

HOMOLOGY STABILITY FOR  $O_{n,n}$

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INTRODUCTION

In algebraic K-theory it is useful to know whether the sequence of homomorphisms

$$\dots \rightarrow H_k(\epsilon O_{n,n}(F)) \xrightarrow{i_*} H_k(\epsilon O_{n+1,n+1}(F)) \rightarrow \dots$$

eventually stabilizes to a sequence of isomorphisms.

An unpublished result of Daniel Quillen states that a similar sequence for  $GL_n(F)$  does stabilize. In this paper we adapt Quillen's method to  $\epsilon O_{n,n}(F)$ , where  $F$  is a field of characteristic zero.

If  $G$  is a group, we let  $H_*(G)$  denote group homology with coefficients in  $\mathbb{Z}$ ;  ${}_{\varepsilon}^0_{n,n}(F)$  denotes the orthogonal group of the quadratic form  $\begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}$  on a  $2n$ -dimensional vector space over  $F$  ( $\varepsilon = \pm 1$ , and  $I$  is the  $n$  by  $n$  identity matrix).

Theorem. If  $\text{char } F = 0$ , then the homomorphism

$$i_* : H_k({}_{\varepsilon}^0_{n,n}(F)) \rightarrow H_k({}_{\varepsilon}^0_{n+1,n+1}(F))$$

is onto for  $n \geq 3k+1$  and an isomorphism for  $n \geq 3k+3$ .

We associate to  ${}_{\varepsilon}^0_{n,n}(F)$  a building-like simplicial complex  $X$ . The action of  ${}_{\varepsilon}^0_{n,n}(F)$  on  $X$  gives rise to a spectral sequence converging to zero which gives information about the homology of  ${}_{\varepsilon}^0_{n,n}(F)$ . The theorem is proved by comparing the spectral sequences for  ${}_{\varepsilon}^0_{n,n}(F)$  and  ${}_{\varepsilon}^0_{n+1,n+1}(F)$  (or, more precisely, by constructing a relative spectral sequence giving information about the relative homology.)

In the first section we define the complex  $X$  and prove that it is homotopy equivalent to a wedge of spheres. In the second section we construct the spectral sequence and prove the theorem.

1. The building-like simplicial complex associated to  $O_{n,n}(F)$  .

A. In this section, let  $F$  be any field, and  $V$  a  $2n$ -dimensional vector space over  $F$ , with a polar basis  $\{e_1, \dots, e_n, f_1, \dots, f_n\}$  (i.e., in this basis the matrix of the quadratic form is  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ , where  $I$  is the  $n$  by  $n$  identity matrix). Let  $\cdot$  denote the inner product associated to the form.

Definition. A subspace  $A$  of  $V$  is called totally isotropic if  $v \cdot w = 0$  for all vectors  $v$  and  $w$  in  $A$  .

The set of all non-trivial totally isotropic subspaces of  $V$  is partially ordered by inclusion. The geometric realization of this partially ordered set is a simplicial complex, which we will call  $X$  ; a  $k$ -dimensional simplex of  $X$  is a chain of  $k + 1$  totally isotropic subspaces of  $V$

$$A_0 \subset A_1 \subset \dots \subset A_k .$$

By Witt's theorem [5], every maximal totally isotropic subspace of  $V$  has dimension  $n$  ; hence our complex  $X$  has dimension  $n - 1$  .

Proposition 1.1.  $X$  is homotopy equivalent to a wedge of  $(n-1)$ -dimensional spheres.

Proof: Let  $E = \langle e_1, \dots, e_n \rangle$  be the subspace of  $V$  spanned by the vectors  $e_i$ .

For each  $k = 0, \dots, n$ , define a subcomplex  $X_k$  of  $X$  by

$$\begin{aligned} X_k &= \text{the union of all maximal simplices } A_1 \subset \dots \subset A_n \\ &\quad \text{such that } \dim(A_n \cap E) \geq n - k . \\ &= \bigcup_{\dim(A_n \cap E) \geq n-k} \overline{\text{st } A_n} . \end{aligned}$$

Thus  $X_0$  is just the closed star of  $E$ ,  $X_n$  is  $X$ , and we have inclusions

$$X_0 \subset X_1 \subset \dots \subset X_n .$$

Claim.  $X_{k-1}$  is a deformation retract of  $X_k$  for  $1 \leq k \leq n - 1$ .

Assuming the claim, we have  $X_{n-1}$  contractible (since  $X_0 = \overline{\text{st } E}$  is contractible).  $X_{n-1}$  is the entire complex minus the stars of maximal isotropic subspaces  $A_n$  such that  $A_n \cap E = 0$ . Since for such  $A_n$ ,  $\text{lk } A_n \subset X_{n-1}$ , by contracting  $X_{n-1}$  to a point we obtain

$$X \simeq \bigvee_{A_n \cap E = 0} \text{susp}(\text{lk } A_n) .$$

( $\simeq$  denotes homotopy equivalence)

But  $\text{lk } A_n$  is just the Tits building for  $A_n$  (since every subspace of a totally isotropic space is clearly totally isotropic). Therefore by the Solomon-Tits theorem [3],  $\text{lk } A_n$  is homotopy equivalent to a wedge of  $(n-2)$ -dimensional spheres.

So

$$X \simeq \bigvee \text{susp}(\bigvee S^{n-2}) \simeq \bigvee S^{n-1} ,$$

and we are done.

Proof of claim. We do this in two steps.

Step 1. Let  $A$  and  $A'$  be two different maximal isotropic subspaces with  $\dim(A \cap E) = \dim(A' \cap E) = n - k$ , (i.e.,  $A, A' \subset X_k \setminus X_{k-1}$ ). Then  $\overline{\text{st } A} \cap \overline{\text{st } A'} \subset X_{k-1}$ .

