & & & \\ H_2(\mathbb{RP}^2) & \longrightarrow & H_2(\mathbb{RP}^3) \oplus H_2(\mathbb{RP}^3) & \longrightarrow & H_2(Y) \\ & \swarrow & & & \\ H_1(\mathbb{RP}^2) & \longrightarrow & H_1(\mathbb{RP}^3) \oplus H_1(\mathbb{RP}^3) & \longrightarrow & H_1(Y) \\ & \swarrow & & & \\ 0 & & & & \end{array}$$

which is

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & H_3(Y) \\ & \swarrow & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H_2(Y) \\ & \swarrow & & & \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \longrightarrow & H_1(Y) \\ & \swarrow & & & \\ 0 & & & & \end{array}$$

So $H_3(Y) \simeq \mathbb{Z}^2$.

To calculate $H_2(Y)$ and $H_1(Y)$, we use the result that if X is a CW complex with k-skeleton X^k then the inclusion $X^k \hookrightarrow X$ induces isomorphisms on H_i for $i < k$. As $\mathbb{RP}^2$ is the 2-skeleton of both copies of $\mathbb{RP}^3$ (Y_+ and Y_-), so $H_1(\mathbb{RP}^2) \to H_1(Y_+)$ and $H_1(\mathbb{RP}^2) \to H_1(Y_-)$ are isomorphisms. Thus the first arrow in the penultimate row is injective, and $H_2(Y) = 0$. Finally the last rows become

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow H_1(Y) \longrightarrow 0$$

so $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$. Since Y is conncted, $H_0(Y) \simeq \mathbb{Z}$.