

A GENERATING FUNCTION FOR THE EULER CHARACTERISTIC OF $\text{Out}(F_n)$

John SMILLIE*

Dept. of Mathematics, Lehman College, CUNY, Bronx, NY 10468, USA

Karen VOGTMANN**

Dept. of Mathematics, Cornell University, Ithaca, NY 14853, USA

Communicated by E.M. Friedlander and S. Priddy

Received 6 November 1985

0. Introduction

Let A be a group satisfying suitable homological finiteness conditions, and $A' \subset A$ any torsion-free subgroup of finite index. Then the rational Euler characteristic $\chi(A)$ of A is defined as $\chi(A')/[A : A']$, where $\chi(A')$ is the usual alternating sum of the ranks of the homology groups $H_i(A', \mathbb{Z})$ (cf. [10]). The rational Euler characteristic has good multiplicative properties which often make it easier to compute and more appropriate for groups with torsion than the usual Euler characteristic. In addition, for many groups there is a strong connection between $\chi(A)$ and number theory. Harder [5] has shown that the rational Euler characteristics of arithmetic groups are closely related to values of zeta functions. In particular, the rational Euler characteristics of the \mathbb{Z} -points of algebraic groups can often be given in terms of Bernoulli numbers. Harer and Zagier [7] have shown that the rational Euler characteristics of mapping class groups can likewise be given as simple formulas involving Bernoulli numbers.

In this paper, we concentrate on the group $\text{Out}(F_n)$ of outer automorphisms of a free group on n generators. In [3] it is shown that $\text{Out}(F_n)$ has homological finiteness properties in common with arithmetic groups and mapping class groups; specifically, it is of type VFL (cf. [1]). In general, however, very little is known about the homology of $\text{Out}(F_n)$. In this paper, we study the rational Euler characteristic $\chi(\text{Out}(F_n))$, and in particular, we give an effective method for computing it. In an interesting departure from the arithmetic case, we do not know whether there is a formula for $\chi(\text{Out}(F_n))$ in terms of Bernoulli numbers, though the denominators share some divisibility properties with the denominators of Bernoulli numbers.

The authors would like to thank Ken Brown for many helpful comments and suggestions.

* Partially supported by NSF grant #DMS-8403639.

** Partially supported by NSF grant #DMS-8514548.

1. The complex K and the action of $\text{Out}(F_n)$

One way to compute $\chi(A)$ is to find a contractible complex X on which A acts with finite stabilizers and finite quotient; then

$$(1.1) \quad \chi(A) = \sum_{\sigma \in \Sigma} \frac{(-1)^{\dim(\sigma)}}{|\text{stab}(\sigma)|}$$

where Σ is a set of representatives for the cells of X modulo the action of A (cf. [1]).

A suitable complex K for the group $\text{Out}(F_n)$ is defined in [3]; we will now recall briefly the definition. For details and the proof that K is contractible, see [3].

Fix a value for n . The complex K will be the geometric realization of a partially ordered set of ‘marked graphs’ G with $\pi_1(G) \cong F_n$.

By a *graph*, we mean a 1-dimensional CW-complex. The 0-cells of a graph are its *vertices*, and the 1-cells are its *edges*. Two graphs G and G' are *isomorphic* if there is a homeomorphism from G to G' sending vertices to vertices.

A graph is *admissible* if it is connected, has no separating edges, and every vertex has valence at least 3. Note that admissible graphs may have loops and multiple edges.

Let R_0 be a fixed graph with one vertex and n edges (an ‘ n -leafed rose’). A *marking* is a homotopy equivalence $g : R_0 \rightarrow G$, where G is a graph. A *marked graph* (g, G) is an equivalence class of markings, where $g : R_0 \rightarrow G$ is equivalent to $g' : R_0 \rightarrow G'$ if there is an isomorphism $h : G \rightarrow G'$ such that $h \circ g$ is homotopic to g' .

A *subforest* φ of a graph G is a subgraph with no cycles, and with the same vertex set as G . The *empty subforest* is the subforest consisting only of the vertices of G , i.e. with no edges. If φ is a subforest of G , we denote by G_φ the graph obtained from G by collapsing each tree in φ to a point. The map $G \rightarrow G_\varphi$ is called a *forest collapse*.

The set of admissible marked graphs with fundamental group F_n form a partially ordered set (poset) under the relation given by $(g, G) > (g', G')$ if there is a forest collapse $f : G \rightarrow G_\varphi$ such that $(f \circ g, G_\varphi) = (g', G')$. We now define K to be the geometric realization of this poset. Thus a k -simplex of K can be represented by a diagram

$$R_0 \xrightarrow{g} G \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k$$

where g is a marking and c_1, \dots, c_k are forest collapses. The vertices of the simplex are the marked graphs $(c_i \circ \dots \circ c_1 \circ g, G_i)$. Two diagrams represent the same cell of K if there are isomorphisms $h : G \rightarrow G'$ and $h_i : G_i \rightarrow G'_i$ which make the following diagram commute up to homotopy:

$$\begin{array}{ccccccc} R_0 & \xrightarrow{g} & G & \xrightarrow{c_1} & G_1 & \xrightarrow{c_2} & \dots & \xrightarrow{c_k} & G_k \\ \parallel & & \downarrow h & & \downarrow h_1 & & & & \downarrow h_k \\ R_0 & \xrightarrow{g'} & G' & \xrightarrow{c'_1} & G'_1 & \xrightarrow{c'_2} & \dots & \xrightarrow{c'_k} & G'_k \end{array}$$

Define $e(G)$ to be the number of edges in G , and $v(G)$ the number of vertices in G . If G is admissible and $\pi_1(G) \cong F_n$, then $e(G) \leq 3n - 3$, and $v(G) \leq 2n - 2$. Since a maximal tree in G has $v(G) - 1 \leq 2n - 3$ edges, the maximum length of a chain of forest collapses in G (and therefore the dimension of K) is equal to $2n - 3$.

The action of $\alpha \in \text{Out}(F_n)$ on K is induced by $\alpha(g, G) = (g \circ A, G)$, where $A : R_0 \rightarrow R_0$ induces α . Thus α sends the cell represented by the diagram

$$R_0 \xrightarrow{g} G \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k$$

to the cell represented by the diagram

$$R_0 \xrightarrow{g \circ A} G \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k.$$

Definition. By a *loop* in a graph G we mean a map from the circle S^1 to G . A loop is *taut* if it is locally injective; thus a taut loop cannot double back on itself in any edge. A *segment* of a taut loop l is an interval of S^1 which is mapped homeomorphically to an edge of G by l . The *length* of l is the number of segments in S^1 .