Proof. A simplex  $\sigma$  in  $\overline{\text{st } A} \cap \overline{\text{st } A'}$  looks like

$$\sigma = A_0 \subset \dots \subset A_s ,$$

with  $A_s \subset A \cap A'$ . So to show  $\sigma \in X_{k-1}$ , it is sufficient to find a maximal isotropic subspace  $B$  with  $A \cap A' \subset B$  and  $\dim(B \cap E) \geq n - k + 1$  (i.e.,  $B \subset X_{k-1}$ ).

Let  $r = \dim(A \cap A' \cap E)$ . Then  $r \leq n - k$  (\*) since  $\dim(A \cap E) = n - k$ .

Let  $r + s = \dim(A \cap A')$  . Then  $r + s < n(**)$  since  $A \neq A'$  .

We can find a basis  $\{u_1, \dots, u_r, v_1, \dots, v_s\}$  for  $A \cap A'$  with the  $u_i$  in  $E$  and  $v_i$  not in  $E$  . Since  $\dim(\langle v_1, \dots, v_s \rangle^\perp \cap E) \geq n - s$  , we can add  $n - s - r$  independent vectors in  $E$  to our basis to obtain a maximal isotropic subspace  $B$  containing  $A \cap A'$  . Then  $\dim(B \cap E) = (n - s - r) + r = n - s$  .

If  $r = n - k$  , then  $(**)$  implies  $n - s > r$  , so  $n - s > n - k$  and we are done.

If  $r < n - k$  , we need to estimate  $s$  :

We notice that the subspace

$$(A \cap E) + (A' \cap E) + (A \cap A')$$

is totally isotropic and hence has dimension  $\leq n$  ; by counting dimensions we have

$$(n - k) + (n - k) + (s + r) - 2r \leq n ,$$

or  $s \leq 2k + r - n$  .

$$\text{Then } n - s \geq n - (2k + r - n)$$

$$= 2n - 2k - r$$

$$> 2n - 2k - (n - k) = n - k$$

and we are done.

Step 2. Let  $A$  be maximal isotropic with  $\dim(A \cap E) = n - k$  . Then  $\overline{\text{st } A} \cap X_{k-1}$  is a deformation retract of  $\overline{\text{st } A}$  .

Proof. First note that  $\overline{\text{st } A} \cap X_{k-1} \neq \emptyset$ , since  $E' = A \cap E \neq 0$ . Let  $A_S$  be a vertex in  $\overline{\text{st } A} \cap X_{k-1}$ , and consider the subspace  $A_S + E'$ . Clearly  $A_S + E' \subset A$ . In fact,  $A_S + E' \subsetneq A$ , since if  $A_S + E' = A$ , then there is a  $k$ -dimensional subspace of  $A_S$  which does not intersect  $E$ , contradicting the assumption that  $A_S \subset X_{k-1}$ .

We can write  $A_S + E'$  as  $A_r \oplus E'$ , where  $A_r \cap E \subset A_r \cap E' = 0$ . Now  $\dim[(A_r \oplus E')^\perp \cap E] = \dim(A_r^\perp \cap E) = n - r$ . Since  $\dim(A_r \oplus E') = r + n - k \leq n - 1$ , we have  $n - r \geq n - k + 1$ , which implies that  $A_r \oplus E' = A_S + E' \subset X_{k-1} \cap \overline{\text{st } A}$ .

Now the simplicial maps

$$\begin{array}{c} A_1 \subset A_2 \subset \dots \subset A_S \\ \downarrow \\ A_1 + E' \subset A_2 + E' \subset \dots \subset A_S + E' \\ \downarrow \\ E' \end{array}$$

induce a contraction of  $\overline{\text{st } A} \cap X_{k-1}$  to the vertex  $E'$ .

Since  $\overline{\text{st } A}$  is itself contractible, we have  $\overline{\text{st } A} \cap X_{k-1}$  is a contractible subcomplex of a contractible complex, so there is a deformation retraction of  $\overline{\text{st } A}$  onto  $\overline{\text{st } A} \cap X_{k-1}$ .

Combining steps 1 and 2, we see that there is a retraction of  $X_k$  to  $X_{k-1}$ , and we are done.  $\square$

B. Now consider a different filtration of  $X$ , namely, let

$Y_i$  = the union of all closed  $i$ -dimensional simplices of the form  $A_1 \subset \dots \subset A_{i+1}$ , with  $\dim A_j = j$ .

Then we have  $\phi \subset Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} = X$ .

Lemma 1.2. 1)  $H_i(Y_j) = 0$  for  $1 \leq i < j \leq n-1$

2)  $H_0(Y_j) \rightarrow H_0(X)$  is onto for  $j = 0$

and an isomorphism for  $j > 0$ .

Proof: This follows from Lemma 1.6 of the book The Discrete Series Representations of  $GL_n$  of a Finite Field, by George Lusztig ([6]), if we use Proposition 1.1 and the Solomon-Tits theorem.  $\square$

Lemma 1.3. The sequence

$$0 \rightarrow H_{n-1}(X) \rightarrow H_{n-1}(X, Y_{n-2}) \rightarrow H_{n-2}(Y_{n-2}, Y_{n-3}) \rightarrow \dots \\ \dots \rightarrow H_2(Y_2, Y_1) \rightarrow H_1(Y_1, Y_0) \rightarrow H_0(Y_0) \rightarrow H_0(X) \rightarrow 0$$

is exact, where the maps are the usual boundary maps  $\partial$ .

