ON THE GEOMETRY OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT

The groups $\mathrm{Aut}\,(F_3)$ and $\mathrm{Out}\,(F_3)$ satisfy strictly exponential isoperimetric inequalities; in particular, they are not automatic. For $n \geqslant 3$, $\mathrm{Aut}\,(F_n)$ and $\mathrm{Out}\,(F_n)$ do not admit bounded bicombings of subexponential length, hence they cannot act properly and cocompactly by isometries on any simply-connected space of non-positive curvature, and they are not biautomatic.

0. *Introduction*

From a geometric viewpoint, a natural way to think of a finitely generated group G is as the vertex set of the Cayley graph Γ associated to a choice of finite generating set for G. One metrizes each edge of the Cayley graph to have length 1, and gives G the induced metric. A normal form for elements in G, relative to the given generating set, can then be thought of as a choice of an edge-path from the identity to each vertex of Γ; when viewed in this way, such a normal form is called a *combing of G*.

In his work on the fundamental groups of hyperbolic surfaces, Max Dehn [5] showed how paths in the Cayley graph of a group can be used to study the algebraic structure of the group; in particular, he bounded the complexity of the word problem in surface groups by a linear isoperimetric inequality (as defined in Section 1 below). The introduction of word-hyperbolic groups [9] and automatic groups [6] has again focused attention on this geometric approach to group theory, and in particular on the idea that one can often deduce much about the structure of a group if one can obtain non-trivial bounds on the isoperimetric function and the types of combing which the group admits.

In this article we shall be concerned with the groups $\mathrm{Aut}\,(F_n)$ and $\mathrm{Out}\,(F_n)$, the automorphism and outer automorphism groups of finitely generated free groups. Hatcher and Vogtmann [10], and independently Gersten (unpublished), have shown that (for all n) $\mathrm{Aut}\,(F_n)$ and $\mathrm{Out}\,(F_n)$ satisfy an exponential isoperimetric inequality. For $n = 2$ it is known that $\mathrm{Aut}\,(F_2)$ satisfies a quadratic isoperimetric inequality (and this is best possible), and $\mathrm{Out}\,(F_2)$ satisfies a linear isoperimetric inequality. Our first result shows that the inequality of [10] is sharp for the generality in which it is stated.

THEOREM A. $\mathrm{Aut}\,(F_3)$ *and* $\mathrm{Out}\,(F_3)$ *satisfy a strictly exponential isoperimetric inequality.*

COROLLARY B. $\mathrm{Aut}\,(F_3)$ *and* $\mathrm{Out}\,(F_3)$ *do not admit bounded combings whose length is sub-exponential. In particular, they are neither automatic nor semihyperbolic.*

Corollary B forms the base step of an inductive process which we use to obtain the following restrictions on the types of combing which $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ admit, for all n. The inductive step in our argument involves an analysis of certain centralizers in $\text{Aut}(F_n)$ and $\text{Out}(F_n)$.

THEOREM C.   *If $n \geqslant 3$, then $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ do not admit bounded bicombings of sub-exponential length.*

This theorem leads to the following restrictions on the algorithmic structure of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$, and on their isometric actions.

COROLLARY D.   *If $n \geqslant 3$, then $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ are not semihyperbolic. In particular, $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ do not act properly and cocompactly by isometries on any simply-connected geodesic metric space of non-positive curvature, and they are not biautomatic.*

Here, we mean semihyperbolic in the sense of [2] and non-positive curvature in the sense of Gromov's $CAT(0)$ inequality [9]. The definition of automatic and biautomatic groups can be found in [6]. Gersten [8] had previously used different techniques to show that $\text{Aut}(F_n)$ ($n \geqslant 3$) and $\text{Out}(F_n)$ ($n \geqslant 4$) do not act properly and cocompactly on any simply-connected geodesic metric space of non-positive curvature.

