

Rational Homology of Bianchi Groups*

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1. Introduction

Let d be a square-free positive integer, and O_{-d} the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Following work of Bianchi [3] and Humbert [10], Swan [18] described fundamental domains for the action of the groups $SL(2, O_{-d})$ on hyperbolic 3-space \mathbb{H} and used these fundamental domains to give presentations for $SL(2, O_{-d})$ for some small values of d . However, the virtual cohomological dimension of $SL(2, O_{-d})$ is only 2, and there is in fact a 2-dimensional cellular retract of \mathbb{H} which is invariant under the action of $SL(2, O_{-d})$ (see Ash [1]). In [12] Mendoza gave an explicit description of such a complex, and computed the cell structure for the cases where O_{-d} is a Euclidean ring ($d = 1, 2, 3, 7, 11$). In [14] these complexes were used to compute the integral homology of $SL(2, O_{-d})$ and related groups in the Euclidean cases. The purpose of the present paper is to describe how to compute the cell structure and homology of Mendoza's complexes for any O_{-d} ; this program is carried out far enough to compute the rational homology of $SL(2, O_{-d})$ for values of d such that the discriminant D of $\mathbb{Q}(\sqrt{-d})$ is greater than -100 . Of particular interest is the exact determination of the rank of the cuspidal cohomology H_{cusp}^1 for these values of d , corresponding to the dimension of the space of cuspidal harmonic automorphic forms for $SL(2, O_{-d})$ (see [16]). It is shown that for

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}, \quad \dim H_{\text{cusp}}^1 = 0.$$

Work of Zimmert [19], Baker [2], Grunewald and Schwermer [7], and Rohlf's [13] implies that for all other values of d , the rank of the cuspidal cohomology is greater than zero; thus the above is a complete list of values of d for which the cuspidal cohomology vanishes.

The computations in this paper also show that for discriminants greater than -100 , all of the torsion in the integral homology of $SL(2, O_{-d})$ comes from the finite subgroups of $SL(2, O_{-d})$. Since Kramer [11] has completely determined the

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partially ordered set of finite subgroups for low discriminants, it is now possible to compute the integral homology of $\mathrm{SL}(2, O_{-d})$ and related groups as in [14].

The paper is organized as follows. Section 2 briefly recalls well-known results about the action of $\mathrm{SL}(2, O_{-d})$ on hyperbolic 3-space \mathbb{H} , the quotient $\mathbb{H}/\mathrm{SL}(2, O_{-d})$, and the rational and cuspidal cohomology of $\mathrm{SL}(2, O_{-d})$. Section 3 describes Mendoza's complex and lists the properties of this complex that we will need. Section 4 describes an algorithm for determining the cell structure of various pieces of the Mendoza complex and uses reduction theory to produce a computable finite bound on the number of steps required to complete the algorithm. Section 5 describes the group of symmetries of the complex, thereby simplifying the process of determining the cell structure. Section 6 contains theorems which aid in determining the quotient of the complex by $\mathrm{SL}(2, O_{-d})$, and finally Sect. 7 contains the results of the computations for $D > -100$.

2. Preliminaries

The ring of integers O_{-d} is a \mathbb{Z} -lattice in the complex plane, generated by 1 and ω , where $\omega = (1 + \sqrt{-d})/2$ if d is congruent to 3 mod 4, and $\omega = \sqrt{-d}$ otherwise. The discriminant D of $\mathbb{Q}(\sqrt{-d})$ is equal to $-d$ if d is congruent to 3 mod 4, and $-4d$ otherwise. Throughout this paper we assume $D < -4$ so that the only units in the ring O_{-d} are 1 and -1 . For a complete discussion of the cases $D = -3, -4$, see [14].

We study the homology of $\Gamma_d = \mathrm{SL}(2, O_{-d})$ by considering Γ_d as a discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ and studying the action of Γ_d on the symmetric space $\mathbb{H} = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$. \mathbb{H} is naturally identified with hyperbolic 3-space, and we will use the upper half-space model for \mathbb{H} . Thus $\mathbb{H} = \{(z, r) : z \in \mathbb{C} \text{ and } r \in \mathbb{R}, r > 0\}$, and the action of an element

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathrm{GL}(2, \mathbb{C})$ on \mathbb{H} is given by the formula

$$(2.1) \quad g(z, r) = \left(\frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}r^2}, \frac{\|\det g\| \cdot r}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}r^2} \right).$$

This extends the standard action by linear fractional transformations of $\mathrm{GL}(2, \mathbb{C})$ on the extended complex plane $\mathbb{C} \cup \infty$ (thought of as the boundary of \mathbb{H}).

The points of $\mathbb{Q}(\sqrt{-d}) \cup \infty$ in $\mathbb{C} \cup \infty$ are called *cusps*. If λ is a cusp, we can write $\lambda = \alpha/\beta$, with α and β in O_{-d} (by convention, $\infty = 1/0$). We can then associate to λ the element of the class group of O_{-d} represented by the ideal $\langle \alpha, \beta \rangle$ of O_{-d} generated by α and β ; this is called the *class* of λ , and denoted $[\lambda]$. The class $[\lambda]$ is independent of the choice of α and β .

We now record some well-known facts about the action of Γ_d on \mathbb{H} :

(2.2) The orbits of the action of Γ_d on the set of cusps are in one-to-one correspondence with the elements of the class group of O_{-d} [15].

(2.3) The orbit space $M = \mathbb{H}/\Gamma_d$ is the interior of a three-manifold with boundary; its boundary is the disjoint union of h toruses, where h is the class number of O_{-d} [6].

$$(2.4) \quad H^*(\Gamma_d; \mathbb{Q}) \cong H^*(M; \mathbb{Q}).$$

Notation. Unless coefficients in homology and cohomology groups are given specifically, we will assume the coefficients are \mathbb{Q} with trivial action.

The contribution to the rational cohomology of Γ_d coming from the boundary of ∞ is easily understood from the long exact sequence of the pair $(M, \partial M)$ and Poincaré duality. We have:

(2.1) **Proposition.** *Let $i: \partial M \rightarrow M$ be the inclusion map. Then $\dim(i_*H_1(\partial M)) = h$ and $\dim(i_*H_2(\partial M)) = h - 1$, where h is the class number of O_{-d} .*

Proof. [15]. \square

(2.6) **Definition.** The *cuspidal cohomology* H_{cusp}^n ($n = 1$ or 2) of Γ_d is the kernel of the map $i^*: H^n(M) \rightarrow H^n(\partial M)$ induced by inclusion.

Although this is not the standard definition of cuspidal cohomology, it agrees with the standard definition in this case (see [7] for the correspondence).

(2.7) **Lemma.** *There are short exact sequences*

$$H_1(\partial M) \xrightarrow{i_*} H_1(M) \rightarrow H_{\text{cusp}}^2 \rightarrow 0$$

and

$$H_2(\partial M) \xrightarrow{i_*} H_2(M) \rightarrow H_{\text{cusp}}^1 \rightarrow 0$$

Proof. By Poincaré duality,

$$H_{\text{cusp}}^1 \cong \ker(\partial_*: H_2(M, \partial M) \rightarrow H_1(\partial M)).$$

By the long exact homology sequence, this is

$$= \text{cok}(i_*: H_2(\partial M) \rightarrow H_2(M)).$$

Similarly, $H_{\text{cusp}}^2 = \text{cok}(i_*: H_1(\partial M) \rightarrow H_1(M))$. \square

(2.8) **Proposition.** $\dim(H_{\text{cusp}}^1) = \dim(H_{\text{cusp}}^2)$.

Proof. Since M is a three-manifold with boundary, we have the Euler characteristic $\chi(M) = \chi(\partial M)/2$. Since ∂M is a disjoint union of toruses, we have $\chi(M) = 0$, i.e.

$$\dim(H_2(M)) - \dim(H_1(M)) + \dim(H_0(M)) = 0$$

or

$$\dim(H_2(M)) = \dim(H_1(M)) - 1.$$

By (2.7), we have

$$\dim(H_2(M)) = \dim(H_{\text{cusp}}^2) + \dim(i_*(H_1(\partial M)))$$

and

$$\dim(H_1(M)) = \dim(H_{\text{cusp}}^1) + \dim(i_*(H_2(\partial M))).$$

The proposition now follows from (2.5). \square

Thus in order to compute the dimension of the rational homology of $\text{SL}(2, O_{-a})$ we need only compute one of the homology groups $H_1(M)$ or $H_2(M)$. For most discriminants, we will actually determine the homotopy type of M ; however, in complicated cases it is easier to simply determine the homology groups. By (2.5) and (2.7), the only uncertainty lies in the dimension of the cuspidal group.