We give a proof of this lemma (Lemma 1.1 of Lusztig's book) even though Lusztig omitted it, since it is not true exactly as Lusztig stated it.

Proof: Associated to the filtration

$\phi \subset Y_0 \subset \dots \subset Y_{n-1} = X$  is a spectral sequence with

$$E_{p,q}^1 = H_{p+q}(Y_p, Y_{p-1}) \Rightarrow H_{p+q}(X) .$$

Since  $Y_p$  is a  $p$ -dimensional complex,  $E_{p,q}^1 = 0$  for  $q > 0$  .

By Lemma 1.2,  $E_{p,q}^1 = 0$  for  $q < 0$  also. Since the sequence converges to  $H(X)$  , which is zero except in degree zero and degree  $n - 1$  , the line  $q = 0$  is exact except at  $H_{n-1}(X, Y_{n-2})$  and at  $H_0(Y_0)$  .

We calculate

$$\text{coker}(\partial: H_1(Y_1, Y_0) \rightarrow H_0(Y_0)) \cong H_0(Y_1) \cong H_0(X)$$

and

$$\begin{aligned} \ker(\partial: H_{n-1}(X, Y_{n-2}) \rightarrow H_{n-2}(Y_{n-2}, Y_{n-3})) &\cong H_{n-1}(X, Y_{n-3}) \\ &\cong H_{n-1}(X) . \end{aligned}$$

We now have the exact sequence in the statement of the lemma.  $\square$

Lemma 1.4. For  $i \geq 2$  ,  $H_i(Y_i, Y_{i-1}) \cong$

$$\begin{aligned} &\oplus_{\substack{A \subset V \\ \dim A = i+1 \\ A \text{ totally isotropic}}} H_{i-1}(T(A)) , \end{aligned}$$

where  $T(A)$  is the Tits building of  $A$  .

$$\begin{aligned} \text{Also } H_1(Y_1, Y_0) &\cong \oplus_{\substack{A \subset V \\ \dim A = 2 \\ A \text{ totally isotropic}}} \ker(H_0(T(A)) \rightarrow Z) \end{aligned}$$

and  $H_0(Y_0) \cong \bigoplus_{\langle v \rangle \subset V} Z$  .  
 $v$  isotropic

Proof: This is a translation of p.11, [6] to our case.  $\square$

If we let  $\tau(A) = H_{i-1}(T(A))$  , the exact sequence of lemma 1.3 becomes

$$\begin{array}{ccccccc}
 0 \rightarrow H_{n-1}(X) & \longrightarrow & \bigoplus_{\substack{A \subset V \\ \dim A = n \\ A \text{ totally isotropic}}} & \tau(A) & \longrightarrow \dots \longrightarrow & \bigoplus_{\substack{A \subset V \\ \dim A = 3 \\ A \text{ totally isotropic}}} & \tau(A) \longrightarrow \\
 & & & & & & \\
 \longrightarrow & \bigoplus_{\substack{A \subset V \\ \dim A = 2 \\ a \text{ totally isotropic}}} & \ker(\tau(A) \rightarrow Z) & \longrightarrow & \bigoplus_{\substack{A \subset V \\ \dim A = 1 \\ A \text{ isotropic}}} & Z \rightarrow Z \rightarrow 0 & \\
 & & & & & & (1)
 \end{array}$$

$0_{n,n}(F)$  acts transitively on this acyclic complex.

## 2. Proof of the theorem in the case char F = 0 .

A. We want to prove that  $i_*: H_k(0_{n,n}(F)) \rightarrow H_k(0_{n+1,n+1}(F))$  is onto for  $n \geq 3k + 1$  and an isomorphism for  $n \geq 3k + 3$  . Actually, we will prove that the relative homology groups vanish for  $n$  large enough with respect to  $k$  . To do this, we want to produce a spectral sequence involving these groups.

Notation. Let  $G_n$  denote  $0_{n,n}(F)$  .

Also, let  $K_i = \bigoplus_{\substack{A \subset V \\ \dim A = i \\ A \text{ totally isotropic}}} \tau(A)$  for  $2 < i \leq n$  ;  
 let  $K_0 = \mathbb{Z}$ ,

$K_1 = \bigoplus_{\substack{A \subset V \\ \dim A = 1 \\ A \text{ isotropic}}} \mathbb{Z}$ ,  $K_2 = \bigoplus_{\substack{A \subset V \\ \dim A = 2 \\ A \text{ totally isotropic}}} \ker(\tau(A) \rightarrow \mathbb{Z})$  and

$K_{n+1} = H_{n-1}(X)$  . In other words, we denote the acyclic complex (1) produced in the last section by

$$0 \rightarrow K_{n+1} \rightarrow K_n \rightarrow \dots \rightarrow K_0 \rightarrow 0 .$$

Let  $EG_n$  be a  $G_n$ -free resolution of  $\mathbb{Z}$  . We form the standard double complex  $K_i \otimes_{G_n} E_j G_n$  by taking maps

$$\begin{array}{ccccc} \rightarrow & K_i \otimes E_{j+1} G_n & \xrightarrow{1 \otimes d} & K_i \otimes E_j G_n & \rightarrow \\ & \downarrow & & \downarrow & \\ & (-1)^{j+1} \partial \otimes 1 & & (-1)^j \partial \otimes 1 & \\ \rightarrow & K_{i+1} \otimes E_{j+1} G_n & \xrightarrow{1 \otimes d} & K_{i+1} \otimes E_j G_n & \rightarrow \\ & \downarrow & & \downarrow & \end{array}$$

where  $\partial$  is the differential in the complex  $K$  and  $d$  is the differential in  $EG_n$  .