Parallels are often drawn between the properties of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$, on the one hand, and the general linear groups $GL_n(\mathbb{Z})$ and the mapping class groups of hyperbolic surfaces, on the other. In this vein, we note that (in unpublished work) Thurston has proved that $GL_n(\mathbb{Z})$ satisfies a quadratic isoperimetric inequality when $n \geqslant 4$. However, these groups are not semihyperbolic, because they contain nilpotent subgroups which are not virtually abelian. The case $n = 3$ is rather different: it is shown in Chapter 10 of [6] that $GL_3(\mathbb{Z})$ satisfies a strictly exponential isoperimetric inequality. We use this fact and the natural map from $\text{Aut}(F_3)$ onto $GL_3(\mathbb{Z})$ to establish the lower bound required for Theorem A.

Recently, Lee Mosher has proved that the mapping class groups of all hyperbolic surfaces of finite type are automatic [12]. In particular, this implies that they satisfy a quadratic isoperimetric inequality. On the other hand, by considering the structure of centralizers in the group, one can show that the mapping class group of a surface of genus at least 3 (or 2, if the surface is not closed) does not act properly and cocompactly on a simply-connected space of non-positive curvature (see [11] or [4]). It remains unknown whether these mappings class groups are biautomatic.

This paper is organized as follows. In Section 1 we recall the definition of an isoperimetric inequality and prove Theorem A. In Section 2 we gather a number of lemmas concerning combings. In Section 3 we prove Theorem C by examining the structure of certain centralizers in $\text{Aut}(F_n)$ and $\text{Out}(F_n)$.

The second author thanks Christophe Pittet for many useful conversations about isoperimetric inequalities.

1. *Isoperimetric inequalities in rank three*

Let us recall the definition of an isoperimetric inequality for a finitely presented group, as introduced in [9]. Given a finite presentation $\langle \mathscr{A} \mid \mathscr{R} \rangle$ for a group G, a reduced word w in the free group $F(\mathscr{A})$ represents the identity in G if and only if w can be expressed in $F(\mathscr{A})$ as a product of conjugates of elements of $\mathscr{R}$. Isoperimetric

inequalities give bounds on the number of factors required for a minimal such expression, and hence on the complexity of the word problem in G; the bounds are given as a function of word length $|w|$.

For example, G is said to satisfy a linear isoperimetric inequality if there exist constants α, β such that every $w \in \ker(F(\mathscr{A}) \to G)$ can be expressed in $F(\mathscr{A})$ as the product of at most $\alpha|w| + \beta$ conjugates of elements of $\mathscr{R}$. One can show that this definition (but not the constants α and β) is independent of the finite presentation chosen [1]. Similarly, one says that G satisfies an exponential (respectively quadratic, sub-exponential, etc.) isoperimetric inequality, if there is an exponential (respectively quadratic, sub-exponential, etc.) function of $|w|$ which bounds the number of factors in a minimal expression showing that $w \in \ker(F(\mathscr{A}) \to G)$. A group is said to satisfy a *strictly* exponential isoperimetric inequality if it satisfies an exponential isoperimetric inequality but does not satisfy a sub-exponential isoperimetric inequality. (A function $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *sub-exponential* if $f(x)/k^x \to 0$ as $x \to \infty$ for every $k > 1$.)

We think of a choice of generators for a group G as an abstract set $\mathscr{A}$ together with a map $\mu \colon \mathscr{A} \to G$ which induces a surjection $\mu_* \colon F(\mathscr{A}) \to G$ from the free group on $\mathscr{A}$ onto G. If $p \colon G \twoheadrightarrow Q$ is a group epimorphism, the induced choice of generators for Q then consists of the same set $\mathscr{A}$ together with the map $p \circ \mu$. A presentation $\langle \mathscr{A} \mid \mathscr{R} \rangle = \langle \mathscr{A} \mid \mathscr{R} \rangle_\mu$ consists of a choice of generators $\mu \colon \mathscr{A} \to G$ together with a subset $\mathscr{R} \subseteq \ker(\mu_*)$ whose normal closure is the whole of $\ker(\mu_*)$. There is a presentation of Q of the form $\langle \mathscr{A} \mid \mathscr{R}' \rangle = \langle \mathscr{A} \mid \mathscr{R}' \rangle_{p \circ \mu}$, with $\mathscr{R} \subset \mathscr{R}'$.

