

AUTOMORPHISMS OF GRAPHS, p -SUBGROUPS OF $\text{Out}(F_n)$ AND THE EULER CHARACTERISTIC OF $\text{Out}(F_n)$

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Finite subgroups of $\text{Out}(F_n)$ are studied by analyzing isometry groups of graphs. The results of this analysis are used to derive information about the powers of primes which divide the denominator of the rational Euler characteristic of $\text{Out}(F_n)$. This information together with a result from a previous paper by the same authors gives information about the kernel of the map from $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$ for n even.

1. Introduction

Let $\text{Out}(F_n)$ be the group of outer automorphisms of a free group on n generators, and let p be a prime. By a result of Culler [3], every finite subgroup of $\text{Out}(F_n)$ can be realized as a group of automorphisms of a connected graph G with no free edges and with Euler characteristic $\chi(G) = 1 - n$. In this paper we study the finite p -subgroups of $\text{Out}(F_n)$ of maximal order for a given prime p , and classify the associated graphs for infinitely many values of n . Together with results from [8], this leads to the determination of a lower bound for the exponent of p which appears in the denominator of the rational Euler characteristic $\chi(\text{Out}(F_n))$ for these values of n . Classical results of Minkowski and of Baumslag and Taylor can be used to establish an upper bound for this exponent of p . For those values of n for which we establish lower bounds, the upper and lower bounds coincide.

Much of the recent work on $\text{Out}(F_n)$ is motivated by apparent analogies between the groups $\text{Out}(F_n)$ and arithmetic groups. For example, in [4] it is shown that $\text{Out}(F_n)$ has strong homological finiteness properties in common with arithmetic groups. The specific motivation for this work came from a comparison of the rational Euler characteristics $\chi(\text{Out}(F_n))$ computed in [8] with the rational

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Euler characteristics of families of arithmetic groups. In the case of arithmetic groups, work of Harder [6] relates these Euler characteristics to Bernoulli numbers, and the Von Staudt theorems describe the denominators of Bernoulli numbers. Our results give corresponding information in the case of $\chi(\text{Out}(F_n))$.

The divisibility properties of numerators of Bernoulli numbers have a number-theoretic interpretation relating to regular primes. (See [2, p. 269] where this connection is made via rational Euler characteristics of arithmetic groups.) It would be interesting to know whether the divisibility properties of numerators of $\chi(\text{Out}(F_n))$ have a similar interpretation.

The paper is organized as follows. In Section 2 we recall results of Minkowski and of Baumslag and Taylor which give an upper bound on the orders of finite subgroups of $\text{Out}(F_n)$, and results of Culler which relate finite subgroups of $\text{Out}(F_n)$ to automorphisms of graphs. In Section 3 we classify the graphs associated to maximal order p -subgroups (' p -maximal' graphs) when n is a multiple of $p - 1$ and the action of the p -subgroup fixes a vertex (Theorem 3.6). In Section 4 we note that for most values of n , the Euler characteristic of the graph guarantees the existence of a fixed vertex. If n is also a multiple of $p - 1$, we have thus classified all of the p -maximal graphs, and we then use [8] to calculate the contribution of the associated groups to $\chi(\text{Out}(F_n))$. In particular, this gives us the exponent of p in the denominator of $\chi(\text{Out}(F_n))$ for these values of n (Theorem 4.3).

The exponent found in Section 4 coincides with the upper bound for the exponent found in Section 2. In contrast, the computations of $\chi(\text{Out}(F_n))$ for small values of n which appear in [8] show that, when n is not a multiple of $p - 1$, the exponent of p in the denominator of $\chi(\text{Out}(F_n))$ is often strictly smaller than the upper bound.

The exponent found for $p = 2$ implies that $\chi(\text{Out}(F_n))$ is non-zero when n is even (Corollary 4.4). This implies that the kernel of the map from $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$ does not have the same homological finiteness properties as $\text{Out}(F_n)$ and $\text{GL}(n, \mathbb{Z})$ (see [8] for details).

2. Finite subgroups of $\text{Out}(F_n)$

An automorphism of the free group F_n induces an automorphism of the abelianization \mathbb{Z}^n . This gives rise to a homomorphism from $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$. Baumslag and Taylor [1] have shown that the kernel of this map is torsion-free, and hence any finite subgroup of $\text{Out}(F_n)$ maps injectively into $\text{GL}(n, \mathbb{Z})$. A classical result of Minkowski [7] computes the maximal order of a p -subgroup of $\text{GL}(n, \mathbb{Z})$ for any prime p as follows. Let ν_p denote the standard p -adic valuation on the rational numbers, i.e. $\nu_p(p^k(r/s)) = k$ if r and s are relatively prime to p . Define a function $\alpha_p(n)$ by

$$\alpha_p(n) = \sum_{i=0}^{\infty} \left[\frac{n}{p^i(p-1)} \right]$$

where square brackets denote the greatest integer function. If P is a maximal order p -subgroup of $\text{GL}(n, \mathbb{Z})$, then $\nu_p(|P|) = \alpha_p(n)$.