3. Mendoza's Construction

In this section we briefly describe the construction of an $\text{SL}(2, O_{-a})$ -invariant, two-dimensional cell complex which is a deformation retract of \mathbb{H} . This construction is due to Mendoza. For details and proofs we refer to [12] or [14].

We first define the notion of the distance between a point (z, r) of \mathbb{H} and a cusp $\lambda \in \mathbb{Q}(\sqrt{-d}) \subset \partial\mathbb{H}$, due to Siegel [17]. Write $\lambda = \alpha/\beta$, with α and β in O_{-a} . Then the distance from (z, r) to λ is

$$(3.1) \quad d((z, r), \lambda) = \frac{\|\beta z - \alpha\|^2 + \|\beta r\|^2}{r \cdot N\langle \alpha, \beta \rangle},$$

where $N\langle \alpha, \beta \rangle$ is the norm of the ideal of O_{-a} generated by α and β , and $\|\cdot\|$ is the standard complex norm.

The set of points of \mathbb{H} which are equidistant from two cusps λ and μ , denoted $S(\lambda, \mu)$, is a totally geodesic plane in \mathbb{H} ; in our model of \mathbb{H} this is a hemisphere or plane perpendicular to the boundary plane \mathbb{C} . In particular, $S(\infty, \alpha/\beta)$ is the hemisphere centered at $(\alpha/\beta, 0)$ with radius the square root of $(N\langle \alpha, \beta \rangle / N\langle \beta \rangle)$.

The following *transformation rule* tells us how the distance function behaves under the action of $\text{GL}(2, \mathbb{Q}(\sqrt{-d}))$.

(3.2) **Proposition.** If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Q}(\sqrt{-d}))$ and $\lambda = \alpha/\beta$ is a cusp, then

$$d(g(z, r), g(\lambda)) = \frac{\|\det g\| \cdot N\langle \alpha, \beta \rangle}{N\langle g(\alpha, \beta) \rangle} d((z, r), \lambda),$$

where $\langle g(\alpha, \beta) \rangle$ is the ideal generated by $a\alpha + b\beta$ and $c\alpha + d\beta$.

Proof. The proof is straightforward using (2.1) and (3.1). \square

As a corollary, we have that the distance function is invariant under elements of $\text{SL}(2, O_{-a})$, i.e.

(3.3) **Proposition** (Mendoza). If $g \in \text{SL}(2, O_{-a})$ then $d(g(z, r), g(\lambda)) = d((z, r), \lambda)$.

Proof. Since $g \in \text{SL}(2, O_{-a})$, it carries the lattice spanned by α and β isomorphically onto the lattice spanned by $a\alpha + b\beta$ and $c\alpha + d\beta$; since $\det g = 1$, the areas of

fundamental domains for these lattices are the same, i.e. $N\langle\alpha, \beta\rangle = N\langle g(\alpha, \beta)\rangle$. \square

(3.4) *Definition.* Let λ be a cusp. The *minimal incidence set* $H(\lambda)$ of λ is the closure of the set of points in \mathbb{H} which are closer to λ than to any other cusp, i.e.

$$H(\lambda) = \{(z, r) : d((z, r), \lambda) \leq d((z, r), \mu) \text{ for all cusps } \mu \neq \lambda\}.$$

(3.5) *Definition.* The *Mendoza complex* X_d is

$$X_d = \bigcup_{\lambda \neq \mu} H(\lambda) \cap H(\mu) = \bigcup_{\lambda} \partial H(\lambda)$$

It is instructive to visualize the case of $SL(2, \mathbb{Z})$ acting on the hyperbolic plane H (upper half space). Given a cusp $p/q \in \mathbb{Q}$ with p and q relatively prime integers, the distance from a point (x, r) of H to $(p/q, 0)$ in ∂H is equal to $((qx - p)^2 + (qr)^2)/r$; the set of points equidistant from p/q and ∞ is the semicircle centered at p/q with radius $1/q$; $H(\infty)$ consists of all points of H which lie above the set of semicircles with radius 1 centered at integer points $(p, 0)$. The complex X is the familiar tree for $SL(2, \mathbb{Z})$. Each minimal set $H(p/q)$ is the closure of a connected component of $H - X$.

We have the following facts about the complex X_d :

(3.6) **Theorem** (Mendoza). (i) X_d is an $SL(2, O_{-d})$ -invariant, two-dimensional CW-complex, with cellular $SL(2, O_{-d})$ -action.

(ii) X_d is a deformation retract of \mathbb{H} by an $SL(2, O_{-d})$ -invariant deformation retraction.

(iii) $X_d/SL(2, O_{-d})$ is a finite CW-complex.

From this theorem it is evident that $X_d/SL(2, O_{-d})$ is a spine for the three-manifold $M = \mathbb{H}/SL(2, O_{-d})$. Thus to determine the homotopy type of $\mathbb{H}/SL(2, O_{-d})$, we need only determine the homotopy type of the finite complex $X_d/SL(2, O_{-d})$.

4. Cell Structure

In order to determine the cell structure of $X_d = \bigcup_{\lambda} \partial H(\lambda)$, we first consider the simpler problem of determining the cell structure of $H(\lambda)$ for a fixed cusp λ . In order to accomplish this we will choose an element L of $GL(2, \mathbb{Q}(\sqrt{-d}))$ which sends ∞ to λ , and then construct $L^{-1}H(\lambda)$. This is equivalent to constructing the set of points closest to ∞ for a few distance function, which depends on λ .

Let n be the smallest positive real integer such that $\lambda \in (1/n)O_{-d}$, and write $\lambda = \alpha/n$ with $\alpha \in O_{-d}$. If $\lambda = \infty$, let $\alpha = 1$ and $n = 0$. The cusp λ and numbers α and n will remain fixed throughout this section

$$\text{Fix } L = \begin{bmatrix} \alpha & 1 \\ n & 0 \end{bmatrix} \in GL(2, \mathbb{Q}(\sqrt{-d})). \text{ If } \lambda = \infty, \text{ take } L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{id}.$$

(4.1) *Definition.* The λ -distance from a point (z, r) in \mathbb{H} to a cusp μ , $d_\lambda((z, r), \mu)$, is equal to $d(L(z, r), L(\mu))$.

Using this “new” distance, we can define the hemisphere $S_\lambda(\mu, \nu)$ to be the set of points equidistant from μ and ν with respect to λ -distance, and the minimal incidence set $H_\lambda(\mu)$ to be the closure of the set of points closer to μ than to any other cusp with respect to λ -distance.

Note that $H_\lambda(\infty)$ is the set of points in \mathbb{H} which lie above all hemispheres $S_\lambda(\infty, \mu)$ for $\mu \in \mathbb{Q}(\sqrt{-d})$. The boundary $\partial H_\lambda(\infty)$ has a natural cell structure coming from the hemispheres $S_\lambda(\infty, \mu)$. To determine the cell determined by $S_\lambda(\infty, \mu)$, note that $S_\lambda(\infty, \mu) \cap S_\lambda(\infty, \nu)$ is either empty or is a semicircle perpendicular to the boundary plane. If the intersection is non-empty, let $l(\mu, \nu)$ be the line determined by the endpoints of this semicircle. Let $\pi : \mathbb{H} \rightarrow \mathbb{C}$ be the projection defined by $\pi(z, r) = z$. A point (z, r) on $S_\lambda(\infty, \mu)$ is closer to ∞ than ν if and only if z lies on the same side of $l(\mu, \nu)$ as μ . Thus $\pi(S_\lambda(\infty, \mu) \cap H_\lambda(\infty))$ is the solution set of a collection of linear inequalities, determined by the hemispheres $S_\lambda(\infty, \nu)$ which intersect $S_\lambda(\infty, \mu)$ non-trivially.

This shows that $S_\lambda(\infty, \mu) \cap H_\lambda(\infty)$ is either empty or is a convex cell in $S_\lambda(\infty, \mu)$. In addition, if $S_\lambda(\infty, \mu) \cap H_\lambda(\infty)$ is a 0-cell or 1-cell, it lies in the boundary of a 2-cell $S_\lambda(\infty, \nu) \cap H_\lambda(\infty)$, where ν is a cusp such that $S_\lambda(\infty, \mu)$ intersects $S_\lambda(\infty, \nu)$ non-trivially.