Note that every homotopy class in $\pi_1(G)$ can be represented by a taut loop. If two taut loops $l_1 : S^1 \rightarrow G$ and $l_2 : S^1 \rightarrow G$ are in the same homotopy class, they are equivalent in the sense that there is a homeomorphism $f : S^1 \rightarrow S^1$ such that $l_1 \circ f = l_2$. In addition, we have

(1.2) **Lemma.** Let G be a graph with no separating edges, and let e_1 and e_2 be two edges of G which are both incident to a vertex v . Then there is a taut loop l which passes successively from e_1 to e_2 , i.e., there are adjacent segments I_1 and I_2 of S^1 such that $l(I_1) = e_1$ and $l(I_2) = e_2$.

Proof. If e_1 and e_2 are in the same component of $G - v$ this is clear. If e_1 and e_2 are in distinct components, then we can find an e'_1 incident to v in the same component as e_1 (since otherwise e_1 would be a separating edge of G). Similarly, we can find an e'_2 incident to v in the same component as e_2 . We construct a loop by starting at e_1 , travelling to e'_1 , then e'_2 and finally e_2 . \square

If l is a taut loop and $c : G \rightarrow G_\varphi$ is a forest collapse, the loop $c \circ l : S^1 \rightarrow G_\varphi$ need not be taut. However, we can recover tautness in a canonical way as follows: $c \circ l$ factors through S^1 / \sim , where $x \sim y$ if x and y are in the same component of $l^{-1}(\varphi)$. Note that S^1 / \sim is homeomorphic to S^1 . The resulting loop $S^1 / \sim \rightarrow G_\varphi$ is taut, and will be called the *projection* l_φ of l .

(1.3) **Lemma.** Let G be an admissible graph. Let $c : G \rightarrow G_\varphi$ and $c' : G \rightarrow G_{\varphi'}$ be two forest collapses, and $h : G_\varphi \rightarrow G_{\varphi'}$ an isomorphism. If $h \circ c$ is homotopic to c' , then $\varphi = \varphi'$.

Proof. For any taut loop l in G and any subforest F , let $\langle l, F \rangle$ denote the number of segments of l which are mapped to edges of F . Note that the length of the projection l_F of l is equal to the length of l minus $\langle l, F \rangle$.

Since the loops $l_{\varphi'}$ and $h \circ l_{\varphi}$ are homotopic taut loops, they are equivalent; in particular, $l_{\varphi'}$ and l_{φ} have the same length. Consequently, $\langle l, \varphi' \rangle = \langle l, \varphi \rangle$ for any taut loop l .

Suppose that $\varphi' \neq \varphi$. Let e be an edge of G which is in one forest but not the other; say $e \in \varphi$ but $e \notin \varphi'$. Let v_1 and v_2 be the vertices of e . Since v_1 and v_2 are at least trivalent, and e is not separating, there are edges a and b at v_1 and c and d at v_2 , with a, b, c, d and e all distinct. By Lemma (1.2), there is a taut loop l_1 passing successively through a and b , and a taut loop l_2 passing successively through c and d . Let l_3 be the taut loop constructed by following l_1 from v_1 to v_1 , then e to v_2 , then l_2 from v_2 to v_2 then e back to v_1 . We then have

$$\begin{aligned} \langle l_1, \varphi' \rangle &= \langle l_1, \varphi \rangle, \\ \langle l_2, \varphi' \rangle &= \langle l_2, \varphi \rangle, \\ \langle l_3, \varphi' \rangle &= \langle l_1, \varphi' \rangle + \langle l_2, \varphi' \rangle + 2, \\ \langle l_3, \varphi \rangle &= \langle l_1, \varphi \rangle + \langle l_2, \varphi \rangle. \end{aligned}$$

Thus $\langle l_3, \varphi' \rangle \neq \langle l_3, \varphi \rangle$. But l_3 is taut, so $\langle l_3, \varphi' \rangle = \langle l_3, \varphi \rangle$. This contradiction proves the lemma. \square

(1.4) **Proposition.** *Let σ and σ' be simplices of K , represented by the diagrams*

$$R_0 \xrightarrow{g} G \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k$$

and

$$R_0 \xrightarrow{g'} G' \xrightarrow{c'_1} G'_1 \xrightarrow{c'_2} \dots \xrightarrow{c'_k} G'_k$$

respectively. Let $\alpha \in \text{Out}(F_n)$ be such that $\alpha\sigma = \sigma'$. Then there is an isomorphism $h : G \rightarrow G'$ that takes the forest collapsed by $c_i \circ \dots \circ c_1$ to the forest collapsed by $c'_i \circ \dots \circ c'_1$ for $i = 1, \dots, k$.

Proof. Since $\alpha\sigma = \sigma'$, we have a homotopy commutative diagram

$$\begin{array}{ccccccccccc} \alpha\sigma: & R_0 & \xrightarrow{g \circ A} & G & \xrightarrow{c_1} & G_1 & \xrightarrow{c_2} & \dots & \xrightarrow{c_k} & G_k \\ & \parallel & & \downarrow h & & \downarrow h_1 & & & & \downarrow h_k \\ \sigma': & R_0 & \xrightarrow{g'} & G' & \xrightarrow{c'_1} & G'_1 & \xrightarrow{c'_2} & \dots & \xrightarrow{c'_k} & G'_k \end{array}$$

where $A : R_0 \rightarrow R_0$ induces α . Let $b_i = c_i \circ \dots \circ c_1$, and $b'_i = c'_i \circ \dots \circ c'_1$. The square

$$\begin{array}{ccc}
 G & \xrightarrow{b_i} & G_i \\
 \downarrow h & & \downarrow h_i \\
 G' & \xrightarrow{b'_i} & G'_i
 \end{array}$$

homotopy commutes. Use h to identify G' with G . The proposition now follows from Lemma (1.3). \square

(1.5) Corollary. *Let Σ be a set of representatives for the simplices of K modulo $\text{Out}(F_n)$. Then there is a one-to-one correspondence between elements of Σ and isomorphism classes $[(G, \Phi)]$ of pairs (G, Φ) , where G is an admissible graph and $\Phi = \varphi_1 \subset \dots \subset \varphi_k$ is a chain of non-empty subforests in G .*

Proof. Let σ be a simplex of K , represented by the diagram

$$R_0 \xrightarrow{g} G \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k.$$

The correspondence is given by associating to σ the class $[(G, \Phi)]$, where Φ is the chain whose i th forest φ_i is the subforest of G collapsed by $c_i \circ \dots \circ c_1$.

Two pairs (G, Φ) and (G', Φ') are isomorphic if there is an isomorphism $h : G \rightarrow G'$ sending φ_i isomorphically to φ'_i for all i . Thus by Proposition (1.4), if two cells are equivalent modulo $\text{Out}(F_n)$, the associated pairs are isomorphic.

Conversely, it is clear that simplices which give rise to isomorphic pairs are equivalent modulo $\text{Out}(F_n)$. \square

Note that the dimension of the simplex corresponding to $[(G, \Phi)]$ is equal to the length of the chain Φ . If Φ is empty, $[(G, \Phi)]$ corresponds to a vertex of K .