The results of Baumslag–Taylor and Minkowski together imply

Proposition 2.1. *Let H be a finite subgroup of $\text{Out}(F_n)$. Then $\nu_p(|H|) \leq \alpha_p(n)$. \square*

In particular, since $\alpha_p(n) = 0$ for $p > n + 1$, we conclude that $\text{Out}(F_n)$ has p -torsion only for $p \leq n + 1$. (A proof of this fact using automorphisms of graphs can be found in [8, Proposition 5.1].)

In order to study the p -subgroups of $\text{Out}(F_n)$ and the Euler characteristic, we will use the connection between finite subgroups of $\text{Out}(F_n)$ and automorphism groups of graphs given in [3]. We will need the following definitions:

Definition 2.2. By a *graph* G we mean a 1-dimensional CW-complex. The 0-cells are the *vertices* $V(G)$ and the 1-cells are the *edges* $E(G)$. An edge e together with a choice of orientation on e is a *directed edge*. Each directed edge \vec{e} has a *terminal vertex* v_1 and an *initial vertex* v_0 (which may be the same vertex of G).

An *automorphism* of G is a permutation of the vertices of G and of the directed edges of G which is induced by a cellular homeomorphism of G . The group of automorphisms of G is denoted $\text{aut}(G)$.

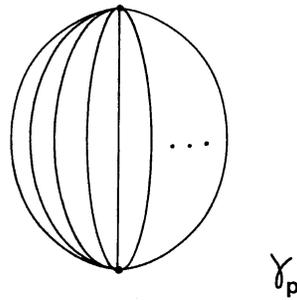
Definition 2.3. The *genus* of a graph G is equal to $e(G) - v(G) + c(G)$, where $e(G)$ is the number of (geometric) edges of G , $v(G)$ is the number of vertices of G , and $c(G)$ is the number of connected components of G . If G is connected, $\text{genus}(G) = 1 - \chi(G) = \text{rank}(\pi_1(G))$.

Definition 2.4. A graph G is *admissible* if it is connected, has no free edges, no bivalent vertices and no separating edges.

Note that if G is admissible, then $\text{genus}(G) > 1$, and any proper subgraph of G has strictly smaller genus. In addition, any homeomorphism of G takes vertices to vertices. A connected graph of genus > 1 may be made admissible in a canonical way by eliminating bivalent vertices and free edges and collapsing separating edges.

Culler's result states that any finite subgroup of $\text{Out}(F_n)$ can be realized as a subgroup of the group of automorphisms of an admissible graph of genus n .

We now fix a prime p . For each graph G we choose a p -Sylow subgroup of $\text{aut}(G)$, denoted $\text{aut}_p(G)$. If x is a vertex of G , we denote by $\text{stab}_p(x)$ the stabilizer of x in $\text{aut}_p(G)$. A graph G has p -value s if $\text{aut}_p(G)$ has order p^s , i.e.

Fig. 1. The graph γ_p , for p odd.

$\nu_p(|\text{aut}(G)|) = s$. We abbreviate this by denoting the p -value of G by $\nu_p(G)$. A graph G of genus n is said to be p -maximal if $\nu_p(G)$ is maximal among all admissible graphs of genus n .

By Proposition 2.1, we have $\nu_p(G) \leq \alpha_p(n)$ for G admissible of genus n . In particular, the cyclic group C_p of order p can act effectively on G only for $p \leq n + 1$. If $p = n + 1$, we have the following characterization of (the unique) p -maximal graph:

Let p be an odd prime. Denote by γ_p the graph with two vertices and p edges, each going from one vertex to the other (see Fig. 1).

Proposition 2.5. *Let p be an odd prime, let C_p be the cyclic group of order p , and let G be an admissible graph of genus n on which C_p acts non-trivially. If $p = n + 1$, then $G \cong \gamma_p$.*

Proof. [8, Corollary 5.2]. \square

3. p -Maximal graphs

In this section we study the structure of p -maximal graphs.

Definition 3.1. A tree S is called a *star* if it has diameter equal to 2 (see Fig. 2). The non-univalent vertex is the *center* of S , and the remaining vertices are the *outer* vertices.

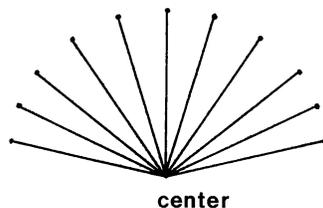


Fig. 2. Star.