Proof of Theorem A. The fact that $\operatorname{Aut}(F_3)$ and $\operatorname{Out}(F_3)$ satisfy an exponential isoperimetric inequality is proved in [10]. We prove here that this inequality is optimal. According to [1], we may work with any finite presentation $\langle \mathscr{A} \mid \mathscr{R} \rangle_\mu$ for $\operatorname{Aut}(F_3)$ (respectively $\operatorname{Out}(F_3)$). For our chosen presentation, we must exhibit a sequence of words w_n whose length grows linearly with n, but whose 'area' grows faster than any sub-exponential function $f \colon \mathbb{N} \to \mathbb{N}$, in the sense that any expression for w_n as a product of conjugates of elements of $\mathscr{R}$ requires more than $f(n)$ terms for n large.

We fix a 3-element set $\{x_1, x_2, x_3\}$ and regard $\operatorname{Aut}(F_3)$ and $\operatorname{Out}(F_3)$ as, respectively, the automorphism and outer automorphism group of the free group on this set. We identify $\operatorname{GL}_3(\mathbb{Z})$ with the automorphism group of the free abelian group on this set. Thus we have natural epimorphisms $\operatorname{Aut}(F_3) \twoheadrightarrow \operatorname{Out}(F_3) \twoheadrightarrow \operatorname{GL}_3(\mathbb{Z})$.

We consider the automorphisms A, B and T of F_3 given by the following.

$$A \colon x_1 \longmapsto x_1 \qquad B \colon x_1 \longmapsto x_1 \qquad T \colon x_1 \longmapsto x_1^2 x_2$$
$$x_2 \longmapsto x_2 \qquad\qquad x_2 \longmapsto x_2 \qquad\qquad x_2 \longmapsto x_1 x_2$$
$$x_3 \longmapsto x_1 x_3 \qquad\quad x_3 \longmapsto x_3 x_2 \qquad\quad x_3 \longmapsto x_3$$

We choose generators $\mu \colon \mathscr{A} \to \operatorname{Aut}(F_3)$, where $\mathscr{A}$ is required to contain elements a, b, t with $\mu(a) = A$, $\mu(b) = B$ and $\mu(t) = T$. We then take the induced choice of generators for $\operatorname{Out}(F_3)$ and $\operatorname{GL}_3(\mathbb{Z})$. Let $\mathscr{R}_{\mathrm{Aut}} \subset \mathscr{R}_{\mathrm{Out}} \subset \mathscr{R}_{\mathrm{GL}}$ be finite sets of $F(\mathscr{A})$ whose normal closures are $\ker(F(\mathscr{A}) \twoheadrightarrow \operatorname{Aut}(F_3))$, $\ker(F(\mathscr{A}) \twoheadrightarrow \operatorname{Out}(F_3))$ and $\ker(F(\mathscr{A}) \twoheadrightarrow \operatorname{GL}_3(\mathbb{Z}))$, respectively.

Consider the words

$$w_n = t^n a t^{-n} b t^n a^{-1} t^{-n} b^{-1}$$

in $F(\mathscr{A})$. In [7] Gersten shows that if $f \colon \mathbb{N} \to \mathbb{N}$ is any sub-exponential function, then for sufficiently large n any expression for w_n as a product of conjugates of relators for

the (finitely presented) subgroup of $GL_3(\mathbb{Z})$ generated by a, b and t requires more than $f(n)$ terms. The argument of [6, 10.4] shows that this remains true even if one is allowed to use any relators from $\mathscr{R}_{GL}$, and thus that the isoperimetric function for $GL_3(\mathbb{Z})$ is at least exponential.

We now observe that with our specific choice of A, B and T, the words w_n actually lie in the kernel of the map $F(\mathscr{A}) \twoheadrightarrow \operatorname{Aut}(F_3)$. Since we have arranged that $\mathscr{R}_{\text{Aut}} \subset \mathscr{R}_{\text{Out}} \subset \mathscr{R}_{GL} \subset F(\mathscr{A})$, this shows that the isoperimetric functions for $\operatorname{Aut}(F_n)$ and $\operatorname{Out}(F_n)$ are at least exponential, as was to be shown.