Definition. For any graph G , define $\text{aut}(G)$ to be the group of path components of the group of isomorphisms from G to itself.

Remark. A graph G can be described combinatorially as a set of vertices, a set of edges, an involution on the edges, and a ‘terminal vertex map’ from the edges to the vertices. In this language, $\text{aut}(G)$ is just the group of automorphisms of G .

(1.6) Proposition. *Let G be an admissible graph, and Φ a chain of subforests of G . Then the stabilizer of the simplex σ of Σ corresponding to $[(G, \Phi)]$ is the subgroup $\text{aut}(G, \Phi)$ of elements of $\text{aut}(G)$ which preserve all subforests in Φ .*

Proof. Any element of $\text{aut}(G, \Phi)$ clearly stabilizes σ . The map from $\text{aut}(G, \Phi)$ to the stabilizer of σ is a homomorphism. By Proposition (1.4) with $\sigma' = \sigma$, this homomorphism is a surjection. The kernel consists of homeomorphisms homotopic to the identity. An argument similar to that of Proposition (1.4) shows that any such homeomorphism of an admissible graph is isotopic to the identity. \square

Let Γ be a set of representatives for the isomorphism classes of admissible graphs G . By Corollary (1.5) and Proposition (1.6), formula (1.1) in our case becomes

$$(1.7) \quad \chi(\text{Out}(F_n)) = \sum_{[(G, \Phi)] \in \mathcal{L}} \frac{(-1)^{\text{length}(\Phi)}}{|\text{aut}(G, \Phi)|} = \sum_{G \in \Gamma} \sum_{[(G, \Phi)]} \frac{(-1)^{\text{length}(\Phi)}}{|\text{aut}(G, \Phi)|}.$$

We now define

$$(1.8) \quad \text{ch}(G) = \sum_{[(G, \Phi)]} \frac{(-1)^{\text{length}(\Phi)}}{|\text{aut}(G, \Phi)|}$$

so that

$$\chi(\text{Out}(F_n)) = \sum_{G \in \Gamma} \text{ch}(G).$$

Now consider the action of $\text{aut}(G)$ on the set of chains of subforests in G . If A is a finite group acting on a finite set S , we have the well-known formula

$$|A| = |\text{orbit}(s)| \cdot |\text{stabilizer}(s)| \quad \text{for any } s \in S.$$

Thus for a chain Φ of subforests in G , we have

$$|\text{aut}(G)| = |\text{orbit}(\Phi)| \cdot |\text{aut}(G, \Phi)|,$$

and (1.8) becomes

$$\text{ch}(G) = \sum_{[(G, \Phi)]} \frac{(-1)^{\text{length}(\Phi)} |\text{orbit}(\Phi)|}{|\text{aut}(G)|}.$$

Summing over chains instead of isomorphism classes of chains gives

$$(1.9) \quad \text{ch}(G) = \frac{1}{|\text{aut}(G)|} \sum_{(G, \Phi)} (-1)^{\text{length}(\Phi)}.$$

(1.10). **Lemma.** *Let P be a finite poset such that for every $x \in P$, the sub-poset $P_{<x}$ of elements less than x is isomorphic to the poset of all proper subsets of a finite set, ordered by inclusion. Then*

$$\chi(P) = \sum_{x \in P} (-1)^{\text{ht}(x)}.$$

Proof. Recall that the *height* $\text{ht}(x)$ of an element $x \in P$ is the length of the longest

chain of elements less than x . Let P_i be the set of elements of P of height $\leq i$. Then we have $P_0 \subset P_1 \subset \dots \subset P_k = P$ for some k . The geometric realization of P_i is obtained from the realization of P_{i-1} by adjoining, for each $x \in P$ of height i , the cone on $P_{<x}$. By hypothesis, $P_{<x}$ is isomorphic to the set of proper subsets of a set of $i+1$ elements; hence its geometric realization is an $(i-1)$ -sphere, and the cone on $P_{<x}$ is homeomorphic to an i -cell. The above formula now follows from the definition of the Euler characteristic of a CW-complex. \square

(1.11) **Corollary.** *Let G be any graph. Then*

$$\sum_{(G, \phi)} (-1)^{\text{length}(\phi)} = \sum_{\substack{\text{subforests} \\ \phi \subset G}} (-1)^{e(\phi)}.$$

Proof. The sum on the left is 1 minus the Euler characteristic of the poset of non-empty subforests in G , partially ordered by inclusion. The extra 1 is contributed by the empty chain. The result now follows by applying Lemma (1.10) to this poset, noting that the height of any subforest ϕ is equal to $e(\phi) - 1$. \square

Definition. For any graph G (not necessarily admissible), let

$$\tau(G) = \sum_{\phi \subset G} (-1)^{e(\phi)}$$

where the sum is over all subforests ϕ of G .

We now combine Corollary (1.11) with (1.9) and (1.7) to obtain:

(1.12) **Proposition.**

$$\chi(\text{Out}(F_n)) = \sum_{G \in \Gamma} \frac{\tau(G)}{|\text{aut } G|}$$

where Γ is a set of representatives for isomorphism classes of admissible graphs G with $\pi_1(G) \cong F_n$.

2. Elementary properties of τ

In this section we describe some elementary properties of the function τ . Let G be an arbitrary graph, and let e be an edge of G . We will denote by $G-e$ the graph obtained from G by removing e , and by G/e the graph obtained by collapsing e to a point. Then we have:

(2.1) **Lemma.** *Let e be an edge of G . If e is a loop, then $\tau(G) = \tau(G-e)$. If e is not a loop, then $\tau(G) = \tau(G-e) - \tau(G/e)$.*

Proof. We can decompose $\tau(G)$ into a sum over the subforests which do not contain e plus a sum over the subforests which contain e .

If a set of edges does not contain e , then it is a subforest in G if and only if it is a subforest in $G-e$. If e is a loop, then e is not in any subforest in G , so $\tau(G) = \tau(G-e)$.

If e is not a loop and a set of edges E contains e , then E is a subforest in G if and only if E/e is a subforest in G/e (with one fewer edge).