Definition 3.2. A p -tree is a graph G obtained from a tree T by replacing each edge of T by a copy of γ_p (see Fig. 3). The tree T is called the *base* of G . If the base of G is a star, G is a p -star.

If G is a p -tree with $m + 1$ vertices, then $\text{aut}(G) \cong (\mathbb{Z}/p)^m \rtimes \text{aut}(T)$, where T is the base of G .

Definition 3.3. A p -tree on $m + 1$ vertices with base T is a *Sylow p -tree* if $\text{aut}_p(T)$ is isomorphic to a p -Sylow subgroup of the symmetric group on m letters.

Let G be a Sylow p -tree on $m + 1$ vertices and base T . If p is odd, $\text{aut}_p(T)$ can be imbedded into the symmetric group on the set of m edges of T , since an automorphism of odd order cannot invert an edge. Thus G is a Sylow p -tree if and only if $\nu_p(T)$ is maximal among p -values of trees with m edges.

By applying the elementary theory of p -groups acting on finite sets we obtain the following characterization of Sylow p -trees:

Proposition 3.4. Let G be a Sylow p -tree with $m + 1$ vertices and base T , and

$$m = a_0 + a_1p + \dots + a_kp^k, \quad \text{with } a_i < p.$$

Then T consists of a subtree T_0 with a_0 edges fixed by $\text{aut}_p(T)$, together with a_i stars, each with p^i edges and center in T_0 , for $i = 1, \dots, k$.

Proof. Consider the action of $\text{aut}_p(T)$ on the set of m edges of T . Since $\text{aut}_p(T)$ is isomorphic to the p -Sylow subgroup of the symmetric group on m letters, there are a_i orbits of order p^i , for $i = 0, \dots, k$. The a_0 fixed edges must form a subtree T_0 . Let C be the closure of a connected component of $T - T_0$. If C consists of more than one edge, we can rearrange the edges of T to form a new tree T' with larger p -value than T : we move all edges in the orbit of C which are at distance at least 2 from T_0 to the vertex $C \cap T_0$ (see Fig. 4). This contradicts the maximality of $\nu_p(T)$. Thus all edges which are not fixed by $\text{aut}_p(T)$ have one vertex in T_0 , and the proposition follows. \square

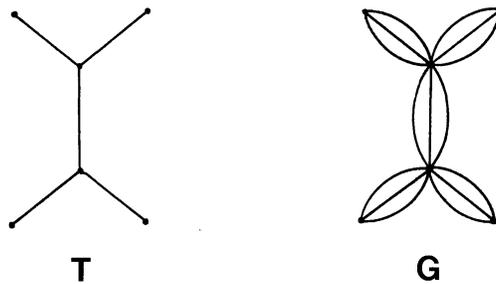


Fig. 3. p -tree G with base T .

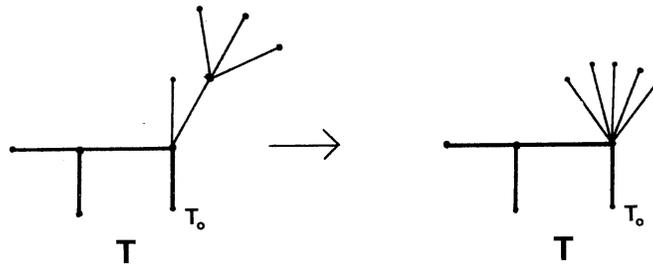


Fig. 4.

Note that if G is a Sylow p -tree on $m + 1$ vertices, then $\nu_p(G) = m + \nu_p(m!) = \alpha_p(m(p - 1))$. It follows that a graph G is p -maximal of genus n if and only if $\nu_p(G) = \alpha_p(n)$.

We now describe an operation on graphs which will be used in the proof of the next theorem.

A partition of a set is said to be *thick* if every subset in the partition has at least 3 elements. Let x be a vertex of a graph G , let $E(x)$ denote the set of directed edges terminating at x , and let Σ be a non-trivial thick partition of $E(x)$. We obtain a new graph G_Σ by *separating* G at x according to Σ , as follows. G_Σ is obtained from G by replacing the vertex x by a set of vertices $\{x_S\}$ indexed by the sets S in Σ . If \vec{e} is a directed edge of G with $\vec{e} \in S \in \Sigma$, then in G_Σ , the terminal vertex of \vec{e} is x_S (see Fig. 5).