2. Combings and quasiconvex subgroups

Let $\mu: \mathscr{A} \to G$ be a choice of finite generating set for the group G. Let $\mathscr{A}^{-1}$ be a set of formal inverses for $\mathscr{A}$, and let $\mu^*: (\mathscr{A} \cup \mathscr{A}^{-1})^* \twoheadrightarrow G$ be the map which $\mu: \mathscr{A} \to G$ induces on the free monoid $(\mathscr{A} \cup \mathscr{A}^{-1})^*$. We shall use the symbol $|w|$ to denote the length of a word $w \in (\mathscr{A} \cup \mathscr{A}^{-1})^*$. We view G as a metric space with the word-metric associated to μ, that is,

$$d(g, h) = \inf\{|w| \,|\, \mu^*(w) = g^{-1}h\}.$$

We identify each word $w \in (\mathscr{A} \cup \mathscr{A}^{-1})^*$ with the discrete, eventually constant path $p_w: \mathbb{N} \to G$ defined by $p_w(t) = \mu^*(w_t)$, where w_t is the prefix of length t in w. It is convenient to adopt the notation $w(t) := p_w(t)$. It is also convenient to assume that there is an element $a_0 \in \mathscr{A}$ such that $\mu(a_0) = 1 \in G$; this ensures that a discrete, eventually constant path $p: \mathbb{N} \to G$ is equal to p_w for some $w \in (\mathscr{A} \cup \mathscr{A}^{-1})^*$ if and only if $p(0) = 1$ and $d(p(t), p(t+1)) \leqslant 1$ for all $t \in \mathbb{N}$.

DEFINITION 2.1. A *combing* of G is a map $\sigma: G \to (\mathscr{A} \cup \mathscr{A}^{-1})^*$ such that $\mu \circ \sigma = \operatorname{id}_G$. Such a map may be thought of as a choice, for each $g \in G$, of an edge-path σ_g in the Cayley graph $\Gamma = \Gamma_\mu(G)$ from 1 to g. A combing is (synchronously) *bounded* if there is a constant C such that the combing paths to adjacent vertices g_1 and g_2 in Γ are within a distance C of each other at each time t, that is, $d(\sigma_{g_1}(t), \sigma_{g_2}(t)) \leqslant C$ for all $t \in \mathbb{N}$. Under translation by the left-action of the group G, a combing gives an edge-path between any two vertices of Γ; when thought of in this way, the combing is called a *bicombing*. A bicombing is *bounded* if, given any two pairs of adjacent vertices of Γ, any two bicombing paths from elements of the first pair to elements of the second are within a bounded distance at each time t, that is, there is a constant C such that whenever $d(g_i, g_k) \leqslant 1$ and $d(h_j, h_l) \leqslant 1$, we have $d(g_i \sigma_{g_i^{-1}h_j}(t), g_k \sigma_{g_k^{-1}h_l}(t)) \leqslant C$ for all t.

The *length* of a combing σ is a function $L_\sigma: \mathbb{N} \to \mathbb{N}$ which bounds the lengths of the combing lines in terms of the lengths of geodesics. Specifically, L_σ is defined by

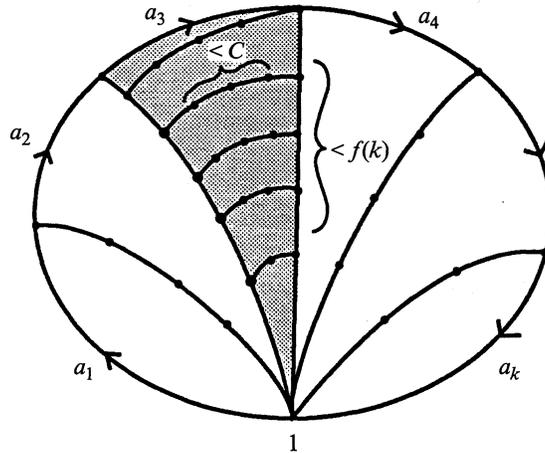$$L_\sigma(n) = \max\{|\sigma_g| \,|\, d(1, g) \leqslant n\}.$$

One says that σ has *sub-exponential* length if L_σ is a sub-exponential function.

It is easy to check that if G admits a bounded bicombing of sub-exponential length with respect to one choice of finite generating set, then it admits such a bicombing with respect to any other choice of finite generating set.

A well-known diagrammatic argument provides the following link between (bi)combings and isoperimetric inequalities. We sketch the proof here; a detailed proof can be found, for example, in [3, 5.1].