This double complex gives a single complex if we let

$$A_p = \bigoplus_{i+j=p} K_i \otimes E_j G_n$$

with differential  $\delta : A_p \rightarrow A_{p-1}$  defined by  
 $\delta = 1 \otimes d + \pm \partial \otimes d$ .

We can filter this single complex in two different ways; namely, we take the filtrations of  $A$  induced by the horizontal and vertical filtrations of the double complex  $K_i \otimes E_j G_n$ .

These two filtrations give us two different spectral sequences both converging to the homology of  $A$ . The vertical filtration gives

$$E_{p,q}^1 = H_q(K_* \otimes E_p G_n, \partial \otimes 1)$$

which is identically zero since the sequence  $\{K_p\}$  is exact.

The horizontal filtration gives the spectral sequence

$$\begin{aligned} E_{p,q}^1 &= H_q(K_p \otimes E_* G_n, 1 \otimes d) \\ &= H_q(G_n; K_p) \end{aligned}$$

This spectral sequence must converge to zero since the vertical one does. We will now calculate the  $E^1$  terms of this sequence.

Recall that

$$K_p = \bigoplus_{\substack{\dim A = p \\ A \text{ tot. isot.} \subset V}} \tau(A).$$

Since  $G_n$  acts transitively on  $p$ -dimensional totally isotropic subspaces of  $V$ , this is

$$= Z[G_n] \otimes_Z [\text{stabilizer of } \langle e_1, \dots, e_p \rangle] \tau(F^p)$$

If we denote the stabilizer of  $\langle e_1, \dots, e_p \rangle$  by  $S_{p,n}$ , we have

$$E_{p,q}^1 = H_q(G_n; K_p) = H_q(G_n; Z[G_n] \otimes_Z [S_{p,n}] \tau(F^p))$$

which, by Shapiro's lemma [7] is

$$= H_q(S_{p,n}; \tau(F^p)) .$$

It is not hard to calculate the stabilizer subgroup  $S_{p,n}$ ; it turns out to be (conjugate to the group of) all matrices of the form

$$\begin{pmatrix} \alpha & * & * \\ 0 & A & * \\ 0 & 0 & t_{\alpha^{-1}} \end{pmatrix}$$

where  $\alpha$  is any matrix in  $GL_p(F)$ ,  $A$  is any matrix in  $G_{n-p}$ , and there are conditions on the  $*$  terms to insure that the whole matrix lies in  $G_n$ . (We have conjugated the quadratic form by the matrix

$$\begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{n-p} & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & I_{n-p} & 0 \end{pmatrix}$$

in order to simplify writing down the

matrices; this does not change the homology.)

We have an obvious inclusion

$$i : (\mathrm{GL}_p \times G_{n-p}) \rightarrow S_{p,n} .$$

Proposition 2.1. If  $n \geq 3t + 1$ , then

$i_* : H_t(\mathrm{GL}_p \times G_{n-p}) \rightarrow H_t(S_{p,n})$  is an isomorphism for  $0 \leq t \leq k$ .

The proposition will be proved in part B. of this section. It should be noted here, however, that the proof of the proposition depends on the fact that  $\mathrm{char} F = 0$ .

In our spectral sequence (\*) we now have

$$E_{p,q}^1 = H_q(\mathrm{GL}_p \times G_{n-p}; \tau(F^P))$$

if we choose  $n \geq 3q + 1$ .

Since we are interested in the relative homology groups, we actually want a "relative" spectral sequence instead of (\*). This is constructed in the following way.

The inclusion  $i : G_n \rightarrow G_{n+1}$  induces an equivariant chain map from the complex  $\{K_i\}$  associated to  $G_n$  to the analogous complex  $\{K_i^!\}$  associated to  $G_{n+1}$ . We form a double complex as follows:

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
 \rightarrow & K_S \otimes E_{t-1} G_n \oplus K'_S \otimes E_t G_{n+1} & \xrightarrow{\partial \otimes 1 \oplus \partial' \otimes 1} & K_{S-1} \otimes E_{t-1} G_n \oplus K'_{S-1} \otimes E_t G_{n+1} & \rightarrow & & \\
 & \downarrow \partial \oplus i_* \partial' & & \downarrow -\partial \oplus -i_* \partial' & & & \\
 \rightarrow & K_S \otimes E_{t-2} G_n \oplus K'_S \otimes E_{t-1} G_{n+1} & \xrightarrow{\partial \otimes 1 \oplus \partial' \otimes 1} & K_{S-1} \otimes E_{t-2} G_n \oplus K'_{S-1} \otimes E_{t-1} G_{n+1} & \rightarrow & & \\
 & \downarrow & & \downarrow & & & 
 \end{array}$$

In this double complex, the columns are just the mapping cones of the chain maps

$$i_* : K_S \otimes E_* G_n \rightarrow K'_S \otimes E_* G_{n+1} .$$

As before, we see that the horizontal filtration gives a spectral sequence which is identically zero since the sequences  $\{K_i\}$  and  $\{K'_i\}$  are exact. The vertical filtration gives a spectral sequence with

$$(**) \quad E_{S,t}^1 = H_t \text{ (mapping cone of } \\
 i_* : K_S \otimes E_* G \rightarrow K'_S \otimes E_* G_{n+1} \text{)}$$

which must also converge to zero.