Note also that $H_\lambda(\infty) = L^{-1}(H(\lambda))$. Since $L \in \text{GL}(2, \mathbb{C})$, $H_\lambda(\infty)$ is isometric to $H(\lambda)$; thus the cell structure of $\partial H(\lambda)$ is the same as the cell structure of $\partial(H_\lambda(\infty))$, which we will now determine.

(4.2) **Proposition.** *Let $\mu = \gamma/\delta$ be a finite cusp, with γ and δ in O_{-d} . Then $S_\lambda(\infty, \mu)$ is a hemisphere centered at μ with radius the square root of*

$$\frac{N\langle L(\gamma, \delta) \rangle}{N\langle \alpha, n \rangle N\langle \delta \rangle}.$$

Proof. By definition, we have

$$\begin{aligned} S_\lambda(\infty, \mu) &= \{(z, r) : d_\lambda((z, r), \infty) = d_\lambda((z, r), \mu)\} \\ &= \{(z, r) : d(L(z, r), L(\infty)) = d(L(z, r), L(\mu))\}. \end{aligned}$$

By the transformation rule (3.2), this is

$$\begin{aligned} &= \left\{ (z, r) : \|\det L\| \frac{N\langle 1, 0 \rangle}{N\langle \alpha, n \rangle} \cdot d((z, r), \infty) = \|\det L\| \frac{N\langle \gamma, \delta \rangle}{N\langle L(\gamma, \delta) \rangle} \cdot d((z, r), \mu) \right\} \\ &= \left\{ (z, r) : \frac{n}{N\langle \alpha, n \rangle} \frac{1}{r} = \frac{n}{N\langle L(\gamma, \delta) \rangle} \frac{\|\delta z - \gamma\|^2 + \delta \bar{\delta} r^2}{r} \right\} \end{aligned}$$

which simplifies to

$$= \left\{ (z, r) : \|z - \mu\|^2 + r^2 = \frac{N\langle L(\gamma, \delta) \rangle}{N\langle \alpha, n \rangle N\langle \delta \rangle} \right\}. \quad \square$$

In order to actually compute $\partial H_\lambda(\infty)$, we need to know that its cell structure can be determined by a known finite number of calculations, i.e. that we can

determine which of the hemispheres $S_\lambda(\infty, \mu)$ have non-empty intersection with $H_\lambda(\infty)$ by considering a known finite collection of cusps μ . The remainder of this section will establish this fact.

(4.3) **Lemma.** *Let μ be a finite cusp. Write $\mu = \gamma/m$, with $m \in \mathbb{Z}$, $|m|$ minimal and $\gamma \in O_{-d}$. If $\lambda = \alpha/n$ is finite, then $N\langle L(\gamma, m) \rangle = N\langle \alpha\gamma + m, n\gamma \rangle \leq n^2m$. If $\lambda = \infty$, then $N\langle L(\gamma, m) \rangle \leq m$.*

Proof. Assume λ is finite. Note first that we can write $mn = n(\alpha\gamma + m) - \alpha(n\gamma)$, showing that mn is in the ideal $\langle \alpha\gamma + m, n\gamma \rangle$.

Let $\alpha = a_0 + a_1\omega$ and $\gamma = c_0 + c_1\omega$, with a_0, a_1, c_0 , and c_1 in \mathbb{Z} . Since $|m|$ is minimal, we have $\gcd(c_0, c_1, m) = 1$. Thus we may choose integers x, y , and z with $xc_0 + yc_1 + zm = 1$.

If d is not congruent to 3 mod 4, then $\omega = \sqrt{-d}$, and we have

$$nz\omega(\alpha\gamma + m) + ((x - a_0)\omega + (y + a_1dz))(n\gamma) = n(\omega + A),$$

where A is some element of \mathbb{Z} . Thus $\langle \alpha\gamma + m, n\gamma \rangle$ contains $n\langle \omega + A, m \rangle$ as a sublattice, and so $N\langle \alpha\gamma + m, n\gamma \rangle$ divides $n^2N\langle \omega + a, m \rangle$, which divides n^2m .

The case d congruent to 3 mod 4 is similar. The statement for $\lambda = \infty$ follows by the same methods. \square

(4.4) **Definition.** An upper reduction constant for $\mathbb{Q}(\sqrt{-d})$ is a real number C , depending only on d , such that for any point (z, r) in \mathbb{H} , there is at least one cusp μ with $d((z, r), \mu) \leq C$.

Note that if C is an upper reduction constant for $\mathbb{Q}(\sqrt{-d})$ for the standard distance function, then it is also an upper reduction constant for d_λ . We have the following result from classical reduction theory [8]:

(4.5) **Theorem.** *Let D be the discriminant of $\mathbb{Q}(\sqrt{-d})$. Then the square root of $-D/2$ is an upper reduction constant for $\mathbb{Q}(\sqrt{-d})$.*

We can now prove the following useful proposition:

(4.6) **Proposition.** *Let μ be a finite cusp, written $\mu = \gamma/m$ with $\gamma \in O_{-d}$, $m \in \mathbb{Z}$ and $|m|$ minimal. If $m > -N\langle \alpha, n \rangle \cdot (D/2)$, then $S_\lambda(\infty, \mu) \cap H_\lambda(\infty) = \emptyset$.*

Proof. Assume λ is finite. By Propositions (4.2) and (4.3), the square of the radius of $S_\lambda(\infty, \mu)$ is equal to

$$\frac{N\langle \alpha\gamma + m, n\gamma \rangle}{N\langle \alpha, n \rangle \cdot m^2} \leq \frac{n^2}{N\langle \alpha, n \rangle \cdot m}.$$

Therefore, if (z, r) is on $S_\lambda(\infty, \mu)$, we have

$$\begin{aligned} d_\lambda((z, r), \infty)^2 &= \left(\frac{n}{N\langle \alpha, n \rangle} \right)^2 \frac{1}{r^2} \geq \frac{n^2}{N\langle \alpha, n \rangle^2} \frac{N\langle \alpha, n \rangle \cdot m}{n^2} \\ &= \frac{m}{N\langle \alpha, n \rangle} > \frac{-D}{2}. \end{aligned}$$

Thus if $m > N\langle\alpha, n\rangle(-D/2)$, we have $d_\lambda((z, r), \infty)^2 = m/N\langle\alpha, n\rangle > (-D/2)$. Since the square root of $-D/2$ is an upper reduction constant for d_λ , there is some cusp v with $d_\lambda((z, r), v) < d_\lambda((z, r), \infty)$, i.e. (z, r) is not in $H_\lambda(\infty)$.

The case $\lambda = \infty$ is similar. \square

As a consequence of this proposition, we know that we can determine $\partial H_\lambda(\infty)$ by considering only cusps of the form γ/m with $\gamma \in O_{-d}$, $m \in \mathbb{Z}$, and $m \leq (-D \cdot N\langle\alpha, n\rangle)/2$. The next step comes from noting that the set $H(\lambda)$ is invariant under the stabilizer $\Gamma(\lambda)$ of λ in $\mathrm{SL}(2, O_{-d})$; thus $H_\lambda(\infty) = L^{-1}H(\lambda)$ is invariant under the group $L^{-1}\Gamma(\lambda)L$, which stabilizes ∞ and hence acts as a group of translations of the boundary plane \mathbb{C} . The following proposition determines a fundamental domain for this group of translations.