Together, these observations give $\tau(G) = \tau(G-e) - \tau(G/e)$. \square

(2.2) **Lemma.** *If G is the disjoint union of G_1 and G_2 , or G is the wedge product of G_1 and G_2 , then $\tau(G) = \tau(G_1) \tau(G_2)$.*

Proof. Any subforest in G is the disjoint union of a subforest in G_1 and a subforest in G_2 (either one of which may be empty). \square

(2.3) **Lemma.** *If G has a separating edge, then $\tau(G) = 0$.*

Proof. Let e be a separating edge, separating the subgraph G_1 from G_2 . Then $\tau(G-e) = \tau(G_1) \tau(G_2) = \tau(G/e)$ by Lemma (2.2). Thus by Lemma (2.1), $\tau(G) = 0$. \square

As a special case of Lemma (2.3), we note

(2.4) **Corollary.** *If G has a free edge, then $\tau(G) = 0$.*

(2.5) **Lemma.** *If E is a set of m edges in G between the same (distinct) vertices, then $\tau(G) = \tau(G-E) - m\tau(G/E)$.*

Proof. By induction on m . The case $m=1$ is Lemma (2.1) above. If $m>1$, let e be an edge in E and apply Lemma (2.1), noting that $\tau(G/e) = \tau(G/E)$. \square

(2.6) **Lemma.** *If G has a bivalent vertex contained in an edge e , then $\tau(G) = -\tau(G/e)$. (Think of G/e as G with the bivalent vertex removed.)*

Proof. Apply Lemma (2.1) to G and e ; since $G-e$ has a free edge, we have $\tau(G-e) = 0$ by (2.4). \square

(2.7) **Proposition.** *Let $f(G)$ be the number of edges in a maximal subforest in G . If G has no separating edges, then $\tau(G) \neq 0$ and $\text{sign}(\tau(G)) = (-1)^{f(G)}$.*

Proof. Let G be a graph with no separating edges. The proof is by induction on $e(G)$. (Recall that $e(G)$ is the number of edges of G .)

If $e(G) = 0$, the only subforest in G is the empty subforest, so $\tau(G) = (-1)^{f(G)} = 1$.

Now assume $e(G) > 0$, and choose a maximal subforest φ of G . If all the edges in $G - \varphi$ are loops, then either φ is the empty subforest and $\tau(G) = 1$, or else G has a separating edge, contradicting our hypothesis.

Let e be an edge in $G - \varphi$ which is not a loop. By (2.1) we have $\tau(G) = \tau(G - e) - \tau(G/e)$. We consider $\tau(G - e)$ and $\tau(G/e)$ separately.

The subforest φ is also a maximal subforest for $G - e$, so by induction we have either $\tau(G - e) = 0$ or $\text{sign}(\tau(G - e)) = (-1)^{f(G - e)} = (-1)^{f(G)}$.

Now note that for any graph H , $f(H)$ is equal to $v(H) - \#\{\text{connected components of } H\}$; thus

$$\begin{aligned} f(G/e) &= v(G/e) - \#\{\text{connected components of } G/e\} \\ &= v(G) - 1 - \#\{\text{connected components of } G\} \\ &= f(G) - 1. \end{aligned}$$

Since G has no separating edges, the same can be said for G/e ; thus by induction $\tau(G/e) \neq 0$ and $\text{sign}(\tau(G/e)) = (-1)^{f(G/e)} = -(-1)^{f(G)}$.

Combining the information about $\tau(G - e)$ and $\tau(G/e)$ gives $\tau(G) \neq 0$ and $\text{sign}(\tau(G)) = (-1)^{f(G)}$. \square

Remark. Let $\Sigma(G)$ be the poset of non-empty subforests of G . Then by Corollary (1.11), we have $\tau(G) = 1 - \chi(\Sigma(G))$. Lemmas (2.1)–(2.6) can be stated as consequences of properties of $\Sigma(G)$. For instance, Lemma (2.1) follows since $\Sigma(G) = \Sigma(G - e)$ if e is a loop, and otherwise $\Sigma(G) = \Sigma(G - e) \cup C\Sigma(G/e)$. Lemma (2.7) is a result of the fact that $\Sigma(G)$ is spherical of dimension $f(G) - 1$. We chose to state these lemmas as statements about $\tau(G)$ instead of $\Sigma(G)$ since that is how we will use them.

Lemmas (2.1)–(2.6) make it easy to compute τ of any graph. In particular, $\tau(G)$ can be expressed in terms of τ of graphs with fewer vertices and edges. Thus we can use Proposition (1.12) to compute the Euler characteristic of $\text{Out}(F_n)$ when there are not too many isomorphism classes of admissible graphs with the right homotopy type. For $n = 2$, there are two classes, and for $n = 3$ there are eight. For $n = 4$, however, this method begins to get cumbersome.

3. Labelled graphs

In this section and the next, we will obtain a general algebraic formula for the Euler characteristic of $\text{Out}(F_n)$.

Let $\Gamma_{v,e}$ be a set of representatives for isomorphism classes of admissible graphs with v vertices and e edges. We define

$$(3.1) \quad \chi_{v,e} = \sum_{G \in \Gamma_{v,e}} \frac{\tau(G)}{|\text{aut}(G)|}.$$

Since an admissible graph G with $\pi_1(G) \cong F_n$ has $\chi(G) = v(G) - e(G) = 1 - n$ and $v(G) \leq 2n - 2$, Proposition (1.12) gives

$$(3.2) \quad \chi(\text{Out}(F_n)) = \chi_{1, n} + \dots + \chi_{2n-2, 3n-3}.$$

Let G be any graph with v vertices and e edges. let V be a set with v elements and E a set with e elements (usually, we set $V = \{1, \dots, v\}$ and $E = \{1, \dots, e\}$). A *labelling* of G consists of an assignment of distinct labels from the set V to the vertices of G , labels from E to the edges of G , and orientations on all edges of G . The set of labellings for G thus has order $v! e! 2^e$. The group $\text{aut}(G)$ acts freely on the set of labellings of G . We define a *labelled graph* to be an orbit of this action. Since the action is free, we have

$$v! e! 2^e = |\text{aut}(G)| \cdot \#\{\text{labelled graphs whose underlying graph is isomorphic to } G\}$$

or

$$\frac{1}{|\text{aut}(G)|} = \frac{1}{v! e! 2^e} \cdot \#\{\text{labelled graphs whose underlying graph is isomorphic to } G\}.$$

Substituting into (3.1), we obtain

$$\chi_{v, e} = \frac{1}{v! e! 2^e} \sum_{G \in \Gamma_{v, e}} \tau(G) \cdot \#\{\text{labelled graphs whose underlying graph is isomorphic to } G\}.$$

We extend the function τ to labelled graphs by defining τ of a labelled graph to be τ of the underlying (unlabelled) graph. By summing over labelled graphs instead of isomorphism classes of graphs, we obtain

$$(3.3) \quad \chi_{v, e} = \frac{1}{v! e! 2^e} \sum_{G \in \text{ALG}_{v, e}} \tau(G)$$

where $\text{ALG}_{v, e}$ denotes the set of admissible labelled graphs with v vertices and e edges.

4. Generating functions

In this section we show how to compute the numbers $\chi_{v, e}$ as the coefficients of a generating function. Specifically, we will produce a two-variable formal power series $C(x, y)$ such that the sum of $\tau(G)$ over all $G \in \text{ALG}_{v, e}$ is the coefficient of $(x^v/v!)(y^e/e!)$ in C . In the language of Goulden and Jackson’s book on combinatorial enumeration [4], we say that C is an exponential generating function for the sum of $\tau(G)$ over admissible labelled graphs, with x marking vertices and y marking edges. Throughout this section, x will mark vertices and y edges whenever they appear in a generating function.