Note that the action of $\text{aut}_p(G)$ on G induces an effective action of $\text{aut}_p(G)$ on G_Σ . In addition, we have

Lemma 3.5. *Let G be a connected graph of genus n with no free edges, let x be a vertex of G , and Σ a non-trivial thick partition of $E(x)$. Then $\text{genus}(G_\Sigma) \leq n$, and each component of G_Σ has genus less than n and greater than 1.*

Proof. The statement that each component of G_Σ has genus greater than 1 follows from the fact that Σ is thick and G has no free edges. Let k be the number of sets in Σ . Then

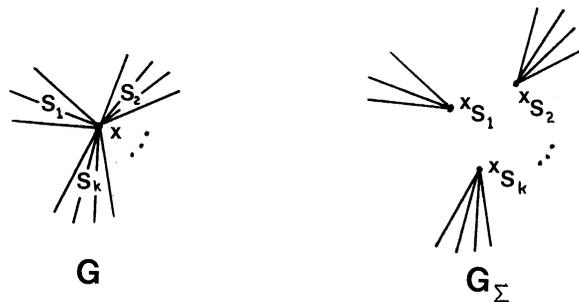


Fig. 5. Separating G at x according to $\Sigma = \{S_1, \dots, S_k\}$.

$$\begin{aligned} \text{genus}(G_\Sigma) &= e(G_\Sigma) - v(G_\Sigma) + c(G_\Sigma) \\ &= e(G) - (v(G) + (k - 1)) + c(G_\Sigma) \\ &= \text{genus}(G) - k + c(G_\Sigma). \end{aligned}$$

The graph G_Σ has at most k components. To see this, connect the new vertices $\{x_s\}$ by a tree with $(k - 1)$ edges; the result is a graph homotopy equivalent to G , and hence connected. Since G_Σ is obtained from this graph by removing $(k - 1)$ edges, G_Σ can have at most k components. Thus the above formula gives $\text{genus}(G_\Sigma) \leq \text{genus}(G)$. The genus of G_Σ is the sum of the genres of its components. If G_Σ has one component, the fact that $k \geq 2$ implies that $\text{genus}(G_\Sigma) < \text{genus}(G)$. If G_Σ has more than one component, the fact that the genus of each component is positive implies that the genus of any one component of G_Σ is strictly less than $\text{genus}(G_\Sigma)$, and hence than $\text{genus}(G)$. \square

The following theorem classifies admissible p -maximal graphs of genus congruent to zero mod $(p - 1)$, when the p -automorphisms fix a point:

Theorem 3.6. *Let p be an odd prime. Let G be an admissible p -maximal graph of genus $n \equiv 0 \pmod{p - 1}$. If $\text{aut}_p(G)$ acts with a fixed vertex, then G is a Sylow p -tree.*

Proof. The proof proceeds by induction on the p -value $s = \nu_p(G)$. If $s = 1$, Proposition 2.5 shows that γ_p is the unique graph of its genus with p -value 1; in this case we have $\text{genus}(\gamma_p) = p - 1$, $\nu_p(G) = 1 = \alpha_p(p - 1)$, and the base tree is a single edge.

Since $n \equiv 0 \pmod{p - 1}$, we have $\alpha_p(n') < \alpha_p(n)$ whenever $n' < n$; thus any admissible graph of strictly smaller genus than G must have strictly smaller p -value, i.e.

$$\text{if } \text{genus}(G') < \text{genus}(G), \text{ then } \nu_p(G') < \nu_p(G). \tag{*}$$

Conversely, since G is p -maximal, condition (*) implies that $n \equiv 0 \pmod{p - 1}$.

Assume now that $s > 1$. We want to split G into subgraphs to which we can apply induction.

Claim 1. There is a vertex $x \in G$ fixed by $\text{aut}_p(G)$ such that $G - \{x\}$ is disconnected.

Proof. By the hypothesis of the theorem, there is some vertex y of G which is fixed by $\text{aut}_p(G)$. Suppose that $G - \{y\}$ is connected.

If $\text{aut}_p(G)$ leaves invariant a non-trivial thick partition Σ of the directed edges terminating at y , then the graph G_Σ obtained from G by separating G at y according to Σ has $\nu_p(G_\Sigma) \geq \nu_p(G)$ but $\text{genus}(G_\Sigma) < \text{genus}(G)$, contradicting (*).

If $\text{aut}_p(G)$ fixes an edge e with y as a vertex, the graph $G - e$ can be made admissible to give a graph of the same or larger p -value and smaller genus, also contradicting (*).

