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END INVARIANTS OF THE GROUP OF OUTER AUTOMORPHISMS OF A FREE GROUP

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1. INTRODUCTION

FREUDENTHAL [4] and Hopf [5] defined the notion of an *end* of a group and proved that every finitely generated group has 0, 1, 2 or infinitely many ends. If the group is finitely presented, the number of ends is equal to the number of ends of any simply connected complex on which the group acts freely with finite quotient. When a group acts in this way there other geometric and topological invariants of the complex which are in fact invariants of the group. In particular, if the ends of the complex are simply connected the group is said to be *simply connected at infinity*; this property is independent of the choice of complex. In this paper we consider the group $Out(F_n)$ of outer automorphisms of a free group of rank n . This group has infinitely many ends for $n = 2$ and one end for $n \geq 3$ (see Section 3 below). We prove the following theorem.

THEOREM 1.1. *$Out(F_n)$ is simply-connected at infinity for $n \geq 5$.*

The natural map from $Out(F_n)$ to the general linear group $GL(n, \mathbf{Z})$ is surjective, and Magnus [8] showed that the kernel Ker_n of this map is finitely generated. The group $GL(n, \mathbf{Z})$ is known to be simply connected at infinity for $n \geq 3$. If Ker_n were finitely presented, then the fact that $Out(F_n)$ is finitely presented would imply that $Out(F_n)$ is simply connected at infinity [6]; however, it is unknown whether Ker_n is finitely presentable for any $n > 2$. There is some evidence that Ker_n may not be finitely presentable, at least for $n = 3$ [9].

Our interest in this question was originally motivated by the question of whether $Out(F_n)$ is a virtual duality group, i.e. whether there is a module D and a duality isomorphism between the cohomology of $Out(F_n)$ and the homology with coefficients in D in a complementary dimension. The property of being a virtual duality group is shared by various classes of groups closely related to $Out(F_n)$, for example $GL(n, \mathbf{Z})$ and mapping class groups of surfaces. If $Out(F_n)$ is $(2n - 5)$ -connected at infinity, it could be shown that $Out(F_n)$ is a virtual duality group; in particular, if $Out(F_3)$ is simply connected at infinity, then it is a virtual duality group. It is intriguing that the methods of this paper give only 0-connectivity for $n = 3$.

The theorem is proved by considering the “outer space” X_n of homothety classes of free minimal actions of F_n on simplicial \mathbf{R} -trees, as defined in [2]. The space X_n is contractible, and $Out(F_n)$ acts on X_n with finite stabilizers. Furthermore, X_n has a simplicial spine K_n ,

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which is an equivariant deformation retract on which $Out(F_n)$ acts with finite quotient. Any torsion-free subgroup of finite index in $Out(F_n)$ therefore acts freely on K_n with finite quotient. Since the end invariants of a group are the same as the end invariants of any subgroup of finite index, we can use this spine K_n to compute the end invariants of $Out(F_n)$.

2. BASIC DEFINITIONS

For the convenience of the reader, we recall briefly constructions of [2] which we need for this paper; further explanations and proofs can be found in [2].

We first recall the definition of the simplicial complex K_n . The vertices of K_n are free minimal actions of F_n on simplicial trees (at the vertices of K_n , all edges have equal length, so we ignore the metrics on the trees). It is convenient to think of an action on a tree in terms of its quotient graph Γ , which comes equipped with an identification of $\pi_1(\Gamma)$ with F_n (a “marking”). More precisely, we fix a graph R_n with one vertex and n edges, and identify F_n with $\pi_1(R_n)$. A vertex of K_n is an equivalence class of marked graphs (g, Γ) , where Γ is a graph with no free edges, no bivalent vertices and no separating edges, $g: R_n \rightarrow \Gamma$ is a homotopy equivalence, and two marked graphs (g, Γ) and (g', Γ') are equivalent if there is a homeomorphism $h: \Gamma \rightarrow \Gamma'$ with $h \circ g$ homotopic to g' . Two vertices (g, Γ) and (g', Γ') are connected by an edge if a marked graph equivalent to (g, Γ) can be obtained from (g', Γ') by collapsing each component of a forest in Γ' to a point. A k -simplex is a chain of k forest collapses.

A marked graph (g, Γ) is a *rose* if Γ has only one vertex. For any finite set W of elements of F_n , we define a norm on roses by setting

$$\|(g, \Gamma)\| = \sum_{w \in W} l(w)$$

where $l(w)$ is the hyperbolic length of w in the action determined by (g, Γ) (see e.g. [1] for background material on hyperbolic length functions of actions on trees).

The complex K_n is the union of contractible “balls”, where the ball of radius k is the union of all stars of roses of norm at most k [2, Section 6]. For judicious choice of W , the balls of radius k are compact [7, Proof of Proposition 6.3]. For the rest of this paper, we fix a set of words W which contains a basis and such that balls of radius k are compact.