We know from computing the terms of the spectral sequence (\*) that

$$\tau(F^S) \otimes_{S_{S,n}} E_* S_{S,n} \rightarrow K_S \otimes_{G_n} E_* G_n$$

and

$$\tau(F^S) \otimes_{S_{S,n+1}} E_* S_{S,n+1} \rightarrow K'_S \otimes_{G_{n+1}} E_* G_{n+1}$$

are homology equivalences, so in (\*\*),

$$\begin{aligned} E_{s,t}^1 &= H_t \text{ (mapping cone of} \\ &\quad i_*: \tau(F^S) \otimes E_* S_{s,n} \rightarrow \tau(F^S) \otimes E_* S_{s,n+1}) \\ &= H_t(S_{s,n+1}, S_{s,n} : \tau(F^S)) \end{aligned}$$

which, by Proposition 2.1 is

$$= H_t(\text{GL}_S \times (G_{n+1-s}, G_{n-s}) ; \tau(F^S))$$

If  $n \geq 3k + 1$ .

We now have our relative spectral sequence, and will prove the theorem by induction on  $k$ . That is, we will assume that for  $q \leq k - 1$  and  $n \geq 3q + 1$ ,

$$H_q(G_{n+1}, G_n) = 0$$

Using the universal coefficient theorem and the Kunneth formula, we calculate

$$\begin{aligned} H_t(\text{GL}_S \times (G_{n+1-s}, G_{n-s}); \tau(F^S)) \\ &= \bigoplus_{p+q=t} H_p(\text{GL}_S) \otimes H_q(G_{n+1-s}, G_{n-s}) \otimes \tau(F^S) \\ &\quad \oplus \bigoplus_{p+q=t-1} \text{Tor}(H_p(\text{GL}_S), H_q(G_{n+1-s}, G_{n-s})) \otimes \tau(F^S) \\ &\quad \oplus \text{Tor}(\bigoplus_{p+q=t-1} H_p(\text{GL}_S) \otimes H_q(G_{n+1-s}, G_{n-s}), \tau(F^S)) \\ &\quad \oplus \text{Tor}(\bigoplus_{p+q=t-2} \text{Tor}(H_p(\text{GL}_S), H_q(G_{n+1-s}, G_{n-s})), \tau(F^S)). \end{aligned}$$

By our induction assumption, all the terms

$H_q(G_{n+1-s}, G_{n-s})$  vanish for  $q \leq k - 1$  and  $n - s \geq 3q + 1$ . If we assume now that  $n \geq 3k$ , it is easily seen that

$$E_{s,t}^1 = H_t(GL_s \times (G_{n+1-s}, G_{n-s}); \tau(F^S)) = 0$$

for  $t = 0, \dots, k - 1$  and  $s \leq k + 1 - t$ .

We also have

$$E_{0,k}^1 = H_k(G_{n+1}, G_n; Z)$$

and

$$\begin{aligned} E_{1,k}^1 &= H_k(GL_1 \times (G_n, G_{n-1}); Z) = H_0(GL_1; Z) \otimes H_k(G_n, G_{n-1}; Z) \\ &= H_k(G_n, G_{n-1}; Z) \end{aligned}$$

since  $H_0(GL_1; Z) = Z/\{n(g - 1) : n \in Z, g \in GL_1\} = Z$ .

Now the  $E^1$  level of the spectral sequence looks like

k	$H_k(G_{n+1}, G_n)$	$\xleftarrow{d_1}$	$H_k(G_n, G_{n-1})$	$\leftarrow * \leftarrow *$		
k-1	0	$\longleftarrow$	0	$\longleftarrow 0 \leftarrow *$		
.	.		.		*	*
.	.		.		*	*
.	.		.			
1					0	*
0	0		0	0	0	0
	0		1	2	...	k k+1

Since  $E_{s,k-(s-1)}^1 = 0$  for  $s = 2, \dots, k + 1$ , the differentials  $d_s : E_{s,k-(s-1)}^s \rightarrow E_{0,k}^s$  are zero. Thus  $E_{0,k}^\infty = E_{0,k}^1 / \text{im } d_1$ . But we know  $E_{0,k}^\infty = 0$  since the

spectral sequence converges to zero, so we must have

$E_{0,k}^1 = \text{im } d_1$ , i.e. the map

$$d_1 : H_k(G_n, G_{n-1}) \rightarrow H_k(G_{n+1}, G_n)$$

is onto for  $n \geq 3k$ .

We now consider the following diagram

$$\begin{array}{ccccccc} H_k(G_n, G_{n-1}) & \longrightarrow & H_{k-1}(G_{n-1}) & \longrightarrow & H_{k-1}(G_n) & \longrightarrow & H_{k-1}(G_n, G_{n-1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & d_1 & & & & & \\ H_k(G_{n+1}, G_n) & \longrightarrow & H_{k-1}(G_n) & \xrightarrow{i_*} & H_{k-1}(G_{n+1}) & \longrightarrow & H_{k-1}(G_{n+1}, G_n) \\ \downarrow & & & & & & \\ & & & & & & 0 \end{array}$$

Our induction assumption implies  $H_{k-1}(G_{n+1}, G_n) = H_{k-1}(G_n, G_{n-1}) = 0$  (since we assumed that  $n \geq 3k$ ).

Thus  $i_*$  is surjective. A diagram chase shows that  $i_*$  is also injective.

We can then construct the diagram

$$\begin{array}{ccccccc} H_k(G_{n+1}) & \longrightarrow & H_k(G_{n+1}, G_n) & \longrightarrow & H_{k-1}(G_n) & \xrightarrow[\cong]{i_*} & H_{k-1}(G_{n+1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & d_1 & & & \\ H_k(G_{n+2}) & \longrightarrow & H_k(G_{n+2}, G_{n+1}) & \longrightarrow & H_{k-1}(G_{n+1}) & \xrightarrow[\cong]{j_*} & H_{k-1}(G_{n+2}) \\ \downarrow & & & & & & \\ & & & & & & 0 \end{array}$$

where  $i_*$  and  $j_*$  are both isomorphisms. Another diagram chase shows that  $H_k(G_{n+2}, G_{n+1}) = 0$  ( $n \geq 3k$ ).

