EXERCISES FOR MA4J7 ALGEBRAIC TOPOLOGY II

WEEK 5

(1) Let S_g be the closed orientable surface of genus g.
  (a) Compute the cohomology, including the ring structure, of S_g.
  (b) Compute the cohomology ring of $S^1 \times S_g$.

(2) If N is a finitely generated free R-module, show that

$$(\prod_\alpha M) \otimes_R N \cong \prod_\alpha (M \otimes_R N).$$

Give an example to show that this is not true for arbitrary N.

(3) Using the ring structure, show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a non-trivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$.

(4) The Borsuk-Ulam theorem says every continuous function $f\colon S^n \to \mathbb{R}^n$ maps some pair of antipodal points to the same point. Fill in the details of the following argument. Suppose on the contrary that $f\colon S^n \to R^n$ satisfies $f(x) \neq (-x)$ for all x . Then define $g\colon S^n \to S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

so $g(-x) = -g(-x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Then apply the previous exercise.