(4.7) Proposition. *Let $\lambda = \alpha/n$ be a finite cusp. Then the group $L^{-1}\Gamma(\lambda)L$ acts on $\partial\mathbb{H} - \{\infty\} = \mathbb{C}$ by translation by elements of the fractional ideal $n\langle\alpha, n\rangle^{-2}$.*

Proof. Choose $c, d \in \langle\alpha, n\rangle^{-1} \subset \mathbb{Q}(\sqrt{-d})$ such that $\alpha c + nd = 1$, and let h be the matrix

$$h = \begin{bmatrix} \alpha & c \\ n & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}(\sqrt{-d})).$$

A straightforward computation shows that a matrix A is in $\Gamma(\lambda)$ if and only if

$$A = h \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} h^{-1} \quad \text{for some } x \in \langle\alpha, n\rangle^{-2}.$$

Thus

$$\begin{aligned} L^{-1}\Gamma(\lambda)L &= \left\{ L^{-1}h \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} h^{-1}L : x \in \langle\alpha, n\rangle^{-2} \right\} \\ &= \left\{ \begin{bmatrix} -n & -n^2x \\ 0 & -n \end{bmatrix} : x \in \langle\alpha, n\rangle^{-2} \right\}. \end{aligned}$$

The action of $\begin{bmatrix} -n & -n^2x \\ 0 & -n \end{bmatrix}$ on $\partial\mathbb{H}$ is the same as the action of $\begin{bmatrix} 1 & nx \\ 0 & 1 \end{bmatrix}$, i.e. translation by $nx \in n\langle\alpha, n\rangle^{-2}$. \square

(4.8) Corollary. *The area of a fundamental domain for the action of $L^{-1}\Gamma(\lambda)L$ on \mathbb{C} is*

$$\frac{n^2}{N\langle\alpha, n\rangle^2} \cdot \frac{\sqrt{-D}}{2} \quad (n \neq 0).$$

Proof. The area of a fundamental domain for O_{-d} is $\sqrt{-D}/2$; the corollary follows from (4.7) because norm is multiplicative. \square

We collect our results so far in the following theorem:

(4.9) **Theorem.** Let λ be a cusp and $L = \begin{bmatrix} \alpha & 1 \\ n & 0 \end{bmatrix}$ the associated matrix. Let P be a fundamental domain for the lattice $n\langle \alpha, n \rangle^{-2}$. Then a fundamental domain for the action of the stabilizer $\Gamma(\lambda)$ of λ on $\partial H(\lambda)$ is given by the union of the cells $H(\lambda) \cap H(L(\gamma/m))$, such that

- (i) m is a real integer, $0 < m < (-D, N\langle \alpha, n \rangle)/2$
- (ii) γ is in O_{-d} and $\gamma/m \in P$, and
- (iii) $\dim[H(\lambda) \cap H(L(\gamma/m))] = 2$.

Notation. The cellular fundamental domain for $\Gamma(\lambda)$ given in the above theorem will be denoted $I(\lambda)$, and its image under L^{-1} will be denoted I_λ or $I_\lambda(\infty)$.

Remarks. For the purpose of finding all 2-cells in $\partial H(\lambda)$, the bound on m given in the theorem is generally much too large. In the following example, we take $d = 23$, and compute I_λ for $\lambda = \infty$, $(1 + \sqrt{-23})/4$, and $(1 - \sqrt{-23})/4$. The theorem implies that $S_\lambda(\infty, \gamma/m)$ may intersect $H_\lambda(\infty)$ only if $m \leq 11$. We will see that in fact $S_\lambda(\infty, \gamma/m)$ intersects $H_\lambda(\infty)$ in a 2-cell only for $m \leq 2$, and hence we can completely determine the cell structure of I_λ by constructing the hemispheres $S_\lambda(\infty, \gamma/m)$ for $m = 1$ and $m = 2$. What the theorem accomplishes is a guarantee that our calculations are complete if we have considered all cusps γ/m with $m \leq 11$.

(4.10) *Example.* Let $d = 23$. The class group of O_{-23} is $\mathbb{Z}/3$; the elements of the class group can be represented by the cusps ∞ , $\omega/2$, and $(\omega + 1)/2$, where $\omega = (1 + \sqrt{-23})/2$. We compute I_λ for each of these cusps.

1. $\lambda = \infty$

By Theorem (4.9), we need only determine $H(\infty) \cap H(\gamma/m)$ for $0 < m \leq 11$ and $\gamma = a + b\omega$ with $0 \leq a, b < m$.

We start with $m = 1$ and $\gamma = 0$. The cell $H(0) \cap H(\infty)$ is the portion of $S(0, \infty)$ which lies outside all hemispheres $S(\gamma/m, \infty)$ which intersect $S(0, \infty)$. After computing the intersections for $\gamma/m = -1/1, 1/1, \omega/2, (\omega - 1)/2, -\omega/2$, and $(-\omega + 1)/2$, we have that $H(0) \cap H(\infty)$ is contained in the hexagon σ on $S(0, \infty)$ with vertices

$$\begin{aligned}
 p_1 &= (z_1, r_1) = \left(\frac{4i}{\sqrt{23}}, \sqrt{7/23} \right) \\
 p_2 &= (z_2, r_2) = \left(\frac{1}{2} + \frac{7i}{\sqrt{23}}, \sqrt{5/23} \right) \\
 p_3 &= (\bar{z}_2, \bar{r}_2) \\
 p_4 &= (-z_1, r_1) \\
 p_5 &= (-z_2, r_2) \\
 p_6 &= (-\bar{z}_2, r_2).
 \end{aligned}$$

The points (z, r) in σ with r smallest are the vertices $p_2, p_3, p_5,$ and $p_6,$ with $1/5 < r^2 < 1/4$. By Proposition (4.3), the square of the radius of $S(\gamma/m, \infty)$ is less than $1/m$; thus $S(\gamma/m, \infty)$ will not intersect σ if $m \geq 5$. We next check that $S(\gamma/m, \infty) \cap \sigma$ is empty or contained in $\partial\sigma$ for all γ/m with $m = 3$ or 4 ; this shows that $\sigma = H(0) \cap H(\infty)$.

We continue with $m = 2$; we must determine $H(\infty) \cap H(\gamma/2)$ for $\gamma = 1, \omega,$ and $\omega + 1$. The set $H(\infty) \cap H(1/2)$ is contained in σ . We have $H(\infty) \cap H(\omega/2) \subset S(\infty, \omega/2)$. By intersecting $S(\infty, \omega/2)$ with $S(\gamma/m, \infty)$ for $\gamma/m = 0/1, \omega/1, (\omega - 1)/2,$ and $(\omega + 1)/2,$ we obtain a parallelogram τ on $S(\infty, \omega/2)$ with vertices $p_1, p_2, q_1 = (-z_1 + \omega, r_1)$ and $q_2 = (-z_2 + \omega, r_2)$. As before, we need only check that no other hemispheres $S(\gamma/m, \infty)$ with $m \leq 4$ intersect the interior of τ . $H(\infty) \cap H((\omega + 1)/2)$ is another parallelogram τ' . Figure 1 is the projection of $\sigma \cup \tau \cup \tau'$ onto the complex plane:

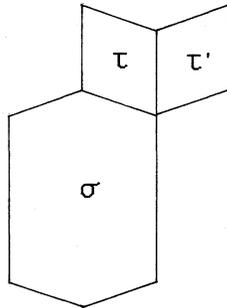


Fig. 1

The translates of $\sigma \cup \tau \cup \tau'$ by $O_{-a} = \Gamma(\infty)$ cover the boundary plane, and we have shown that $H(\infty) \cap H(\gamma/m)$ is either empty or is a cell in the union of these translates for all cusps γ/m . Thus $\sigma \cup \tau \cup \tau'$ is a fundamental domain for the action of $\Gamma(\infty)$ on $\partial H(\infty)$; i.e. by definition, $I(\infty) = \sigma \cup \tau \cup \tau'$.

2. $\lambda = \omega/2$

The ideal $2\langle \omega, 2 \rangle^{-2}$ is generated as a \mathbb{Z} -module by 2 and $(\omega + 1)/2$. As before, we begin with $m = 1$; then γ may be 0 or 1 . We first compute $H_\lambda(\infty) \cap H_\lambda(0)$; this lies on $S_\lambda(0, \infty)$, which has radius $1/\sqrt{2}$. If we compute the intersections of $S_\lambda(\gamma/m, \infty)$ with $S_\lambda(0, \infty)$ for $\gamma/m = 1/1, -1/1, (\omega - 1)/2,$ and $(1 - \omega)/2,$ we obtain a parallelogram $\bar{\tau}$, whose lowest points (z, r) have $r^2 = 5/23$. By Proposition (4.6), the square of the radius of $S_\lambda(\gamma/m, \infty)$ is less than or equal to $2/m$, so we need only consider cusps γ/m with $m \leq 9$. We check that none of these spheres $S_\lambda(\gamma/m, \infty)$ intersect the interior of $\bar{\tau}$ for such cusps γ/m , so $\bar{\tau} = H_\lambda(\infty) \cap H_\lambda(0)$. [The name $\bar{\tau}$ is deliberate; in fact $H_\lambda(\infty) \cap H_\lambda(0) = L^{-1}(H(\infty) \cap H(\omega/2))$, so we are just looking at our previous 2-cell τ from the other side.]