LEMMA 2.2. *If a group G admits a bounded combing of sub-exponential length, then G is finitely presented and satisfies a sub-exponential isoperimetric inequality.*

Sketch of Proof. The idea is to construct a van Kampen diagram for a word of length k which represents the identity in G as follows. The word is represented by a loop in the Cayley graph starting at 1. From each vertex g of this loop, draw the combing line from 1 to g (see figure).

van Kampen diagram for $w = a_1...a_k$

Since the combing is bounded, the area between combing lines to adjacent vertices can be filled by relators of length at most $2C + 2$, where C is the bound on the width of the combing. Since the length of the combing lines is bounded by a sub-exponential function $f(k)$, the entire disk can be filled by a sub-exponential number $kf(k)$ of these relators.

Combining this with Theorem A, we obtain the following.

COROLLARY B. Aut (F_3) *and* Out (F_3) *do not admit bounded combings whose length is sub-exponential. In particular, they are neither automatic nor semihyperbolic.*

DEFINITION 2.3. A subgroup $H \subseteq G$ is said to be *quasiconvex* with respect to a bounded bicombing $\sigma\colon G \to (\mathscr{A} \cup \mathscr{A}^{-1})^*$ if there exists a constant $C > 0$ such that for every $h \in H$, the image of the path in G determined by the word σ_h is entirely contained in the C-neighbourhood of H.

Quasiconvex subgroups play an important role in the study of bicombings for two reasons. First, a subgroup which is quasiconvex with respect to a given bicombing σ inherits a bicombing with much the same geometry as that of σ. Secondly, many subgroups, in particular centralizers of finite sets of elements, are quasiconvex with respect to arbitrary bounded bicombings of the ambient group (see [13] and [2]).

LEMMA 2.4. *If G admits a bounded bicombing $\sigma\colon G \to (\mathscr{A} \cup \mathscr{A}^{-1})^*$, and $H \subseteq G$ is quasiconvex with respect to σ, then there is a choice of finite generating set $v\colon \mathscr{B} \to H$ and a bounded bicombing $\tau\colon H \to (\mathscr{B} \cup \mathscr{B}^{-1})^*$ whose length satisfies $L_\tau(n) \leqslant L_\sigma(Kn)$ for some constant $K > 0$.*

Proof. Let the constant C be as in the definition of quasiconvexity. It is straightforward to check that the set $\mathscr{B} = \{h \in H \mid d(1, h) \leq 2C+1\}$ generates H (see [13] or [2]). In other words, the inclusion $\nu: \mathscr{B} \hookrightarrow H$ is a choice of generators for H. To avoid confusion, we write d_H for the word-metric on H associated to this choice of generators, and write d_G for the given word-metric on G. Notice that $d_G(h, h') \leq K d_H(h, h')$ for all $h, h' \in H$, where $K = 2C+1$.

In order to define the desired bicombing $\tau: H \to (\mathscr{B} \cup \mathscr{B}^{-1})^*$, for all $h \in H$ and all positive integers $t \leq |\sigma_h|$ we choose a point $p^h(t) \in H$ which is closest to $\sigma_h(t)$. Such a choice yields a discrete, eventually constant path $p^h: \mathbb{N} \to H$ which begins at the identity, ends at h, and has the property that $d_H(p^h(t), p^h(t+1)) \leq 1$ for all $t \in \mathbb{N}$. As we noted above (since $1 \in \mathscr{B}$), each such path in H is determined by a word $\tau_h \in (\mathscr{B} \cup \mathscr{B}^{-1})^*$. Thus we obtain an assignment of words $h \mapsto \tau_h$ which is a bounded bicombing with $|\tau_h| = |\sigma_h|$ for all $h \in H$. Since $d_G(h, h') \leq K d_H(h, h')$ for all $h, h' \in H$, we have that $L_\tau(n) \leq L_\sigma(Kn)$.

Combining Lemma 2.4 with the fact that the centralizer of an element in a finitely generated group is quasiconvex with respect to an arbitrary bounded bicombing (see [13] or [2]), we obtain the following.