Thus we have shown that  $H_k(G_{n+1}, G_n) = 0$  for  $n \geq 3k + 1$ , completing the induction step. Note that we needed only  $n \geq 3k = 3(k - 1) + 3$  to prove that  $i_* : H_{k-1}(G_n) \rightarrow H_{k-1}(G_{n+1})$  is an isomorphism; if we repeat the argument for  $k+1$  instead of  $k$ , we see that

$$i_* : H_k(G_n) \rightarrow H_k(G_{n+1})$$

is an isomorphism for  $n \geq 3k + 3$ .  $\square$

B. We will now prove Proposition 2.1. Actually, we will prove a stronger statement than what we need, namely

Proposition 2.2. Let  $F$  be a field of characteristic zero and  $S_{p,n}$  and  $G_n$  as above. Then for all  $p, n$  and  $t$ ,

$$i_* : H_t(GL_p \times G_{n-p}) \rightarrow H_t(S_{p,n})$$

is an isomorphism.

This proposition depends on the fact that  $\text{char } F = 0$ . It is not true, for example, for a finite field, since then the groups involved are finite groups, and a map inducing an isomorphism on the homology of two finite groups is itself an isomorphism [2].

If we can prove this proposition for homology with

coefficients in any algebraically closed field  $k$ , it will be true for homology with coefficients in  $Z$  by the following standard argument.

Lemma 2.3. If  $H_*(A; k) = 0$  for any algebraically closed field  $k$ , then  $H_*(A; Z) = 0$ .

Proof: Suppose  $\text{char } k = 0$ . The universal coefficient theorem gives

$$H_*(A; k) \cong H_*(A; Q) \otimes_Q k \oplus \text{Tor}(H_*(A; Q), k).$$

The torsion term is zero since  $k$  is a free  $Q$ -module. Thus  $H_*(A; k) = 0$ , if and only if  $H_*(A; Q) = 0$ .

If  $\text{char } k = p$ , we have

$$H_*(A; k) \cong H_*(A; A/p) \otimes_{Z/p} k \oplus \text{Tor}(H_*(A; Z/p), k)$$

Here, too,  $H_*(A; k) = 0$  implies that  $H_*(A; Z/p) = 0$ .

Now consider the Bockstein homology sequence for the short exact sequence

$$0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0$$

which is

$$\dots \rightarrow H_n(A; Q/Z) \rightarrow H_{n-1}(A; Z) \rightarrow H_{n-1}(A; Q) \rightarrow \dots$$

Since  $H_{n-1}(A; Q) = 0$ , we have  $H_n(A; Q/Z) \cong H_{n-1}(A; Z)$ .

Now  $Q/Z$  is a torsion group; it is the direct limit

$$Q/Z = \varinjlim_{n,p} Z/p^n.$$

Since homology commutes with direct limits, we need only show that  $H_*(A; \mathbb{Z}/p^n) = 0$  for all  $p$  and  $n$ . To do this, we consider the short exact sequence

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^n \rightarrow 0$$

which gives the homology sequence

$$\dots \rightarrow H_{i+1}(\mathbb{Z}/p^{n-1}) \rightarrow H_i(\mathbb{Z}/p) \rightarrow H_i(\mathbb{Z}/p^n) \rightarrow H_i(\mathbb{Z}/p^{n-1}) \rightarrow \dots$$

By induction we may assume  $H_*(\mathbb{Z}/p^{n-1}) = H_*(\mathbb{Z}/p) = 0$ , which implies  $H_*(\mathbb{Z}/p^n) = 0$ .  $\square$

Corollary. If  $\omega : X \rightarrow Y$  induces an isomorphism  $\omega_* : H_*(X; k) \rightarrow H_*(Y; k)$  on homology with coefficients in any algebraically closed field  $k$ , then  $\omega_* : H_*(X; \mathbb{Z}) \rightarrow H_*(Y; \mathbb{Z})$  is an isomorphism.

Proof: In Lemma 2.3, let  $A$  be the mapping cone of  $\omega$ .  $\square$

Thus we may assume that we have coefficients in an algebraically closed field  $k$ .

Consider the short exact sequence of subgroups of  $G_n$  consisting of matrices of the form

$$I \rightarrow \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & * & * \\ 0 & A & * \\ 0 & 0 & t_\alpha^{-1} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & t_\alpha^{-1} \end{pmatrix} \rightarrow I$$

where  $A \in G_{n-p}$  and  $\alpha \in GL_p$ ; i.e. this is

$$1 \rightarrow N \rightarrow S_{p,n} \rightarrow GL_p \times G_{n-p} \rightarrow 1$$

where  $S_{p,n}$  and  $G_{n-p}$  are the groups defined in 2A., and  $N$  is the kernel of the projection

$$S_{p,n} \rightarrow GL_p \times G_{n-p} .$$

This short exact sequence gives a first quadrant spectral sequence (the Lyndon-Hochschild-Serre spectral sequence, [4]) with

$$E_{s,t}^2 = H_s(GL_p \times G_{n-p}; H_t(N)) \Rightarrow H_{s+t}(S_{p,n}) .$$

We will examine this spectral sequence in the following two cases.

Case 1.  $\text{char } F = 0$  and  $\text{char } k = \ell$  (so  $\ell$  is invertible in  $F$ ).