To compute $H_\lambda(\infty) \cap H_\lambda(1)$, we intersect $S_\lambda(1, \infty)$ with the spheres $S_\lambda(\gamma/m, \infty)$ with $\gamma/m = 0/1, 2/1, (\omega - 1)/2, (\omega + 1)/2, (\omega - 3)/2, (3 - \omega)/2, (5 - \omega)/2$ to obtain an octagon ϱ on $S_\lambda(1, \infty)$. The lowest point (z, r) on this octagon again has $r^2 = 5/23$, so we must check that no other hemispheres $S_\lambda(\gamma/m, \infty)$ intersect the interior of ϱ for $m \leq 9$.

In the process of this checking, we have shown that every hemisphere $S_\lambda(\gamma/m, \infty)$ lies below the “roof” of translates of $\bar{\tau}$ and q by $L^{-1}\Gamma(\omega/2)L$; thus $I_{\omega/2} = q \cup \bar{\tau}$ (Fig. 2).

3. $\lambda = (\omega + 1)/2$

The process is nearly identical to the process for $\lambda = \omega/2$, yielding Fig. 3. The labels \bar{q} and $\bar{\tau}'$ are deliberate.

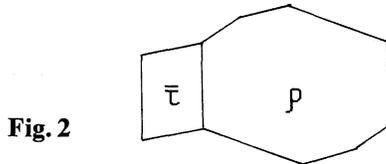


Fig. 2

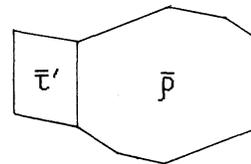


Fig. 3

Note that the 2-cells in these pictures involve only cusps γ/m with $m \leq 2$. We have in fact shown that $\sqrt{23/5}$ is the infimum of the set of upper reduction constants for $\mathbb{Q}(\sqrt{-23})$.

We will explain the apparent symmetries in these pictures in the next section.

5. Symmetries

In this section we describe the symmetries of the complex $X_d = \bigcup \partial H(\lambda)$, which will simplify our computations.

(5.1) *Definition.* $\partial H(\lambda)$ is isomorphic to $\partial H(\mu)$ if there is an orientation preserving isometry of \mathbb{H} taking $H(\lambda)$ to $H(\mu)$.

Let $\lambda = \alpha/\beta$ be a cusp. Recall that the class of λ , denoted $[\lambda]$, is the element of the ideal class group of O_{-d} represented by the ideal $\langle \alpha, \beta \rangle$ generated by α and β .

(5.2) **Proposition.** *If λ and μ are cusps with $[\lambda] = [\mu]$, then $\partial H(\lambda)$ is isomorphic to $\partial H(\mu)$.*

Proof. By (2.1), we can find $g \in \text{SL}(2, O_{-d})$ with $g(\lambda) = \mu$. By (3.3), g preserves the standard distance function, so sends $H(\lambda)$ isomorphically onto $H(\mu)$. \square

(5.3) **Proposition.** *Let λ be a cusp. Then for any cusp μ and any (z, r) in \mathbb{H} we have*

- (i) $d((-z, r), -\mu) = d((z, r), \mu)$ and
- (ii) $d_{\bar{\lambda}}((\bar{z}, r), \bar{\mu}) = d_{\lambda}((z, r), \mu)$.

Proof. Part (i) is immediate from the definition of the distance function (3.1). Part (ii) follows from the definition of λ -distance (4.1), the transformation rule (3.2) and the fact that the norm of an ideal is the same as the norm of the conjugate ideal. \square

(5.4) **Corollary.** $\partial H(\infty)$ is symmetric with respect to the origin and the plane $\text{Im}(z) = 0$.

(5.5) **Corollary.** $H_\lambda(\infty)$ is the reflection of $H_{\bar{\lambda}}(\infty)$ through the plane $\text{Im}(z) = 0$.

Note that (5.2)–(5.5) can be observed in the example $d = 23$ (Figs. 1–3 at the end of Sect. 4) if we take into account the fact that $[(\omega - 1)/2] = [\bar{\omega}/2]$.

There is one less obvious type of symmetry, given by the following theorem.

(5.6) Theorem. *Let λ , μ , and ν be cusps such that $2[\lambda] = 0$ and $[\mu] + [\lambda] = [\nu]$. Then $\partial H(\mu)$ is isomorphic to $\partial H(\nu)$.*

Proof. Write $\lambda = \alpha/\beta$ with $\alpha, \beta \in O_{-d}$. Since $2[\lambda] = 0$, there is an isomorphism

$$g: O_{-d} \oplus O_{-d} \rightarrow \langle \alpha, \beta \rangle \oplus \langle \alpha, \beta \rangle,$$

given by a matrix

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

with $\langle x, y \rangle = \langle z, w \rangle = \langle \alpha, \beta \rangle$ and $\|\det g\| = N\langle \alpha, \beta \rangle$.

For any $(s, t) \in O_{-d} \oplus O_{-d}$, we have

$$g(s, t) = (xs + yt, zs + wt) \in \langle \alpha, \beta \rangle \langle s, t \rangle \oplus \langle \alpha, \beta \rangle \langle s, t \rangle.$$

Since g is an isomorphism, $xs + yt$ and $zs + wt$ actually generate $\langle \alpha, \beta \rangle \langle s, t \rangle$, i.e.

$$(*) \quad \langle g(s, t) \rangle = \langle \alpha, \beta \rangle \langle s, t \rangle.$$

Since norm is multiplicative, this gives

$$(**) \quad N\langle g(s, t) \rangle = N\langle \alpha, \beta \rangle N\langle s, t \rangle = \|\det g\| \cdot N\langle s, t \rangle.$$

Write $\mu = \gamma/\delta$ with $\gamma, \delta \in O_{-d}$. We can apply the transformation rule (3.2) to calculate

$$\begin{aligned} d(g(z, r), g(\mu)) &= \left(\frac{\|\det g\| \cdot N\langle \gamma, \delta \rangle}{N\langle g(\gamma, \delta) \rangle} \right) \cdot d((z, r), \mu) \\ &= d((z, r), \mu) \text{ by } (**). \end{aligned}$$

Thus the map on \mathbb{H} induced by g preserves the distance function, so takes $H(\mu)$ isomorphically to $H(g(\mu))$.

By (*), we have

$$[g(\mu)] = [g(\gamma, \delta)] = [\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle] = [\langle \alpha, \beta \rangle] + [\langle \gamma, \delta \rangle] = [\lambda] + [\mu] = [\nu].$$

Therefore, by Proposition (5.2), we have $\partial H(g(\mu))$ isomorphic to $\partial H(\nu)$. \square

The converse of this theorem is also true, i.e.:

(5.7) Proposition. *If $\partial H(\mu)$ is isomorphic to $\partial H(\nu)$, then $2[\mu] = 2[\nu]$.*

Proof. Any orientation-preserving isometry of \mathbb{H} is given by an element $g \in (P)\mathrm{SL}(2, \mathbb{C})$. If $gH(\mu) = H(\nu)$, then Proposition (3.7) implies that $g^{-1}\Gamma(\mu)g = \Gamma(\nu)$, where $\Gamma(x)$ denotes the stabilizer in $\mathrm{SL}(2, O_{-d})$ of the cusp x . As we noted in the proof of (3.7), this stabilizer $\Gamma(x)$ is isomorphic to $\langle a, b \rangle^{-2}$, for any

$a, b \in O_{-d}$ with $x = a/b$. Since $[\langle a, b \rangle^{-2}] = -2[\langle a, b \rangle] = -2[x]$, we have $\Gamma(\mu)$ is isomorphic to $\Gamma(v)$ if and only if $2[\mu] = 2[v]$. \square

Theorem (5.6) is of special interest when the subgroup C_2 of elements of order two in the class group of O_{-d} is large. This group C_2 is isomorphic to $(\mathbb{Z}/2)^{t-1}$, where t is the number of distinct prime divisors of the discriminant D of $\mathbb{Q}(\sqrt{-d})$ (see, e.g. [9]). If t is large, the group of symmetries of X_d is large and we need to determine relatively few of the fundamental domains I_λ to obtain complete information on the cell structure of X_d .