COROLLARY 2.5. *If a finitely generated group G admits a bounded bicombing of sub-exponential length, then the centralizer of every element in G admits a bounded bicombing of sub-exponential length.*

We need one final fact concerning bicombings. Recall that a subgroup $H \subseteq G$ is said to be a *retract* if there is a homomorphism $\pi: G \to H$ such that $\pi|_H = \mathrm{id}_H$.

LEMMA 2.6. *If G admits a bounded bicombing of sub-exponential length, and H is a retract of G, then H admits a bounded bicombing of sub-exponential length.*

Proof. Given any choice of finite generating set $\mu: \mathscr{A} \to G$ for G, if one gives H the induced choice of generators $\pi\mu: \mathscr{A} \to H$, then with respect to the corresponding word-metrics, π is (weakly) distance decreasing and restricts to an isometry on H. Thus if σ is the given bicombing of G, then the assignment $\sigma^H: h \mapsto \sigma_h$ is a bounded bicombing of H with respect to the generators $\pi\mu: \mathscr{A} \to H$, and $L_{\sigma^H} \leq L_\sigma$.

3. *Centralizers and retractions*

It follows from Lemma 2.6 and Corollary 2.5 that the class of finitely generated groups which admit a bounded bicombing of sub-exponential length is closed under passage to retracts and centralizers. In order to apply this to the groups $\mathrm{Aut}\,(F_n)$ and $\mathrm{Out}\,(F_n)$, we consider the centralizer of a particular automorphism of F_n.

We first fix some notation to be used throughout this section. Let $\{x_1, \ldots, x_n\}$ be a basis for F_n, and identify F_{n-1} with the subgroup of F_n generated by $\{x_1, \ldots, x_{n-1}\}$. Let $A \cong \mathrm{Aut}\,(F_{n-1})$ denote the subgroup of $\mathrm{Aut}\,(F_n)$ consisting of automorphisms which fix x_n and send F_{n-1} to itself, and let $F \cong F_{n-1}$ denote the subgroup of $\mathrm{Aut}\,(F_n)$ consisting of automorphisms which fix F_{n-1} and send x_n to $wx_n w^{-1}$, for some $w \in F_{n-1}$. Finally, let I be the automorphism which fixes F_{n-1} and sends x_n to its inverse x_n^{-1}, let $\mathrm{Fix}\,(I) = \{w \in F_n \mid I(w) = w\}$ and set $\mathrm{Flip}\,(I) = \{w \in F_n \mid I(w) = w^{-1}\}$.

LEMMA 3.1.   Fix $(I) = F_{n-1}$ and Flip (I) is equal to the set of reduced words in F_n of the form $I(u)^{-1} x_n^{\pm 1} u$ or $I(u)^{-1} u$, where $u \in F_n$.

Proof.   The fact that Fix$(I) = F_{n-1}$ is clear.

Suppose $w \in$ Flip(I). If w has odd word length, write w as a reduced word $v x_i^{\pm 1} u$, with $|u| = |v|$. Since I preserves word length, the equation $I(w) = w^{-1}$ forces $I(v) = u^{-1}$ and $I(x_i) = x_i^{-1}$, so $v = I(u)^{-1}$ and $i = n$. If w has even word length, write w as a reduced word vu with $|u| = |v|$; again we obtain $v = I(u)^{-1}$. Since any word of this form is in Flip(I), we are done.

PROPOSITION 3.2.   *The centralizer* $C(I)$ *of* I *in* Aut(F_n) *is equal to* $\langle I \rangle \times (F \rtimes A)$. *The natural map from* Aut(F_n) *to* Out(F_n) *maps* $C(I)$ *onto the centralizer of the image of* I *in* Out(F_n).

Proof.   It is easy to check that the subgroups F, A and $\langle I \rangle$ are in the centralizer $C(I)$. These subgroups together generate a subgroup of Aut(F_n) of the form $\mathbb{Z}_2 \times (F \rtimes A)$, which we claim is all of the centralizer.