Given the power series $C(x, y)$, note that $\chi_{v, e}$ is the coefficient of $x^v(2y)^e$ in C . Formulas (3.2) and (3.3) then imply that the Euler characteristic $\chi(\text{Out}(F_n))$ is equal to the coefficient of z^{n-1} in the one-variable function $C(1/z, z/2)$, i.e.

$$(4.1) \quad \chi(z) = C(1/z, z/2) \text{ is an ordinary (non-exponential) generating function for the Euler characteristic of } \text{Out}(F_{n+1}).$$

To obtain a first approximation to the power series C , we will lift the restriction that graphs must be admissible, i.e., we will compute an exponential generating function $A(x, y)$ for the sum of $\tau(G)$ over all labelled graphs G . Since this counts $\tau(G)$ for disconnected graphs and graphs with bivalent vertices, we must eliminate these terms later. It also ostensibly counts $\tau(G)$ for graphs with free or separating edges, but in view of Lemmas (2.3) and (2.4), these terms are zero.

Let $A_{v, e}$ denote the coefficient of $(x^v/v!)(y^e/e!)$ in the power series $A(x, y)$, and $\text{LG}_{v, e}$ the set of all labelled graphs with v vertices and e edges. We want

$$A_{v, e} = \sum_{G \in \text{LG}_{v, e}} \tau(G) = \sum_{G \in \text{LG}_{v, e}} \sum_{\varphi \subset G} (-1)^{e(\varphi)}.$$

We will now change the order of summation above; i.e., instead of first fixing a labelled graph G with v vertices and e edges and adding up $(-1)^{e(\varphi)}$ over the (labelled) subforests of G , we will instead first fix a labelled forest φ on v vertices (with edge labels selected from e possibilities) and add up $(-1)^{e(\varphi)}$ over all labelled graphs G on v vertices and e edges which contain φ as a subforest. Let LF_v denote the set of labelled forests on v vertices. Then

$$A_{v, e} = \sum_{\substack{\varphi \in \text{LF}_v \text{ with} \\ \text{edge labels in a} \\ \text{subset of } \{1, \dots, e\}}} \sum_{\substack{G \in \text{LG}_{v, e} \text{ with} \\ G \supset \varphi}} (-1)^{e(\varphi)}.$$

If for each φ we assume that the labels come from the set $\{1, \dots, e(\varphi)\}$, we get

$$\begin{aligned} A_{v, e} &= \sum_{\varphi \in \text{LF}_v} \binom{e}{e(\varphi)} \sum_{G \in \text{LG}_{v, e}, G \supset \varphi} (-1)^{e(\varphi)} \\ &= \sum_{\varphi \in \text{LF}_v} (-1)^{e(\varphi)} \binom{e}{e(\varphi)} \cdot \#\{G \in \text{LG}_{v, e} : G \supset \varphi\}. \end{aligned}$$

Given a labelled forest φ with v vertices and edge labels $\{1, \dots, e(\varphi)\}$, we construct a labelled graph G with φ as a subforest by adding $m = e(G) - e(\varphi)$ oriented edges labelled with the set $\{e(\varphi) + 1, \dots, e\}$ to φ . Since we have v choices for where to put each end of each edge, there are v^{2m} labelled graphs containing φ as a subforest. Note that this depends only on $e(\varphi)$ and not on the forest φ itself. Thus we have

$$(4.2) \quad A_{v, e} = \sum_{i=0}^{\max(v-1, e)} \#\{\varphi \in \text{LF}_v : e(\varphi) = i\} \cdot (-1)^i \binom{e}{i} \cdot v^{2(e-i)}.$$

To complete our determination of the coefficients of $A(x, y)$, we must count the number of labelled forests with v vertices and i edges. We will do this using exponential generating functions. The following lemmas are adapted from [4].

Definition. Let A_1 and A_2 be two sets of labelled graphs. Let $H_1 \in A_1$ and $H_2 \in A_2$. Let φ_i ($i=1, 2$) be order-preserving maps from $\{1, \dots, v(H_i)\}$ to $\{1, \dots, v(H_1) + v(H_2)\}$ such that φ_1 and φ_2 have disjoint images; similarly, let ψ_i ($i=1, 2$) be order-preserving maps from $\{1, \dots, e(H_i)\}$ to $\{1, \dots, e(H_1) + e(H_2)\}$ such that ψ_1 and ψ_2 have disjoint images. Define $A_1 * A_2$ to be the set of ordered sextuples $(H_1, H_2, \varphi_1, \varphi_2, \psi_1, \psi_2)$. Note that an element of $A_1 * A_2$ gives us a labelled graph whose underlying graph is the disjoint union $H_1 \dot{\cup} H_2$. The label on the j th edge of H_i is $\psi_i(j)$, and the vertices are similarly relabelled.

We extend the function τ to $A_1 * A_2$ by defining $\tau(H_1, H_2, \varphi_1, \varphi_2, \psi_1, \psi_2) = \tau(H_1 \dot{\cup} H_2)$. By Lemma (2.2), $\tau(H_1, H_2) = \tau(H_1) \tau(H_2)$.

(4.3) Lemma. *Let A and B be two sets of labelled graphs. Let $C = A * B$. Let f_A, f_B and f_C be the exponential generating functions for the sum of $\tau(G)$ over A, B and C respectively. Then $f_C = f_A \cdot f_B$.*

Proof. This follows from [4, Lemma 3.2.11] and the disjoint union property of τ (Lemma (2.2)). \square

(4.4) Lemma. *Let A_0 be a set of connected graphs, and let A be the set of labelled graphs whose underlying graphs are isomorphic to elements of A_0 . Let f_A be the exponential generating function for the sum of $\tau(G)$ over A . Then $\exp(f_A) - 1$ is the exponential generating function for the sum of $\tau(G)$ over labelled graphs all of whose connected components are isomorphic to graphs in A_0 .*

Proof. By Lemma (4.3), the exponential generating function for the sum of $\tau(G)$ over labelled graphs with k components, each of which is isomorphic to a graph in A_0 , is $(f_A)^k/k!$ (we divide by $k!$ because the components are not ordered). Summing over k proves the lemma. \square