(5.8) *Example.* Let $d = 5$. The class group of O_{-5} is $\mathbb{Z}/2$; thus $I(\lambda) \cup I(\infty)$ contains a fundamental domain for the action of $\text{SL}(2, O_{-d})$ on X_d , where λ is any non-principal cusp. The cell structure of $I(\lambda)$ is identical to that of $I(\infty)$, since $[\lambda] + [\lambda] = [\infty]$. Let g denote the isomorphism $g: H(\infty) \rightarrow H(\lambda)$ as in Proposition (5.7). If two points (z_1, r_1) and (z_2, r_2) in $H(\infty)$ are equivalent by an element $h \in \text{SL}(2, O_{-d})$, then ghg^{-1} identifies $g(z_1, r_1)$ and $g(z_2, r_2)$ in $H(\lambda)$.

(5.9) *Example.* Let $d = 87$. The class group of O_{-87} is cyclic of order 6, generated by the class $a = [\langle \omega, 2 \rangle]$. We can determine the cell structures of $I(\infty)$ and $I(\omega/2)$ by the methods of Sect. 4. The class $3a$ is of order two in the class group; hence if λ is a cusp of this class, we have $I(\lambda)$ isomorphic to $I(\infty)$. The class $4a$ is $a + 3a$; if μ is a cusp of class $4a$, then $I(\mu)$ is isomorphic to $I(\omega/2)$. Since $5a = (-a)$, and since conjugate ideals represent inverse elements of the class group, a cusp v of class $5a$ has $I(v)$ isomorphic to the reflection of $I(\omega/2)$. Finally, $2a$ is the inverse of $4a$, so the respective complexes are reflections of each other. Notice that in order to determine the structure of all complexes $\partial H(\lambda)$, we needed to determine I_λ for only two of the six classes of cusps.

6. Quotient

In Sect. 4 we found a cellular fundamental domain $I(\lambda)$ for the action of the stabilizer $\Gamma(\lambda)$ on $\partial H(\lambda)$. The quotient $I(\lambda)/\Gamma(\lambda) = \partial H(\lambda)/\Gamma(\lambda)$ is a torus. In this section we find all additional identifications needed to determine the quotient $X_d/\text{SL}(2, O_{-d}) = (\bigcup \partial H(\lambda))/\text{SL}(2, O_{-d})$.

(6.1) *Notation.* Let $\lambda_1, \dots, \lambda_h$ be a set of cusps representing the set of ideal classes of O_{-d} . We may assume $\lambda_1 = \infty$ and $\lambda_i = (\omega - k_i)/n_i$ for $i > 1$, where $k_i \in \mathbb{Z}$ and $n_i = N\langle \omega - k_i, n_i \rangle$ [5].

Let $I = \bigcup_{i=1}^h I(\lambda_i)$. Since $I(\lambda_i)$ is a fundamental domain for the action of $\Gamma(\lambda_i)$ on $\partial H(\lambda_i)$, we have $X_d/\text{SL}(2, O_{-d}) = I/\text{SL}(2, O_{-d})$.

By well-known results in reduction theory [8], there are only a finite number of cusps within a given finite distance of a point in \mathbb{H} . Thus we can make the following definitions:

(6.3) *Definition.* If $(z, r) \in \mathbb{H}$, the *minimal cusp distance* $d(z, r)$ is equal to

$$\min \{d((z, r), \lambda) : \lambda \text{ is a cusp}\}.$$

(6.4) *Definition.* If σ is a cell of X_d , the *minimal cusp set* $\text{cusp}(\sigma)$ is the set of cusps λ such that $d((z, r), \lambda) = d(z, r)$ for all points (z, r) in the interior of σ .

The following proposition is useful in determining which cells can be identified by elements of $\text{SL}(2, O_{-d})$:

(6.4) **Proposition.** *Let σ be a cell of X_d and $g \in \text{SL}(2, O_{-d})$. Then $\text{cusp}(g(\sigma))$ is the set of cusps λ such that $g^{-1}\lambda \in \text{cusp}(\sigma)$; i.e.*

$$\text{cusp}(g(\sigma)) = g(\text{cusp}(\sigma)).$$

Proof. This follows immediately from the $\text{SL}(2, O_{-d})$ -invariance of the distance function [Proposition (3.3)]. \square

The next proposition gives a necessary condition for cells to be equivalent under $\text{SL}(2, O_{-d})$. See Sect. 4 for notation.

(6.5) **Proposition.** *If $\sigma \in I(\lambda_i)$ and $\tau \in I(\lambda_j)$ are two 2-cells which are equivalent under $\text{SL}(2, O_{-d})$, then their images $\sigma' \in I_{\lambda_i}$ and $\tau' \in I_{\lambda_j}$ are congruent by an orientation-reversing Euclidean motion of upper half-space.*

Proof. Since σ is a 2-cell of $I(\lambda_i) \subset \partial H(\lambda_i)$ we can write

$$\sigma = H(\lambda_i) \cap H(\mu) \subset S(\lambda_i, \mu) \text{ for some cusp } \mu.$$

Similarly,

$$\tau = H(\lambda_j) \cap H(\nu) \subset S(\lambda_j, \nu) \text{ for some cusp } \nu.$$

Choose an element g of $\text{SL}(2, O_{-d})$ with $g(\sigma) = \tau$. By Proposition (6.4) we have

$$\{g(\lambda_i), g(\mu)\} = \{\lambda_j, \nu\}.$$

Claim 1. $g(\lambda_i) = \nu$ and $g(\mu) = \lambda_j$.

Proof. If $i \neq j$, then since any element of $\text{SL}(2, O_{-d})$ preserves the class of a cusp, we have $[g(\lambda_i)] = [\lambda_i] \neq [\lambda_j]$; thus $g(\lambda_i) = \nu$ and $g(\mu) = \lambda_j$.

Now consider the case $i = j$. Since both σ and τ are cells of $I(\lambda_i)$, and $I(\lambda_i)$ is a fundamental domain for the action of $\Gamma(\lambda_i)$ on $\partial H(\lambda_i)$, g cannot stabilize λ_i . Therefore in this case too, $g(\lambda_i) = \nu$ and $g(\mu) = \lambda_j$. \square

Let $L_i = \begin{bmatrix} \omega - k_i & 1 \\ n_i & 0 \end{bmatrix}$ and $L_j = \begin{bmatrix} \omega - k_j & 1 \\ n_j & 0 \end{bmatrix}$. Then $\sigma' = L_i^{-1}(\sigma)$ and $\tau' = L_j^{-1}(\tau)$. Let

$$S_i = L_i^{-1}(S(\lambda_i, \mu)) = S_{\lambda_i}(\infty, L_i^{-1}(\mu))$$

and

$$S_j = L_j^{-1}(S(\lambda_j, \nu)) = S_{\lambda_j}(\infty, L_j^{-1}(\nu)).$$

Note that $L_j^{-1}g L_i(\sigma') = \tau'$, and in fact $L_j^{-1}g L_i(S_i) = S_j$.

Claim 2. Let $h = L_j^{-1}g L_i$. Then $h|_{S_i} = \Phi|_{S_i}$ for some euclidean motion Φ of upper half-space.

Proof. Since h is a hyperbolic isometry, it suffices to show that S_i and $S_j = h(S_i)$ have the same radius.

Let $(z', r') \in S_i$ and $(w', s') = h(z', r') \in S_j$. Set $(z, r) = L_i(z', r')$, then

$$\begin{aligned} 1/r' &= d((z', r'), \infty) \\ &= d(L_i^{-1}(z, r), L_i^{-1}(\lambda_i)). \end{aligned}$$

By the transformation rule, this is

$$\begin{aligned} &= (1/n_i) \cdot n_i \cdot d((z, r), \lambda_i) \\ &= d((z, r), \lambda_i). \end{aligned}$$

Let $(w, s) = g(z, r)$; thus $(w', s') = L_j^{-1}(w, s)$. As above, we obtain

$$1/s' = d((w, s), \lambda_j).$$

Since distance is invariant under $\mathrm{SL}(2, O_{-d})$, this is

$$\begin{aligned} &= d(g^{-1}(w, s), g^{-1}(\lambda_j)) \\ &= d((z, r), \mu) \text{ by Claim 1.} \end{aligned}$$

Since $(z, r) \in S(\lambda_i, \mu)$, this is

$$= d((z, r), \lambda_i)$$

which by our first calculation is

$$= 1/r'.$$

Thus $s' = r'$ whenever $(w', s') = h|_{S_i}(z', r')$. This implies that the radius of S_i is equal to the radius of S_j , as was to be shown. \square

It remains only to show that the Euclidean isometry Φ which agrees with h on S_i must reverse orientation. Since Φ preserves upper halfspace, this is equivalent to showing that $\Phi|_{\partial\mathbb{H}}$ reverses orientation. We know that $h(\infty) = L_j(v)$ and $h(L_i(\mu)) = \infty$ by Claim 1, i.e. h reverses the vertical direction. However, h preserves the orientation of \mathbb{H} ; therefore the projection of $h|_{S_i}$ must reverse orientation on the boundary plane. \square

The next theorem tells us that in order to determine all identifications on I modulo $\mathrm{SL}(2, O_{-d})$, we need only decide which 2-cells are identified. All other identifications are consequences of these and of identification by the stabilizers $\Gamma(\lambda_i)$.