Let ϕ be any automorphism which commutes with I. Applying ϕI and $I\phi$ to x_i, for $i < n$, we see that $\phi(x_i) = I\phi(x_i)$, that is, $\phi(x_i)$ is in the fixed subgroup F_{n-1} of I. The same computation with x_n shows that $\phi(x_n)$ is in Flip(I). Since $\phi(x_n)$ is the only element of the basis $\{\phi(x_i)\}$ which is not in F_{n-1}, the letter x_n must occur exactly once in the reduced word for $\phi(x_n)$; therefore, by Lemma 3.1, $\phi(x_n) = u^{-1} x_n^{\pm 1} u$ for some $u \in F_{n-1}$. Thus ϕ is in the subgroup $\mathbb{Z}_2 \times (F \rtimes A)$ described above, and the claim is proved.

It remains to show that $C(I)$ maps onto the centralizer of the image of I in Out(F_n), that is, an automorphism which commutes with I up to inner automorphism is an element of $C(I)$ composed with an inner automorphism. Let ϕ be such an automorphism, so $\phi^{-1} I \phi I = \iota_w$, where ι_w denotes the inner automorphism $x \mapsto w^{-1} x w$. Since I has order two, so does $\phi^{-1} I \phi = \iota_w I$. Applying $\iota_w I$ twice to x_i, for $i \leqslant n-1$, we see that $x_i = (\iota_w I)^2 x_i = (I(w) w)^{-1} x_i (I(w) w)$, so $I(w) w$ commutes with x_i for all $i \leqslant n-1$. Since $n-1 \geqslant 2$, this implies that $I(w) w = 1$, that is, $w \in$ Flip(I). Hence, by Lemma 3.1, $w = I(u)^{-1} x_n^{\pm 1} u$ or $w = I(u)^{-1} u$, according to whether w has odd or even length. Let $\psi = \phi \iota_u$. Then

$$\psi^{-1} I \psi I = \iota_u^{-1}(\phi^{-1} I \phi) \iota_u I$$
$$= \iota_{u^{-1}}(\iota_w I) \iota_u I$$
$$= \iota_{u^{-1}} \iota_w \iota_{I(u)}$$
$$= \iota_{(I(u) w u^{-1})}.$$

If w has even length, then $I(u) w u^{-1} = 1$, so the preceding calculation shows that ψ is an element of the centralizer of I, and we are done.

If w has odd length, then $I(u) w u^{-1} = x_n^{\pm 1}$; however, we claim that in fact there is no automorphism ψ such that $\psi^{-1} I \psi I$ is conjugation by x_n (or its inverse). Suppose, to the contrary, that $\psi I = I \psi \iota_{x_n}$. Applying both sides to x_n, we see that $\psi(x_n) \in$ Flip(I), so that $\psi(x_n) = I(u)^{-1} x_n u$ or $\psi(x_n) = I(u)^{-1} u$ for some $u \in F_n$. Since $\psi(x_n)$ is primitive, the vector of its exponent sums is unimodular in $\mathbb{Z}^n$; since the exponent sum of x_i is zero for $i < n$, the exponent sum of x_n must be ± 1. We may assume that the exponent sum is 1, replacing x_n by x_n^{-1} if necessary, so $\psi(x_n) = I(u)^{-1} x_n u$.

Applying $I\psi I = \psi\iota_{x_n}$ to $(x_1)^m$ for any m, we obtain

$$I\psi(x_1)^m = \psi(x_n)^{-1}\psi(x_1)^m\psi(x_n)$$
$$= u^{-1}x_n^{-1}I(u)\psi(x_1)^m I(u)^{-1}x_n u.$$

Choose m large enough so that $\psi(x_1)^m$ has word length longer than $\psi(x_n)$. Since I preserves word length, there must be exactly $|\psi(x_n)|$ cancellations on the right-hand side of the equation. Therefore exactly one of the x_n in the above expression cancels. If only the second one cancels, then $\psi(x_1)^m$ cannot begin with the reduced string $I(u)^{-1}x_n$. Now apply I to both sides of the equation to see that $\psi(x_1)^m$ does begin with the reduced string $I(u)^{-1}x_n$, giving the necessary contradiction. If only the first x_n cancels, we obtain a similar contradiction.

The following proposition completes the proof of Theorem C, since we already know (Corollary B) that $\text{Aut}(F_3)$ and $\text{Out}(F_3)$ are not in the class of groups which admit bounded bicombings of sub-exponential length.