We can calculate the homology  $H_t(N)$  by looking at the short exact sequence

$$1 \rightarrow [N, N] \rightarrow N \rightarrow N/[N, N] \rightarrow 1 \quad (1)$$

A short calculation shows that  $N$  is the set of matrices of the form

$$X = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & -t_b \\ 0 & 0 & 1 & -t_a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $a$  and  $b$  are any  $p \times (n-p)$  matrices and  $c$  is a  $p \times p$  matrix such that  $-(c + {}^t c) = a {}^t b + b {}^t a$ .

The inverse of  $X$  is

$$X^{-1} = \begin{pmatrix} 1 & -a & -b & {}^t c \\ 0 & 1 & 0 & {}^t b \\ 0 & 0 & 1 & {}^t a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the commutators  $XYX^{-1}Y^{-1}$  are matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & d - {}^t d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $d$  is any  $p \times p$  matrix. Thus the commutator subgroup  $[N, N]$  is isomorphic to the additive abelian group of  $p \times p$  skew-symmetric matrices over  $F$ , and  $N/[N, N]$  is isomorphic to the additive abelian group of  $p \times 2(n-p)$  matrices over  $F$ .

The sequence (1) gives a spectral sequence with

$$E_{p,q}^2 = H_p(N/[N, N]; H_q([N, N])) \quad (2)$$

converging to the homology of  $N$ .

We have the following formula for the homology  $H_*(G; k)$  of an abelian group  $G$  with coefficients in a field  $k$ . Let  $\ell = \text{char } k$  and let  ${}_{\ell}G$  be the

subgroup of elements of  $G$  annihilated by  $\ell$  if  $\ell > 0$ ; if  $\ell = 0$ , set  ${}_0G = 0$ . Let  $\Lambda(V)$  and  $\Gamma(V)$  be the exterior and divided power algebras respectively of a  $k$ -vector space  $V$ .

Lemma 2.4.  $\Lambda(G \otimes_Z k) \otimes_k \Gamma({}_\ell G \otimes_Z k) = H_\star(G; k)$ .

Proof: [1].  $\square$

Since  $\ell$  is invertible in  $F$  and  $[N, N]$  is abelian, we can compute

$$H_q([N, N]; k) = \begin{cases} 0 & \text{for } q > 0 \\ k & \text{for } q = 0 \end{cases}$$

Thus in the spectral sequence (2) we have  $E_{p,q}^2 = 0$  for  $q > 0$  and  $E_{p,0}^2 = H_p(N/[N, N]; k)$ . But  $N/[N, N]$  is also abelian; we can compute

$$H_p(N/[N, N]; k) = \begin{cases} 0 & \text{for } q > 0 \\ k & \text{for } q = 0. \end{cases}$$

Therefore the entire spectral sequence is zero except that  $E_{0,0}^2 = k$ . Thus

$$H_t(N; k) = \begin{cases} 0 & \text{for } q > 0 \\ k & \text{for } q = 0. \end{cases}$$

We now return to the spectral sequence associated to the short exact sequence  $1 \rightarrow N \rightarrow S_{p,n} \rightarrow GL_p \times G_{n-p} \rightarrow 1$ . By the above, we now have

$$E_{s,t}^2 = \begin{cases} 0 & \text{for } t > 0 \\ H_s(\mathrm{GL}_p \times G_{n-p}) & \text{for } t = 0 \end{cases}$$

Since this spectral sequence converges to the homology of  $S_{p,n}$ , we have

$$H_s(\mathrm{GL}_p \times G_{n-p}) = H_s(S_{p,n}) \quad \text{for all } s .$$

Case 2.  $\mathrm{char} F = \mathrm{char} k = 0$  .

Since  $\mathrm{char} F = 0$ , there is an imbedding of the rational numbers  $\mathbb{Q}$  into  $F$ . We identify an element  $d \in \mathbb{Q}^*$  with the matrix

$$D = \begin{pmatrix} (d) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (d^{-1}) \end{pmatrix}$$

in  $\mathrm{GL}_p \times G_{n-p}$ , where  $(d)$  is the  $p \times p$  diagonal matrix with  $(d)_{ii} = d$ . We can then define an action of  $\mathbb{Q}^*$  on  $S_{p,n}$  by  $d \cdot A = D^{-1}AD$ . We will try to identify the induced action of  $\mathbb{Q}^*$  on the spectral sequence with

$$E_{s,t}^2 = H_s(\mathrm{GL}_p \times G_{n-p}; H_t(N)) \Rightarrow H_{s+t}(S_{p,n}) . \quad (3)$$

The action of  $\mathbb{Q}^*$  on  $\mathrm{GL}_p \times G_{n-p}$  is trivial since  $F$  is commutative.

The action of  $Q^*$  on  $H_*(N)$  is more complicated. To identify it we again look at the short exact sequence (1) and the resulting spectral sequence (2) with

$$E_{p,q}^2 = H_p(N/[N, N]; H_q([N, N])) = H_{p+q}(N) .$$

A short computation shows that  $Q^*$  acts on  $[N, N]$  as multiplication by  $d^2$  and on  $N/[N, N]$  as multiplication by  $d$ . Note that  $[N, N]$  is in the center of  $N$ , so the action of  $[N, N]$  on  $H_*([N, N])$  induced by conjugation is trivial.

By the universal coefficient theorem, we have

$$H_p(N/[N, N]; H_q([N, N])) = H_p(N/[N, N]) \otimes H_q([N, N])$$

(the torsion is zero since we have coefficients in a field).