The only remaining possibility (since $p \geq 3$) is that there are p edges terminating at y , none of which are fixed by $\text{aut}_p(G)$. Let $V(y)$ be the set of vertices in G which are connected to y by an edge. If $V(y)$ has one element x , then x is fixed by $\text{aut}_p(G)$ and $G - \{x\}$ is disconnected. If $V(y)$ has p elements, then $\text{aut}_p(G)$ acts effectively on the graph C obtained from G by removing y and the edges containing y . (Note that $\text{genus}(C) > 0$, since if $\text{genus}(C) = 0$, G is a ‘wheel’ with p spokes, so $\nu_p(G) = 1$.) Thus C can be made into an admissible graph C' , $\text{genus}(C') < \text{genus}(G)$ and $\text{aut}_p(G)$ acts effectively on C' , contradicting (*). \square

By Claim 1, we can choose a vertex x fixed by $\text{aut}_p(G)$ with $G - \{x\}$ disconnected. Let C_0 be a connected component of $G - \{x\}$, and $C = C_0 \cup \{x\}$ the corresponding subgraph of G . If $\text{genus}(C) = 1$, C is a loop and the orbit of C can be replaced by a p -star to produce a graph G' of strictly lower genus and the same or larger p -value, contradicting (*). Thus $1 < \text{genus}(C) < \text{genus}(G)$, and we may assume that C is admissible. Condition (*) now implies that $\nu_p(C) < \nu_p(G)$. In addition, condition (*) on G and the p -maximality of G imply similar conditions for C . Since (*) for C implies that $\text{genus}(C) \equiv 0 \pmod{p-1}$, in order to apply induction to C we need only check that $\text{aut}_p(C)$ fixes a vertex of C .

Claim 2. $\text{aut}_p(C)$ has a fixed vertex.

Proof. Let Γ_p be the stabilizer of C in $\text{aut}_p(G)$. It suffices to show that Γ_p is a p -Sylow subgroup of $\text{aut}(C)$, since Γ_p fixes x .

Assume $\nu_p(C) > \nu_p(|\Gamma_p|)$. Let C' be the p -star of the same genus as C . Then $\nu_p(C') \geq \nu_p(C)$ and $\text{aut}_p(C')$ has a fixed point. Let G' be the graph obtained by replacing every subgraph in the $\text{aut}_p(G)$ -orbit of C by a copy of C' , attached to x at the fixed point of $\text{aut}_p(C')$. Then $\text{genus}(G') = \text{genus}(G)$ and $\nu_p(G') > \nu_p(G)$, contradicting the fact that G is p -maximal. \square

We can now apply induction to the graph C to conclude that C is a Sylow p -tree. Since C_0 was an arbitrary component of $G - \{x\}$, the entire graph G is a wedge product of p -trees, and is hence a p -tree. Let T be the base of G . Since $\text{aut}_p(G) \cong (\mathbb{Z}/p)^m \rtimes \text{aut}_p(T)$, the maximality of $\nu_p(G)$ implies that $\nu_p(T)$ is maximal, i.e. G is a Sylow p -tree. \square

We now consider the case $p = 2$. The following relative notions will be useful.

Let H be a subgraph of a graph G . We denote by $\text{aut}_p(G, H)$ the subgroup of $\text{aut}_p(G, H)$ which fixes H pointwise.

Definition 3.7. Let G be a connected graph of genus ≥ 2 with no free edges and let v be a vertex of G . The pair (G, v) is p -maximal if $|\text{aut}_p(G, v)| \geq |\text{aut}_p(G', v')|$ for all pairs (G', v') with G' admissible and $\text{genus}(G') = \text{genus}(G)$. If G has genus 1 and no free edges, (G, v) will be considered to be 2-maximal.

Definition 3.8. A graph G is a *rose* if it has only one vertex. We denote the rose of genus n by R_n .

Proposition 3.9. *Let G be an admissible graph, and let v be a vertex of G . If (G, v) is 2-maximal, then G is a rose.*

Proof. The proof proceeds by induction on $n = \text{genus}(G)$. If $n = 2$ or $n = 3$ the proposition is true by inspection.

Assume now that (G, v) is 2-maximal, but G is not a rose. Then we can find a vertex $w \neq v$ which is connected to v by an edge. Let $\Omega(w)$ be the orbit of w under $\text{aut}_2(G, v)$, and let H be the subgraph of G consisting of all edges from $\Omega(w)$ to v .

Suppose $H = G$, and let $|\Omega(w)| = 2^a$. Since G is admissible, there are $b > 2$ edges between v and w . Thus $\text{genus}(G) = 2^a(b - 1)$, and a straightforward calculation shows that $\nu_2(G) < \nu_2(R)$, where R is the rose of genus $2^a(b - 1)$. But this contradicts the 2-maximality of (G, v) . Thus $H \neq G$.