In order to count labelled forests, we begin with Cayley’s classical theorem on vertex-labelled trees, which says that there are n^{n-2} vertex-labelled trees on n vertices (for a proof see, e.g. [4]). Since a tree with n vertices has $n - 1$ edges, the number of labelled trees we obtain by adding edge labels and orientations is $n^{n-2}(n - 1)! 2^{n-1}$. Thus the function

$$\begin{aligned} T(x, y) &= \sum_{n=1}^{\infty} n^{n-2}(n - 1)! 2^{n-1} \frac{x^n y^{n-1}}{n! (n - 1)!} \\ &= \frac{1}{2y} \sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} (2xy)^n \end{aligned}$$

is an exponential generating function for the number of labelled trees, with x marking vertices and y marking edges. By Lemma (4.4), we obtain the exponential generating function $F(x, y)$ for the number of labelled forests with x marking vertices and y marking edges by formally exponentiating $T(x, y)$:

$$F(x, y) = \exp(T(x, y)) - 1.$$

Let F_v be the series consisting of the terms in $F(x, y)$ with x -degree equal to v . Then formula (4.2) can be expressed more succinctly in terms of power series as follows:

(4.5) Proposition.

$$A(x, y) = \sum_{v=1}^{\infty} F_v(x, -y) \cdot \exp(v^2 y)$$

is an exponential generating function for the sum of $\tau(G)$ over all graphs G , with x marking vertices and y marking edges. \square

We must now pare down $A(x, y)$ so that it only counts $\tau(G)$ for admissible graphs G . We start by computing exponential generating functions for the sum of $\tau(G)$ over certain non-admissible graphs. Recall that an admissible graph has negative Euler characteristic.

(4.6) Proposition. *Let $P(x, y)$ be the exponential generating function for the sum of $\tau(G)$ over labelled graphs G all of whose components have positive Euler characteristic. Then $P(x, y) = \exp(x) - 1$.*

Proof. By Corollary (2.4) a graph with free edges has $\tau = 0$. Therefore, the only connected graph with $\tau \neq 0$ and positive Euler characteristic is an isolated vertex. Since $\tau(\text{vertex}) = 1$, the exponential generating function for the sum of $\tau(G)$ over connected labelled graphs G with positive Euler characteristic is x . By Lemma (4.4), the exponential generating function for the sum of $\tau(G)$ over disjoint unions of vertices is $\exp(x) - 1$. \square

(4.7) Proposition. *Let $Z(x, y)$ be the exponential generating function for the sum of $\tau(G)$ over labelled graphs G all of whose components have zero Euler characteristic. Then $Z(x, y) = (1 + 2xy)^{1/2} - 1$.*

Proof. If a connected graph G with $\tau(G) \neq 0$ has zero Euler characteristic, G must be homeomorphic to a circle. If G has n vertices, it also has n edges and there are $((n-1)!/2) n! 2^n$ labelled graphs isomorphic to G . By Lemma (2.6), $\tau(G) = (-1)^{n-1}$. Thus the exponential generating function for the sum of $\tau(G)$ over connected labelled graphs G with Euler characteristic zero is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)! n! 2^n}{2} \frac{x^n y^n}{n! n!} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1} x^n y^n}{n} = \frac{1}{2} \log(1 + 2xy).$$

By Lemma (4.4), $Z(x, y) = \exp(\frac{1}{2} \log(1 + 2xy)) - 1 = (1 + 2xy)^{1/2} - 1$, as was to be shown. \square

(4.8) Proposition. *Let $N(x, y)$ be the exponential generating function for the sum of $\tau(G)$ over labelled graphs G all of whose components have negative Euler characteristic. Then*

$$1 + A(x, y) = (1 + N(x, y))(1 + Z(x, y))(1 + P(x, y)).$$

Proof. Any graph can be decomposed uniquely as the disjoint union of (possibly empty) graphs G_- , G_0 and G_+ , where all the components of G_- have negative Euler characteristic, all the components of G_0 have zero Euler characteristic, and all the components of G_+ have positive Euler characteristic. The lemma now follows from Lemma (4.3). The ‘1’s’ are contributed by the empty graph. \square

Using Propositions (4.6)–(4.8), we can now solve for $N(x, y)$:

$$N(x, y) = \exp(-x) \cdot (1 + 2xy)^{-1/2} \cdot (1 + A(x, y)) - 1.$$

Remark. Any generating function that counts something over graphs with negative Euler characteristic will only involve terms with $v < e$, by the definition of $\chi(G)$. Thus the coefficient of $x^v y^e$ in $N(x, y)$ will be zero unless $v < e$.

We now turn to the problem of eliminating the contribution to N coming from graphs with bivalent vertices. Any graph G counted in the generating function N can be obtained from a graph G_0 with no bivalent vertices by adding $k \geq 0$ bivalent vertices (since G has no circle components). We will call G a *k-fold subdivision* of G_0 . We will need the following lemma:

(4.9) Lemma. *let G_0 be a graph with $v - k$ (non-bivalent) vertices and $e - k$ edges. Let $r(G_0)$ be the number of labelled graphs whose underlying graph is a k -fold subdivision of G_0 , divided by the number of labelled graphs whose underlying graph is G_0 . Then*

$$r(G_0) = a_{k, e, v} = \frac{e! v! 2^k (e-1)!}{(e-k)! (v-k)! k! (e-1-k)!}.$$

In particular, this ratio depends only on k, e and v , and not on G_0 .

Proof. Let S denote the set of pairs consisting of a k -fold subdivision G of G_0 together with a labelling of G . The group $\text{aut}(G_0)$ acts freely on S , and the orbits

of this action are the labelled graphs whose underlying graph is a k -fold subdivision of G_0 . Thus the number of labelled graphs whose underlying graph is a k -fold subdivision of G_0 is equal to $\#S/|\text{aut}(G_0)|$. The number of labelled graphs whose underlying graph is isomorphic to G_0 is equal to $\#\{\text{labellings of } G_0\}/|\text{aut}(G_0)|$. We have

$$r(G_0) = \frac{\#S/|\text{aut}(G_0)|}{\#\{\text{labellings of } G_0\}/|\text{aut}(G_0)|} = \frac{\#\{k\text{-fold s.d.'s of } G_0\} \cdot \#\{\text{labellings of a } k\text{-fold s.d. of } G_0\}}{\#\{\text{labellings of } G_0\}}$$

A subdivision of G_0 can be described by an ordered $(e-k)$ -tuple (a_1, \dots, a_{e-k}) , where a_i is the number of vertices added to the i -th edge of G_0 . Since a k -fold subdivision of G_0 has v vertices and e edges, the above expression becomes

$$= \#\{(a_1, \dots, a_{e-k}) : \sum a_j = k\} \cdot \frac{v! e! 2^e}{(v-k)! (e-k)! 2^{e-k}}$$

The number of $(e-k)$ -tuples of non-negative integers which sum to k is equal to the coefficient of x^k in the series expansion of $1/(1-x)^{e-k}$; since this coefficient is equal to $\binom{e-k}{k}$, we have

$$r(G_0) = \frac{(e-1)!}{k! (e-k-1)!} \cdot \frac{v! e! 2^e}{(v-k)! (e-k)! 2^{e-k}} = a_{k, e, v}$$

as was to be shown. \square

Let $H(x, y)$ denote the exponential generating function for the sum of $\tau(G)$ over labelled graphs with no bivalent vertices and no isolated vertices. Let $N_{v, e}$ and $H_{v, e}$ be the coefficients of $(x^v/v!)(y^e/e!)$ in N and H respectively. As we noted above, any graph G which contributes to $N_{v, e}$ is a k -fold subdivision of some graph G_0 with no bivalent vertices, for some $k \geq 0$. By Lemma (2.6), we have $\tau(G) = (-1)^k \tau(G_0)$. Thus

$$(4.10) \quad N_{v, e} = \sum_{k=0}^{v-1} (-1)^k a_{k, v, e} H_{v-k, e-k}$$

where $a_{k, v, e}$ is defined in Lemma (4.9).