By the formula for the homology of an abelian group (Lemma 2.4), this is equal to

$$\Lambda^p(N/[N, N] \otimes_{\mathbb{Z}} k) \otimes \Lambda^q([N, N] \otimes_{\mathbb{Z}} k) .$$

Thus  $d \in Q^*$  acts on the  $E_{p,q}^2$  term as multiplication by  $d^{p+2q}$ . The action commutes with the differentials, so the action on  $E_{p,q}^\infty$  is multiplication by  $d^{p+2q}$ . Thus on the filtration of  $H_t(N)$  we have  $d$  acting as multiplication by the following powers of  $d$  :

$$\left( \begin{array}{c} H_{0,t} \\ \hline d^{2t} \end{array} \right) \left( \begin{array}{c} H_{1,t-1} \\ \hline d^{2t-1} \end{array} \right) \cdots \left( \begin{array}{c} H_{t-1,1} \\ \hline d^{t+1} \end{array} \right) \left( \begin{array}{c} H_{t,0} = H_t(N) \\ \hline d^t \end{array} \right)$$

(here  $E_{i,t-1}^\infty = H_{i,t-i}/H_{i-1,t-i+1}$ ).

Returning to the spectral sequence (3), we note that the action of  $Q^*$  on  $E_{s,t}^2$  is just the action on  $H_t(N)$ , since the action on  $GL_p \times G_{n-p}$  is trivial. The action again commutes with the differentials, so the action on  $E_{s,t}^\infty$  is the same as the action on  $E_{s,t}^2$ , i.e. the action on  $H_t(N)$ . Thus in the filtration of  $H_r(S_{p,n})$  with

$$E_{k,r-k}^\infty = H_{i,t-i}/H_{i-1,t-i+1}$$

we have  $d \in Q^*$  acting by multiplication by the following powers of  $d$ :

$$\begin{array}{c} \left( \begin{array}{c} d^{2(r-k)} \\ \hline \end{array} \right) \cdots \left( \begin{array}{c} d^{r-k+1} \\ \hline \end{array} \right) \left( \begin{array}{c} d^{r-k} \\ \hline \end{array} \right) \\ \swarrow \quad \searrow \\ \left( \begin{array}{c} H_{0,r} \\ \hline \end{array} \right) \cdots \left( \begin{array}{c} H_{k,r-k} \\ \hline \end{array} \right) \cdots \left( \begin{array}{c} H_{r-1,1} \\ \hline \end{array} \right) \left( \begin{array}{c} H_{r,0} = H_r(S_{p,n}) \\ \hline \end{array} \right) \end{array}$$

Since  $d^m \neq 1$  for  $d \neq 1$  or  $-1$  and  $m > 0$ , the action of  $Q^*$  on  $H_{k,r-k}$  is not trivial unless  $r - k = 0$ . However, we know that the action of  $Q^*$  on  $H_r(S_{p,n}) = H_{r,0}$  is trivial since  $Q^*$  acts on  $S_{p,n}$

by inner automorphism. Therefore we must have

$H_{k,r-k} = 0$  for  $r - k > 0$ , which means  $E_{s,t}^{\infty} = 0$  for  $t > 0$  and  $E_{s,0}^{\infty} = H_s(S_{p,n})$ . Since this is a first quadrant spectral sequence, the map  $E_{s,0}^{\infty} \rightarrow E_{s,0}^2$  is injective, i.e.

$$1 \rightarrow H_s(S_{p,n}) \xrightarrow{\pi_*} H_s(GL_p \times G_{n-p}) .$$

But this map splits, so  $\pi_*$  is also onto, i.e.

$$\pi_* : H_s(S_{p,n}) \xrightarrow{\cong} H_s(GL_p \times G_{n-p}) .$$

Remark. (The symplectic case). If instead of  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$  we take the form  $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ , we get the symplectic group  $Sp_{2n}$  instead of  $O_{n,n}$ . This proof works for the symplectic group, i.e. the homology stabilizes for  $n \geq 3k + 1$ . In this case the simplicial complex of flags of totally isotropic subspaces is the building associated to  $Sp_{2n}$ , and since the Weyl group can be shown to be a finite euclidean reflection group, this building is homotopy equivalent to a wedge of spheres [3]. The proof in section 1 also shows that the building is a wedge of spheres. The stabilizer of the subspace  $\langle e_1, \dots, e_p \rangle$  has exactly the same form as in the  $O_{n,n}$  case; namely it consists of matrices of the form

$$\begin{pmatrix} \alpha & * & * \\ 0 & A & * \\ 0 & 0 & t_{\alpha}^{-1} \end{pmatrix}$$

where  $\alpha \in GL_p(F)$  and  $A \in Sp_{2(n-p)}$ . The kernel of the projection

$$\begin{pmatrix} \alpha & * & * \\ 0 & A & * \\ 0 & 0 & t_\alpha^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & t_\alpha^{-1} \end{pmatrix}$$

is all matrices of the form  $\begin{pmatrix} I_p & a & b & c \\ 0 & I_{n-p} & 0 & t_b \\ 0 & 0 & I_{n-p} & -t_a \\ 0 & 0 & 0 & I_p \end{pmatrix}$ , and

the commutator subgroup  $[N, N]$  is the matrices

$$\begin{pmatrix} I_p & 0 & 0 & x \\ 0 & I_{n-p} & 0 & 0 \\ 0 & 0 & I_{n-p} & 0 \\ 0 & 0 & 0 & I_p \end{pmatrix}, \text{ where } x \text{ is any } p \times p \text{ matrix,}$$

so  $[N, N]$  = the additive abelian group of  $p \times p$  matrices and  $N/[N, N]$  = the additive abelian group of  $p \times 2(n-p)$  matrices. The action of  $Q^*$  on these groups is the same as in the  $O_{n,n}$  case, and the rest of the proof is identical.

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