An automorphism of G fixing v induces an automorphism of H fixing v , so we have a homomorphism $\text{aut}(G, v) \rightarrow \text{aut}(H, v)$ with kernel $\text{aut}(G, H)$. Thus

$$|\text{aut}(G, v)| \leq |\text{aut}(H, v)| \cdot |\text{aut}(G, H)|. \quad (*)$$

Let G_H be the quotient graph G/H and $v_H \in G_H$ the image of v . Then $\text{aut}(G, H)$ acts effectively on (G_H, v_H) , so $|\text{aut}(G, H)| \leq |\text{aut}(G_H, v_H)|$, and $(*)$ gives

$$|\text{aut}(G, v)| \leq |\text{aut}(H, v)| \cdot |\text{aut}(G_H, v_H)|.$$

Case 1. H is not a tree. If H is not a tree, we must have both (G_H, v_H) and (H, v) 2-maximal, since otherwise we could replace (G_H, v_H) and (H, v) by 2-maximal pairs whose wedge product (G', v') has genus n and strictly larger 2-automorphism group than (G, v) .

Since $H \neq G$, we have $\text{genus}(H) < \text{genus}(G)$. Since H is 2-maximal, by induction H must be homeomorphic to a rose; i.e. there are exactly two edges from v to w . Furthermore, $\text{genus}(G_H) < \text{genus}(G)$, so by induction again, G_H must be a rose.

Choose a maximal tree T in H . Since G_H is a rose, so is G/T , and G/T can be described as the quotient of G by the following equivalence relation. For each directed edge t of T with initial vertex v , let t' be the other oriented edge of H with the same initial and terminal vertices, and set $t \sim \bar{t}'$, where \bar{t}' denotes the edge t' with the opposite orientation. The action of $\text{aut}_2(G, v)$ induces an effective action on $G/\sim = \widehat{G/T}$.

Let f be an edge of $G - H$, and h an edge of $H - T$. Let α be the element of $\text{aut}(G/T, v)$ which interchanges the images of f and h , fixing all other edges. Then α cannot be realized by an element of $\text{aut}_2(G, v)$, so $|\text{aut}_2(G/T, v)| > |\text{aut}_2(G, v)|$, contradicting the 2-maximality of (G, v) .

Case 2. H is a tree. Since H is a tree, $\text{aut}_2(G, v)$ acts effectively on the quotient graph (G_H, v_H) . Since G_H is of the same genus as G , (G_H, v_H) must also be 2-maximal. If G_H is not a rose, we can choose $w \neq v_H$ connected to v_H by an edge, and form an invariant graph H_1 as above. If H_1 is not a tree, we can achieve a contradiction as in Case 1. If H_1 is a tree, we can repeat the process until the quotient graph is a rose, i.e. isomorphic to G/T , for some invariant maximal tree T in G . Let x be a univalent vertex of T . Since G is admissible, we can find distinct edges e and f of $G - T$ at x such that e is not a loop. Let α be the element of $\text{aut}_2(G/T)$ which sends the image of e to its opposite \bar{e} and fixes all other edges. Then α cannot be realized by an automorphism of G , so $|\text{aut}_2(G/T)| > |\text{aut}_2(G, v)|$, contradicting the 2-maximality of (G, v) . \square

Corollary 3.10. *If G is an admissible 2-maximal graph with an odd number of vertices, then G is a rose.*

Proof. The group $\text{aut}_2(G)$ acts on the set of vertices of G . If $v(G)$ is odd, then $\text{aut}_2(G)$ must fix a vertex. By Proposition 3.9, G is a rose. \square

Corollary 3.11. *If G is an admissible 2-maximal graph of genus n , with n even, then G is a rose.*

Proof. Since n is even, $\chi(G) = v(G) - e(G)$ is odd. $\text{Aut}_2(G)$ acts on $v(G)$ and on $e(G)$. If $\text{aut}_2(G)$ fixes a vertex, then G is a rose by Proposition 3.9.

Assume now that $\text{aut}_2(G)$ fixes no vertices. Then $v(G)$ must be even by Corollary 3.10, so $e(G)$ is odd and $\text{aut}_2(G)$ fixes an edge e . Since e does not separate G , $\text{aut}_2(G)$ acts effectively on G/e , which is an admissible graph of genus n with an odd number of vertices, and is hence a rose by Corollary 3.10. Thus G is a graph with two vertices v and w with $b \geq 2$ edges between v and w , and $a \geq 0$ loops at each of v and w . Since $e(G) = 2a + b$ is odd, b is odd, and $b \geq 3$. Let f be an edge of G between v and w , with $f \neq e$. Let α be the element of $\text{aut}_2(G/e)$ which sends f to \bar{f} and fixes all other edges of G/e , including all other images of edges from v to w . Then α cannot be realized as an automorphism of G , so $|\text{aut}_2(G/e)| > |\text{aut}_2(G)|$ contradicting the 2-maximality of G . \square

4. Euler characteristic

In this section we apply the results of Sections 2 and 3 to the Euler characteristic $\chi(\text{Out}(F_n))$. We will need the following formulas from [8, Proposition 1.12]:

For any graph G , let

$$\tau(G) = \sum_{F \subset G} (-1)^{e(F)} \quad (4.1)$$

where the sum is over all subforests F of G with the same vertex set as G , and $e(F)$ is the number of edges in F .