The relation between N and H can be expressed in terms of formal power series as follows. For any formal power series

$$Q(x, y) = \sum q_{i, j} x^i y^j$$

define a new power series $\bar{Q}(x, y)$ by

$$\bar{Q}(x, y) = \sum q_{i, j} \frac{x^i y^j}{(j-1)!}$$

Then Lemma (4.9) and Formula (4.10) can be restated as

$$\bar{N} = \bar{H} \cdot \exp(-2xy)$$

allowing us to solve for H .

We have now eliminated the contributions of all graphs with vertices of valence less than three from the generating function. It remains only to eliminate disconnected graphs to obtain $C(x, y)$. We have

(4.11) **Proposition.** $C(x, y) = \log(1 + H(x, y))$.

Proof. Apply Lemma (4.4). \square

We have now completely determined a generating function for $\chi(\text{Out}(F_{n+1}))$. We collect the steps in the following theorem:

(4.12) **Theorem.** *Let $\chi(z)$ be the ordinary generating function for $\chi(\text{Out}(F_{n+1}))$. Then $\chi(z) = C(1/z, z/2)$, where*

$$\begin{aligned} C(x, y) &= \log(1 + H(x, y)), \\ \bar{H}(x, y) &= \bar{N}(x, y) \cdot \exp(2xy), \\ N(x, y) &= e^{-x} (1 + 2xy)^{-1/2} (1 + A(x, y)) - 1, \\ A(x, y) &= \sum_{v=1}^{\infty} F_v(x, -y) \cdot \exp(v^2 y), \\ F(x, y) &= \exp(T(x, y)) - 1, \end{aligned}$$

and

$$T(x, y) = \frac{1}{2y} \sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} (2xy)^n.$$

For computational purposes, we notice that $H(x, y)$ has only finitely many terms on each diagonal $v - e = \text{constant}$, since there are only a finite number of graphs, all of whose vertices are at least trivalent, with a given Euler characteristic. Thus we can substitute z *before* taking the logarithm, so that we need only to take the formal logarithm of a one-variable power series instead of a two-variable power series, i.e., $\chi(z) = \log(1 + H(1/z, z/2))$.

The calculation can now be performed with a computer, using Theorem 4.12. Up to degree 11, the polynomial $\chi(z)$ is:

$$\begin{aligned} &-\frac{1}{24}z - \frac{1}{48}z^2 - \frac{161}{5760}z^3 - \frac{367}{5760}z^4 - \frac{120257}{580608}z^5 - \frac{39793}{45360}z^6 \\ &-\frac{6389072441}{1393459200}z^7 - \frac{993607187}{34836480}z^8 - \frac{5048071877071}{24524881920}z^9 \end{aligned}$$

$$-\frac{9718190078959}{5748019200}z^{10} - \frac{375393773534736899347}{24103053950976000}z^{11}.$$

These numbers factor as follows:

$$\begin{aligned} \frac{1}{24} &= \frac{1}{2^3 \cdot 3}, & \frac{1}{48} &= \frac{1}{2^4 \cdot 3}, & \frac{161}{5760} &= \frac{7 \cdot 23}{2^7 \cdot 3^2 \cdot 5}, \\ \frac{367}{5760} &= \frac{367}{2^7 \cdot 3^2 \cdot 5}, & \frac{120257}{580608} &= \frac{53 \cdot 2269}{2^{10} \cdot 3^4 \cdot 7}, \\ \frac{39793}{45360} &= \frac{13 \cdot 3061}{2^4 \cdot 3^4 \cdot 5 \cdot 7}, & \frac{6389072441}{1393459200} &= \frac{31 \cdot 6379 \cdot 32309}{2^{15} \cdot 3^3 \cdot 5^2 \cdot 7}, \\ \frac{993607187 \cdot 1181 \cdot 841327}{34836480} &= \frac{1181 \cdot 841327}{2^{12} \cdot 3^5 \cdot 5 \cdot 7}, & \frac{5048071877071}{24524881920} &= \frac{43 \cdot 163 \cdot 1787 \cdot 403037}{2^{18} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11}, \\ \frac{9718190078959}{5748019200} &= \frac{23 \cdot 137 \cdot 3084160609}{2^{12} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11}, \\ \frac{375393773534736899347}{24103053950976000} &= \frac{29 \cdot 41 \cdot 113 \cdot 433 \cdot 1753 \cdot 3680924879}{2^{22} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13}. \end{aligned}$$

5. Divisibility

In this section we prove a divisibility property for the denominator of $\chi(\text{Out}(F_n))$.

(5.1) Proposition. *Let p be an odd prime, let C_p be the cyclic group of order p , and let G be an admissible graph on which C_p acts non-trivially. Then $\chi(G) \leq 2 - p$. If $\chi(G) = 2 - p$, G is the graph with 2 vertices and p edges, none of which are loops, and C_p acts by fixing the vertices and permuting the edges.*

Proof. Let F be the fixed subgraph of C_p . Let $m = \chi(G) - \chi(F)$. We prove three properties of m .

(1) $m < 0$.

This follows from the fact that F is a proper subgraph and G has no free edges.

(2) p divides m .

Since C_p acts freely on $G - F$, p divides the numbers of edges and vertices in $G - F$.

(3) $\chi(F) + m/p \leq 1$.

The quotient of G by C_p is a graph with Euler characteristic $\chi(F) + m/p$. The inequality follows since the quotient graph is connected.

Now $\chi(G) = \chi(F) + m$. Using (3) we have

$$\chi(G) \leq 1 - (m/p) + m = 1 + m(1 - (1/p)).$$

By (1) and (2), m is less than or equal to $-p$. Thus we have

$$\chi(G) \leq 1 - p(1 - (1/p)) = 2 - p,$$

as was to be shown.