Note that for any $\lambda \in \{\lambda_1, \dots, \lambda_h\}$, the stabilizer of a 2-cell σ in $I(\lambda)$ is either trivial or $\mathbb{Z}/2$. If the stabilizer is $\mathbb{Z}/2$, we can subdivide σ into two 2-cells, each with trivial stabilizer. We will call the resulting subdivided complex $I'(\lambda)$. Let I' be the disjoint union of the $I'(\lambda_i)$. Let $p: I' \rightarrow X_d$ be the map induced by the inclusions $I'(\lambda_i) \rightarrow X_d$, and Ψ the composition of p with the quotient map $q: X_d \rightarrow X_d/\mathrm{SL}(2, O_{-d})$.

(6.6) Theorem. *Let q and q' be i -cells ($i=0$ or 1) of I' which are equivalent modulo $\mathrm{SL}(2, O_{-d})$, i.e. $\Psi(q) = \Psi(q')$. Then there is a sequence of pairs (σ_k, ϱ_k) , $1 \leq k \leq n$, with $\varrho_1 = q$, $\varrho_n = q'$, σ_k a 2-cell of I' and $\varrho_k \subset \partial\sigma_k$ such that either*

(i) $q_{k+1} = g(q_k)$ for an element g in the stabilizer of some λ_i (g may be the identity) or

(ii) (σ_k, q_k) is equivalent to (σ_{k+1}, q_{k+1}) modulo $SL(2, O_{-d})$.

Proof. Let M denote the quotient $\mathbb{H}/SL(2, O_{-d})$, and $M(k)$ the image in M of the k -skeleton of I' , i.e. $M(k) = \Psi(I'(k))$.

Since the stabilizers of 2-cells in I' are trivial, the map Ψ gives a two-to-one correspondence between 2-cells of I' and 2-cells in $M(2)$, which can be thought of a one-to-one correspondence between 2-cells of I' and “sides” of 2-cells in $M(2)$.

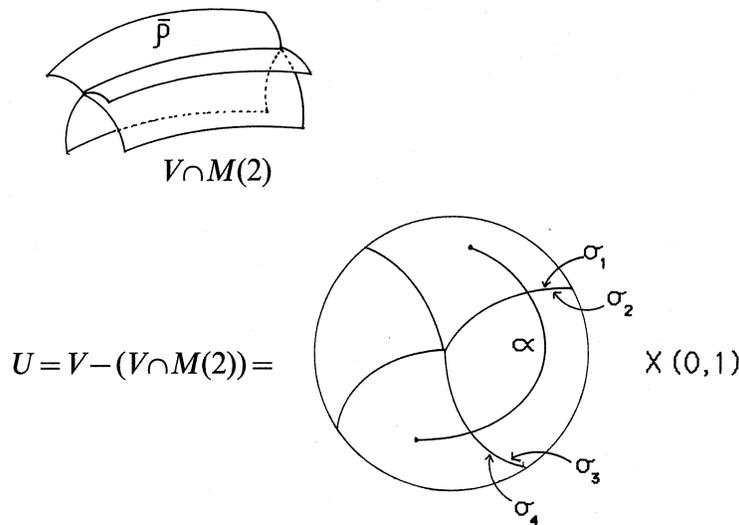
Let $\bar{q} = \Psi(q) = \Psi(q') \subset M(i)$. Let U be a small neighborhood of \bar{q} in M which does not intersect any other i -cell of $M(i)$, and let $V = U - (U \cap M(2))$.

Claim. The connected components of V are in one-to-one correspondence with the orbits of $\Psi^{-1}(\bar{q})$ under the stabilizers $\Gamma(\lambda_i)$.

Proof. Let $\tau \in \Psi^{-1}(\bar{q})$. Then τ is an i -cell of $I(\lambda)$ for some $\lambda \in \{\lambda_1, \dots, \lambda_h\}$. Let γ_τ be the (unique) geodesic in \mathbb{H} from λ to the barycenter of τ . Then $\Psi(\gamma_\tau)$ intersects U in exactly one component of V . If $\Psi(\gamma_{\tau'})$ enters the same connected component of V for some $\tau' \neq \tau$ in $\Psi^{-1}(\bar{q})$, then the path $\gamma_{\tau'}^{-1}\gamma_\tau$ is homotopic to a path which doesn't intersect $M(2)$, i.e. $\gamma_{\tau'}^{-1}\gamma_\tau$ represents an element of the stabilizer $\Gamma(\lambda)$ of λ , and the i -cell τ' is equivalent to τ modulo $\Gamma(\lambda)$.

To obtain the sequence (σ_k, q_k) , connect the component of V corresponding to q with the component corresponding to q' by a path α in U which intersects only 2-cells of $M(2)$ (see Fig. 4). Then σ_k is the 2-cell corresponding to the k^{th} side of a 2-cell in $M(2)$ encountered along α . If g_k is an element of $SL(2, O_{-d})$ which identifies σ_{2k-1} with σ_{2k} , then $q_{2k} = g(q_{2k-1})$. The 1-cell $q_{2k+1} \subset \partial\sigma_{2k+1}$ which maps to \bar{q} is equivalent to q_{2k} modulo a cusp stabilizer, since they correspond to the same component of V . \square

$i = 1$



q_4 is equivalent to q' modulo a cusp stabilizer.

Fig. 4

(6.7) *Example.* Let $d=23$. In Example (3.10) we computed I_λ for $\lambda = \infty, \omega/2$, and $(\omega+1)/2$. After translating I_λ back to $I(\lambda) \subset X_d$, we have:

$$\begin{aligned} I(\infty) &= (H(\infty) \cap H(0)) \cup (H(\infty) \cap H(\omega/2)) \cup (H(\infty) \cap H((\omega+1)/2)) \\ &= \sigma \cup \tau \cup \tau' \\ I(\omega/2) &= (H(\infty) \cap H(\omega/2)) \cup (H(\omega+1)/2 \cap H(\omega/2)) \\ &= \tau \cup \varrho \\ I((\omega+1)/2) &= (H((\omega+1)/2) \cap H(\omega/2)) \cup (H((\omega+1)/2) \cap H(\infty)) \\ &= \tau' \cup \varrho. \end{aligned}$$

by Proposition (6.4), the only possible identifications are by the stabilizers, and the identification of σ to itself given by the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

In this case, it is easiest to determine the quotient by simply drawing the pictures:

Figure 5 shows $I(\omega/2)/\Gamma(\omega/2)$, where the shaded region is the image of τ and the unshaded region is the image of ϱ :

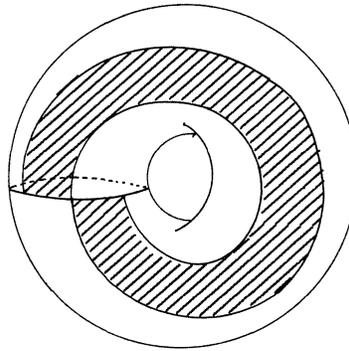


Fig. 5

Figure 6 shows $I((\omega+1)/2)/\Gamma((\omega+1)/2)$, where the shaded region is τ' and the unshaded region is again ϱ :

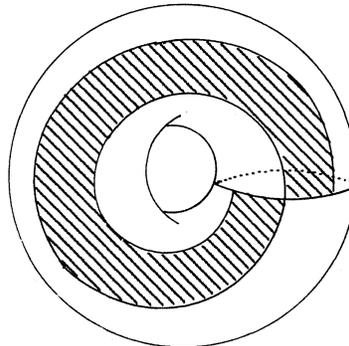


Fig. 6

Figure 7 shows $I(\infty)/\langle \Gamma(\infty), A \rangle$; the shaded region is $\tau \cup \tau'$ and the unshaded region is the image of σ . It is evident after staring for a moment that when glued together, the three pictures form a 2-complex homotopy equivalent to $T^2 \vee S^2 \vee S^1$, with the torus T^2 generated by ϱ , the 2-sphere S^2 by $\tau \cup \tau'$ and the circle S^1 by σ . In

particular, we have

$$\dim(H^i(\mathrm{SL}(2, O_{-23}); \mathbb{Q}) = \begin{cases} 1 & i=0 \\ 3 & i=1 \\ 2 & i=2. \end{cases}$$

Note that $H_{\mathrm{cusp}}^1(\mathrm{SL}(2, O_{-23})) = 0$.