PROPOSITION 3.3. *Let $\mathscr{G}$ be a class of groups which is closed under passage to retracts and centralizers. If for some $n \geqslant 3$, $\text{Aut}(F_n) \in \mathscr{G}$ or $\text{Out}(F_n) \in \mathscr{G}$, then $\text{Aut}(F_m) \in \mathscr{G}$ for all $m < n$.*

Proof. The case $\text{Aut}(F_n) \in \mathscr{G}$ follows from Proposition 3.2, since $A \cong \text{Aut}(F_{n-1})$ is clearly a retract of the centralizer $C(I) \cong \mathbb{Z}_2 \times (F \rtimes A)$.

We now consider the case $\text{Out}(F_n) \in \mathscr{G}$. Let $\pi: \text{Aut}(F_n) \to \text{Out}(F_n)$ be the natural map. By Proposition 3.2, the centralizer $C(\pi(I))$ is equal to $\pi(C(I))$. The projection π sends the subgroup A isomorphically onto its image $\pi(A)$. Define a retraction $f: \pi(C(I)) \to \pi(A)$ by $f(\pi(t, w, \phi)) = \pi(\iota_w \phi)$. This map is well-defined, since $\pi(t, w, \phi) = \pi(s, u, \psi)$ if and only if $s = t$ and, for some $v \in F_{n-1}$, $w = uv$ and $\phi = \iota_v \psi$. Thus $\pi(A) \cong \text{Aut}(F_{n-1})$ is a retract of $\pi(C(I)) = C(\pi(I))$, concluding the proof of the proposition.

Finally, we remark that the techniques in this paper also apply to the group $\text{GL}_n(\mathbb{Z})$. Let J be the automorphism of $\mathbb{Z}^n$ which fixes the first $n-1$ basis vectors $e_1, \ldots, e_{n-1}$ and sends e_n to $-e_n$. The centralizer $C(J)$ of J in $\text{GL}_n(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2 \times \text{GL}_{n-1}(\mathbb{Z})$. Since $\text{GL}_3(\mathbb{Z})$ satisfies a strictly exponential isoperimetric inequality, we have the following.

PROPOSITION 3.4. *$\text{GL}_n(\mathbb{Z})$ does not admit a bounded bicombing of sub-exponential length for any $n \geqslant 3$.*

References

1. J. M. ALONSO, 'Inégalités isopérimétriques et quasi-isométries', *C. R. Acad. Sci. Paris Sér. I* 311 (1990) 761–764.
2. J. M. ALONSO and M. R. BRIDSON, 'Semihyperbolic groups', *Proc. London Math. Soc.* 70 (1995) 56–114.
3. M. R. BRIDSON, 'On the geometry of normal forms in discrete groups', *Proc. London Math. Soc.* 67 (1993) 596–616.
4. M. R. BRIDSON and A. HAEFLIGER, 'Metric spaces of non-positive curvature', Preprint.
5. M. DEHN, *Papers on group theory and topology* (translated and introduced by John Stillwell, Springer, Berlin, 1987).
6. D. B. A. EPSTEIN, J. W. CANNON, D. F. HOLT, S. V. F. LEVY, M. S. PATERSON and W. P. THURSTON, *Word processing in groups* (Bartlett and Jones, Boston, 1992).

7. S. M. GERSTEN, 'Bounded cohomology and combings of groups', Preprint, version 5.5, University of Utah.
8. S. M. GERSTEN, 'The automorphism group of a free group is not a $CAT(0)$ group', *Proc. Amer. Math. Soc.* 121 (1994) 999–1002.
9. M. GROMOV, 'Hyperbolic groups', *Essays in group theory*, Math. Sci. Res. Inst. Publ. 8 (ed. S. M. Gersten, Springer, New York, 1987).
10. A. HATCHER and K. VOGTMANN, 'Isoperimetric inequalities for automorphism groups of free groups', *Pacific J. Math.*, to appear.
11. M. KAPOVICH and B. LEEB, 'Metrics of nonpositive curvature on graph-manifolds', Preprint.
12. L. MOSHER, 'Mapping class groups are automatic', *Ann. of Math.*, to appear.
13. H. SHORT, 'Groups and combings', Preprint, ENS Lyons, 1990.

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