If $\chi(G) = 2 - p$, then $m = -p$ and $\chi(F) = 2$. It follows that $\chi(G/C_p) = 1$. Thus G/C_p is a tree and F is a pair of subtrees. Any subtree of G/C_p which intersects F in two components corresponds to a graph with the same Euler characteristic as G . Since G has no free edges, no subgraph has the same Euler characteristic. The only possibility is that G/C_p is an interval and F is the pair of endpoints. This implies the last statement of the proposition. \square

(5.2) Corollary. *If p is an odd prime and p divides the denominator of $\chi(\text{Out}(F_n))$ then $n \geq p - 1$. If $n = p - 1$, then p divides the denominator of $\chi(\text{Out}(F_n))$ exactly once.*

Proof. By Proposition (1.12),

$$\chi(\text{Out}(F_n)) = \sum_{G \in \Gamma} \frac{\tau(G)}{|\text{aut}(G)|}.$$

If p divides the denominator of $\chi(\text{Out}(F_n))$, then it must divide the order of some automorphism group of some graph G of genus n . By Proposition (5.1), this can only happen if $\chi(G) \leq 2 - p$, i.e., if $n \geq p - 1$.

If $n = p - 1$ and G has an automorphism of order p , then, by Proposition (5.1), G is determined up to isomorphism. Hence only one term in the above sum has a denominator divisible by p . This term is equal to $(1 - p)/2p!$. Since $1 - p$ is not divisible by p , this term makes a non-trivial contribution to the denominator of the sum. Since $2p!$ is divisible by p exactly once, p divides the denominator of $\chi(\text{Out}(F_n))$ exactly once. \square

Remark. By a result of Culler [2], every finite subgroup of $\text{Out}(F_n)$ is a group of isometries of some graph G . Thus Proposition (5.1) implies that p -torsion occurs in $\text{Out}(F_n)$ only for $n \geq p - 1$.

The m th Bernoulli number B_m can be defined as the coefficient of $z^{2m}/(2m)!$ in the power series expansion of $z/(e^z - 1)$. We remark that the Bernoulli numbers also have the property that the denominator is divisible only by primes p with $p \leq 2m + 1$, and that p does divide the denominator when $p = 2m + 1$ [6].

Further results about the power of p which divides the denominator of $\chi(\text{Out}(F_n))$ can be found in [9].

6. Related groups

The rational Euler characteristic of groups behaves well with respect to group ex-

tensions. Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups. If N and Q are of finite homological type and G is virtually torsion-free, then G is of finite type and $\chi(G) = \chi(N) \cdot \chi(Q)$ (cf. [1, p. 250]).

Applying this to the sequence $1 \rightarrow F_n \rightarrow \text{Aut}(F_n) \rightarrow \text{Out}(F_n) \rightarrow 1$, we note that $\chi(\text{Aut}(F_n)) = (1 - n) \cdot \chi(\text{Out}(F_n))$. In particular, the following is a generating function for $\chi(\text{Aut}(F_{n+1}))$ for $n < 12$:

$$\begin{aligned} & \frac{1}{24}z + \frac{1}{24}z^2 + \frac{161}{1920}z^3 + \frac{367}{1440}z^4 + \frac{601285}{580608}z^5 + \frac{39793}{7560}z^6 \\ & + \frac{6389072441}{199065600}z^7 + \frac{993607187}{4354560}z^8 + \frac{5048071877071}{272498688}z^9 \\ & + \frac{9718190078959}{574801920}z^{10} + \frac{375393773534736899347}{2191186722816000}z^{11}. \end{aligned}$$

We now consider the kernel Ker_n of the natural map from $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$. Ker_2 is trivial, and Ker_n is known to be torsion-free and finitely generated for $n \geq 3$ [8]. Furthermore, since $\text{Out}(F_n)$ has virtual cohomological dimension $2n - 3$, the cohomological dimension of Ker_n is less than or equal to $2n - 3$. No further homological finiteness properties are known, however; for instance it is not known whether Ker_n is finitely presented, which would imply that $H_2(\text{Ker}_n)$ is finitely generated. The following proposition shows that information about the Euler characteristic of $\text{Out}(F_n)$ gives information about finite generation of the homology of Ker_n :

(6.1) Proposition. *Let $n \geq 3$ be an integer such that $\chi(\text{Out}(F_n))$ is not zero. Then the kernel Ker_n of the natural map from $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$ does not have finitely generated integral homology.*

Proof. Let Γ be a torsion-free subgroup of finite index in $\text{GL}(n, \mathbb{Z})$. Note that Γ is of finite type. Let \mathcal{A} be the inverse image of Γ under the map $\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$. Since Ker_n is torsion-free, \mathcal{A} is torsion-free of finite index. The short exact sequence $1 \rightarrow \text{Ker}_n \rightarrow \mathcal{A} \rightarrow \Gamma \rightarrow 1$ gives rise to a spectral sequence

$$(*) \quad E_{pq}^2 = H_p(\Gamma; H_q(\text{Ker}_n)) \Rightarrow H_{p+q}(\mathcal{A}).$$

Assume now that $H_i(\text{Ker}_n)$ is finitely generated for all i . Since Γ is torsion-free and of finite type, we have

$$\sum_p (-1)^p H_p(\Gamma; H_q(\text{Ker}_n)) = \chi(\Gamma) \cdot \text{rk}_{\mathbb{Z}}(H_q(\text{Ker}_n))$$

(cf. [1, p. 253]). We now compute the Euler characteristic of \mathcal{A} using the spectral sequence (*); we obtain

$$\chi(\mathcal{A}) = \sum_{p,q} (-1)^{p+q} H_p(\Gamma; H_q(\text{Ker}_n))$$

$$= \sum_q \chi(\Gamma) \cdot (-1)^q \text{rk}_{\mathbb{Z}}(H_q(\text{Ker}_n)) = \chi(\Gamma) \cdot \chi(\text{Ker}_n).$$

Harder's formula for the Euler characteristic of $\text{GL}(n, \mathbb{Z})$ gives $\chi(\Gamma) = \chi(\text{GL}(n, \mathbb{Z})) = 0$ for $n \geq 3$ [5]. Since $\chi(\text{Out}(F_n))$ (and therefore $\chi(\Lambda)$), is non-zero by assumption, the above multiplicative formula gives a contradiction. \square

Corollary (5.2) and our computations for $n \leq 12$ now give

(6.2) Corollary. *If $n + 1$ is prime or $n \leq 12$, Ker_n does not have finitely generated homology.*

In [9] it is shown that $\chi(\text{Out}(F_n))$ is non-zero whenever n is even. Thus in these cases too, Ker_n does not have finitely generated homology.

The fact that Ker_n is finitely generated shows that $H_1(\text{Ker}_n)$ is finitely generated. Thus the above corollary implies that $H_i(\text{Ker}_n)$ is not finitely generated for some $i > 1$. If $n = 3$, for example, we must have either $H_2(\text{Ker}_n)$ or $H_3(\text{Ker}_n)$ is not finitely generated, i.e., either Ker_n is not finitely presented, or $H_3(\text{Ker}_n)$ is not finitely generated (or both).

We conclude with a conjecture suggested by our calculations.

Conjecture. $\chi(\text{Out}(F_n)) < 0$ for all n , and $|\chi(\text{Out}(F_n))|$ grows exponentially with n .

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