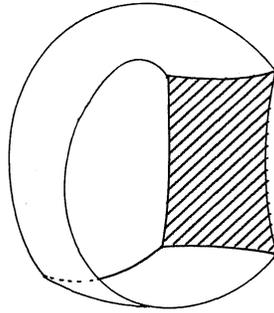


Fig. 7

(6.8) *Remark.* When D is large, it is not always so easy to see what the quotient looks like. If we are only interested in computing the homology, it is easiest to compute H_2 , using the boundary map from 2-cells to 1-cells. In this example ($d=23$), the 2-cells and 1-cells in the quotient are identified as follows:

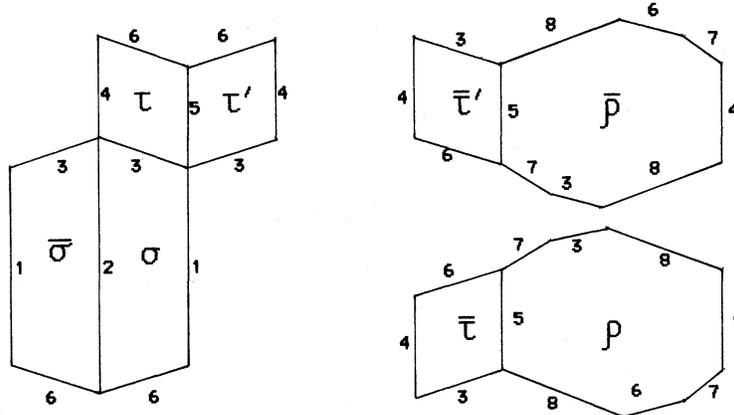


Fig. 8

The boundary matrix $\mathbb{Z}^4 \rightarrow \mathbb{Z}^8$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After row-reducing this matrix, we see that its kernel has dimension 2, i.e. $\dim(H_2 \text{SL}(2, O_{-23}))=2$.

7. Results

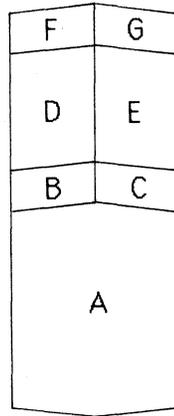
This section contains the results of the computations of homology of $\text{SL}(2, O_{-d})$ for discriminants $D > -100$. The quotient $\mathbb{H}/\text{SL}(2, O_{-d})$ in every case computed has the homotopy type of a wedge produce of circles S^1 , 2-spheres S^2 and toruses T^2 . The subcolumns under the heading ‘‘Homotopy Type’’ list the number of each of these which occurs in the wedge product.

(7.1) Table of results

$-D$	d	Class group of O_{-d}	$\dim(H_{\text{cusp}})$	Homotopy type of $\mathbb{H}/\text{SL}(2, O_{-d})$		
				S^1	S^2	T^2
3	3	$\langle 1 \rangle$	0	0	0	0
4	1	$\langle 1 \rangle$	0	0	0	0
7	7	$\langle 1 \rangle$	0	1	0	0
8	2	$\langle 1 \rangle$	0	1	0	0
11	11	$\langle 1 \rangle$	0	1	0	0
15	15	$\mathbb{Z}/2$	0	2	1	0
19	19	$\langle 1 \rangle$	0	1	0	0
20	5	$\mathbb{Z}/2$	0	2	1	0
23	23	$\mathbb{Z}/3$	0	1	1	1
24	6	$\mathbb{Z}/2$	0	2	1	0
31	31	$\mathbb{Z}/3$	0	1	1	1
35	35	$\mathbb{Z}/2$	1	3	2	0
39	39	$\mathbb{Z}/4$	0	2	2	1
40	10	$\mathbb{Z}/2$	1	3	2	0
43	43	$\langle 1 \rangle$	1	2	1	0
47	47	$\mathbb{Z}/5$	0	1	2	2
51	51	$\mathbb{Z}/2$	1	3	2	0
52	13	$\mathbb{Z}/2$	1	3	2	0
55	55	$\mathbb{Z}/4$	1	3	3	1
56	14	$\mathbb{Z}/4$	1	3	3	1
59	59	$\mathbb{Z}/3$	1	2	2	1
67	67	$\langle 1 \rangle$	3	4	3	0
68	17	$\mathbb{Z}/4$	1	3	3	1
71	71	$\mathbb{Z}/7$	0			
79	79	$\mathbb{Z}/5$	1			
83	83	$\mathbb{Z}/3$	2	5	4	0
84	21	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	3	7	6	0
87	87	$\mathbb{Z}/6$	2			
88	22	$\mathbb{Z}/2$	2	4	3	0
91	91	$\mathbb{Z}/2$	3	5	4	0
95	95	$\mathbb{Z}/8$	1			

(7.2) *Example.* Let $d=47$. The class group of O_{-47} is $\mathbb{Z}/5 = \{1, a, a^2, a^3, a^4\}$, represented by the cusps $\{\infty, \bar{\omega}/3, \omega/2, \bar{\omega}/2, \omega/3\}$ respectively. Figures 8–10 show computer-generated fundamental domains I_λ for the action of the stabilizers of the cusps $\lambda = \infty, \omega/2$, and $\omega/3$ on the boundary of the minimal sets H_λ . Each cell in I_λ is equal to $H_\lambda \cap H_\mu$ for some cusp μ , and the chart to the right of the figure identifies μ and the class $[\mu]$ for each cell. The labels on the 2-cells indicate which cells are identified modulo $SL(2, O_{-47})$. Note that the fundamental domains for the remaining cusp classes ($\lambda = \bar{\omega}/2$ and $\lambda = \bar{\omega}/3$) can be deduced from these pictures by the methods of Sect. 6.

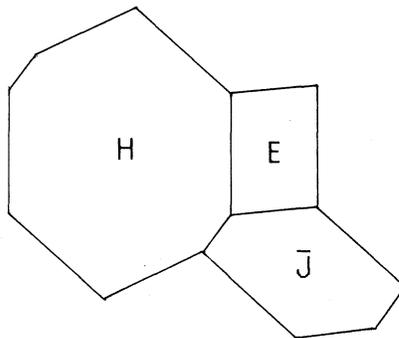
$\lambda = \infty$



2-cell	μ	$[\mu]$
A	0	1
B	$(\omega-1)/3$	a
C	$\omega/3$	a^4
D	$(\omega-1)/2$	a^3
E	$\omega/2$	a^2
F	$(2\omega-2)/3$	a
G	$2\omega/3$	a^4

Fig. 9

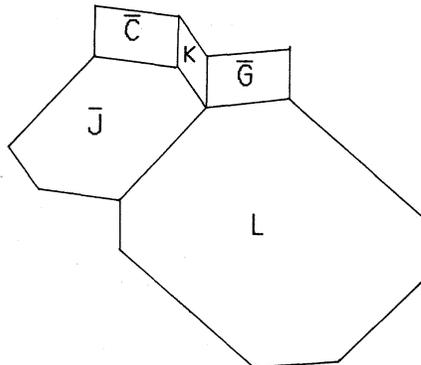
$\lambda = \omega/2$



2-cell	μ	$[\mu]$
\bar{E}	∞	1
H	$(\omega-1)/2$	a^3
J	$2\omega/3$	a^4

Fig. 10

$\lambda = \omega/3$



2-cell	μ	$[\mu]$
\bar{C}	∞	1
K	$(\omega+2)/3$	a
\bar{G}	$(\omega+1)/3$	1
\bar{J}	$\omega/2$	a^2
L	$(\omega-1)/3$	a

Fig. 11

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