Then

$$\chi(\text{Out}(F_n)) = \sum_{G \in \Gamma} \frac{\tau(G)}{|\text{aut}(G)|} \quad (4.2)$$

where Γ is a set of representatives for homeomorphism classes of admissible graphs of genus n .

Remark 4.1. Let $\bar{n} = \prod_p p^{\alpha_p(n)}$. By Proposition 2.1, every finite subgroup of $\text{Out}(F_n)$ has order dividing \bar{n} . Since $\text{aut}(G)$ is isomorphic to a finite subgroup of $\text{Out}(F_n)$ for G admissible of genus n , the above formula implies that the denominator of $\chi(\text{Out}(F_n))$ also divides \bar{n} . An interesting connection with Bernoulli numbers was noted by Minkowski [7]. Let B^{2i} denote the i th Bernoulli number; then $\bar{n} = 2^n b_1 \dots b_{\lfloor n/2 \rfloor}$, where b_i is the denominator of $B_{2i}/2i$. In other words,

$$2^n \prod_{2i \leq n} \frac{B_{2i}}{2i} \chi(\text{Out}(F_n)) \text{ is an integer for } n \geq 2.$$

Formula 4.2 for $\chi(\text{Out}(F_n))$ gives the following corollary of Proposition 2.5:

Proposition 4.2. *If p is an odd prime and p divides the denominator of $\chi(\text{Out}(F_n))$, then $p \leq n + 1$. If $p = n + 1$, then $v_p(\chi(\text{Out}(F_n))) = -1$.*

Proof. [8, Corollary 5.2.] \square

The following theorem generalizes this result:

Theorem 4.3. *For any prime p , $v_p(\chi(\text{Out}(F_n))) \geq -\alpha_p(n)$. If $p - 1$ divides n , and p does not divide $n - 1$, then $v_p(\chi(\text{Out}(F_n))) = -\alpha_p(n)$.*

Proof. The inequality is immediate from Proposition 2.1 and (4.2), using the ultrametric inequality for v_p .

We first consider the case $p = 2$. Since 2 does not divide $n - 1$, n must be even. By Corollary 3.11, the only 2-maximal graph for n even is the rose R_n . We have $|\text{aut}(R_n)| = 2^n n!$, so $v_2(R_n) = \alpha_2(n)$. Since $\tau(R_n) = 1$, Formula (4.2) implies the statement for $p = 2$.

Now let p be an odd prime, and let G be an admissible p -maximal graph of genus n . Since G is connected, $\chi(G) = 1 - n$. The vertices and edges of G break up into $\text{aut}_p(G)$ -orbits, so the assumption that $(p, n - 1) = 1$ implies that $\text{aut}_p(G)$ must have a fixed edge or vertex in G . Since p is odd, $\text{aut}_p(G)$ acts without

inverting edges, so in fact there must be a fixed vertex. Since $p - 1$ divides n , Theorem 3.6 implies that G is a Sylow p -tree.

Let T be the base tree of G . The action of $\text{aut}_p(G)$ on G induces an action on T . Let $m = n/(p - 1)$ be the number of edges of T . If we write out the p -adic expansion of m ,

$$m = a_0 + a_1p + a_2p^2 + \cdots + a_kp^k,$$

then by Proposition 3.4 there are a_i edge-orbits of size p^i , for $i = 0, \dots, k$. The a_0 edges fixed by $\text{aut}_p(G)$ form a subtree T_0 of T . Each non-trivial orbit is a star with (fixed) center in T_0 . A partition of the edges of T into orbits in this way will be called a p -Sylow structure on T . Note that the set of p -Sylow structures on T is in one-to-one correspondence with the set of p -Sylow subgroups of $\text{aut}(T)$.

The contribution to $\chi(\text{Out}(F_n))$ from graphs G of maximal p -value is

$$\sum_{\nu_p(G) = \alpha_p(n)} \frac{\tau(G)}{|\text{aut}(G)|}.$$

By [8, Lemma 2.2], $\tau(G) = (\tau(\gamma_p))^m = (1 - p)^m$. Furthermore, $\text{aut}(G)$ is the semidirect product of $(\mathbb{Z}/p)^m$ with $\text{aut}(T)$. Thus the above sum is equal to

$$\frac{(1 - p)^m}{p^m} \sum \frac{1}{|\text{aut } T|}$$

where the sum is over all p -Sylow trees T with m edges.

To compute the sum of $1/|\text{aut}(T)|$, we will label the vertices of the trees T . For each tree T with m edges (and $m + 1$ vertices), choose a set of $m + 1$ labels, and let S be the set of possible labellings of the vertices of T ; thus $\#S = (m + 1)!$. The group $\text{aut}(T)$ acts freely on S , giving

$$\#S = \#\{\text{orbits of } \text{aut}(T)\} \cdot |\text{aut}(T)|.$$

We define a *vertex-labelled tree* to be an orbit of $\text{aut}(T)$. Then the above equation can be written

$$\frac{1}{|\text{aut}(T)|} = \frac{1}{(m + 1)!} \cdot \#\{\text{vertex-labelled trees homeomorphic to } T\}.$$

Thus

$$\sum \frac{1}{|\text{aut}(T)|} = \frac{1}{(m + 1)!} \cdot \#\{\text{vertex-labelled } p\text{-Sylow trees on } m + 1 \text{ vertices}\}.$$

Thus the contribution to $\chi(\text{Out}(F_n))$ coming from graphs with maximal p -value is equal to

$$\frac{(1-p)^m \cdot \#\{\text{vertex-labelled } p\text{-Sylow trees on } m+1 \text{ vertices}\}}{p^m(m+1)!}.$$

Since p does not divide $n-1$, p does not divide $m+1$. Therefore $\alpha_p(n) = m + \nu_p(m!) = m + \nu_p((m+1)!)$. Thus to prove that the exponent of p in the denominator of $\chi(\text{Out}(F_n))$ is equal to $\alpha_p(n)$, it suffices to show that the number of vertex-labelled p -Sylow trees on $m+1$ vertices is prime to p .

We will actually count the number of pairs consisting of a vertex-labelled p -Sylow tree T together with a p -Sylow orbit structure on the edges of T . As we noted above, the set of p -Sylow structures on T is in one-to-one correspondence with the set of p -Sylow subgroups of $\text{aut}(T)$, and is thus congruent to 1 mod p by the Sylow subgroup theorem. Therefore we will obtain the same answer mod p by counting trees with a p -Sylow structure as by counting trees alone.

Each vertex-labelled p -Sylow tree T with a p -Sylow structure is determined uniquely by its labelled fixed tree T_0 , the label sets for each non-trivial vertex orbit, and the vertices in T_0 to which each labelled orbit is glued. Thus

$$\begin{aligned} & \#\{\text{vertex-labelled } p\text{-Sylow trees on } m+1 \text{ vertices with a } p\text{-Sylow structure}\} \\ &= \#\{\text{labelled fixed trees } T_0\} \\ & \quad \cdot \#\{\text{label sets for non-trivial vertex orbits}\} \\ & \quad \cdot \#\{\text{vertices in } T_0\}^{\#\{\text{non-trivial vertex orbits}\}}. \end{aligned}$$

Claim 1. The number of possible labelled fixed trees T_0 is equal to

$$\binom{m+1}{a_0+1} (a_0+1)^{(a_0-1)}.$$

Proof. By Cayley's theorem [5], there are $(a_0+1)^{(a_0-1)}$ vertex-labelled trees on a_0+1 vertices. For each of these, we choose a label set of a_0+1 labels out of a possible $m+1$. \square

Claim 2. The number of possible label sets for the non-trivial vertex orbits is congruent to 1 mod p .

Proof. After labelling T_0 , there are $m-a_0$ labels left. The number of ways of dividing $(m-a_0)$ labels into a_i groups of size p^i for $i=1, \dots, k$ is equal to the number of p -Sylow subgroups of the symmetric group on $m-a_0$ letters, and is hence congruent to 1 mod p . \square

Since T_0 has $a_0 + 1$ vertices and there are $a_1 + \cdots + a_k$ non-trivial vertex orbits, Claims 1 and 2 imply that

$$\begin{aligned} & \#\{\text{vertex-labelled } p\text{-Sylow trees on } m + 1 \text{ vertices with a } p\text{-Sylow structure}\} \\ & \equiv \binom{m + 1}{a_0 + 1} (a_0 + 1)^{\Sigma - 1} \pmod{p}, \quad \text{where } \Sigma = a_0 + \cdots + a_k. \end{aligned}$$

It is now easy to check that this number is prime to p , since $m + 1$ is prime to p . Thus the number of vertex-labelled p -Sylow trees on m edges is prime to p , which concludes the proof of the theorem. \square

As an immediate corollary, we have

Corollary 4.4. $\chi(\text{Out}(F_n))$ is non-zero whenever n is even. \square

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