Let (g, Γ) be a marked graph. The *star graph* of (g, Γ) with respect to W is defined as follows. For each $w \in W$, represent the conjugacy class of $g_*(w)$ by a cyclically reduced closed edge-path in Γ . The star graph has one vertex for each oriented edge of Γ , and one edge from e to f for each occurrence of $e\bar{f}$ in each edge-path representing an element of W .

The norm of (g, G) with respect to W can be computed using the star graph of (g, G) with respect to W : the norm $\|(g, \Gamma)\|$ is one-half of the sum of the valences of the vertices of the star graph.

Let $\rho = (r, R)$ be a rose in K_n . An *ideal edge* of ρ is a partition of the oriented edges of R into two subsets, each with at least two elements, such that some oriented edge is separated from its opposite. The subsets determined by an ideal edge are the *components* of the ideal edge. We will often specify an ideal edge by naming the elements of one component.

Two ideal edges α and β are said to *cross* if each of the components of α has nonempty intersection with each of the components of β ; otherwise they are *compatible*. A set of distinct, pairwise-compatible ideal edges is called an *ideal tree*.

Given an ideal tree $\{\alpha_1, \dots, \alpha_k\}$ in a rose $\rho = (r, R)$, we obtain a new marked graph with $k + 1$ vertices, denoted $\rho^{\alpha_1, \dots, \alpha_k}$ by *blowing up* the ideal tree (see [2, Section 2.2]). The new

marked graph has $k + n$ edges, one for each edge of R and one for each ideal edge α_i . The edges $\{\alpha_1, \dots, \alpha_k\}$ form a maximal tree in $\rho^{\alpha_1, \dots, \alpha_k}$, which can be collapsed to recover ρ .

If an ideal edge α of ρ separates an oriented edge e of R from its opposite \bar{e} , then e corresponds to an edge of ρ^α which has distinct endpoints. This edge can be collapsed to obtain a new rose, denoted ρ_e^α . The rose ρ_e^α is said to be obtained from ρ by the *elementary Whitehead move* (α, e) . Elementary Whitehead moves correspond to Whitehead automorphisms of the free group. The roses ρ and ρ_e^α are connected by an edge-path of length two in K_n , called an *elementary Whitehead path*. We will denote edge-paths in K_n by listing their vertices; the elementary Whitehead path from ρ to ρ_e^α is then $(\rho, \rho^\alpha, \rho_e^\alpha)$.

Definition 2.1. Let S and T be subsets of the set of vertices of a graph. The *dot product* $S \cdot T$ is the number of (unoriented) edges with one vertex in S and one vertex in T . The *absolute value* $|S|$ of S is the dot product of S with its complement.

Let $\rho = (r, R)$ be a rose. Each component of an ideal edge α of ρ can be thought of as a set of oriented edges of R or as a set of vertices of the star graph of ρ with respect to W . We denote by $|\alpha|$ the absolute value of (either) component considered as a set of vertices in the star graph. A single oriented edge e of R corresponds to a single vertex of the star graph. Since the star graph has no loops, the absolute value $|e| = |\{e\}|$ is simply the valence of e as a vertex of the star graph.

The following lemma is a formal consequence of the definitions.

LEMMA 2.2. *Let ρ be a rose. If W contains a basis, then the absolute value of any edge or any ideal edge of ρ is nonzero.*

Proof. There is an edge path in the star graph from e to \bar{e} for every edge e of R , since W contains a basis. This edge path must cross an ideal edge which separates e from \bar{e} . □

Let ρ be a rose, $\{\alpha_1, \dots, \alpha_k\}$ an ideal tree in ρ , and $\{e_1, \dots, e_k\}$ a set of edges in ρ which forms a forest in $\rho^{\alpha_1, \dots, \alpha_k}$. The following relationship between norms is included in Proposition 3.3.1 of [2], and is central to this paper:

$$\|\rho_{e_1, \dots, e_k}^{\alpha_1, \dots, \alpha_k}\| = \|\rho\| + \sum_{i=1}^k |\alpha_i| - \sum_{i=1}^k |e_i|. \tag{*}$$

A Whitehead move (α, e) is *reductive* if $\|\rho_e^\alpha\| \leq \|\rho\|$, or, equivalently, if $|\alpha| \leq |e|$. It is *strictly reductive* if $\|\rho_e^\alpha\| < \|\rho\|$.

3. CONNECTIVITY AT INFINITY

There are several ways to see that K_n is connected at infinity for $n \geq 3$, i.e. that $Out(F_n)$ has one end. For example, one can show that $Out(F_n)$ has Serre's property FA for $n \geq 3$ (see, e.g. [3]), i.e. any action of $Out(F_n)$ on a tree has a fixed point, and therefore $Out(F_n)$ has one end by Stallings' theorem [10]. We will present a direct proof that K_n is connected at infinity for $n \geq 3$, to motivate and set up the proof that K_n is simply connected at infinity for $n \geq 5$.

For each k , the ball B_k of radius k is the union of a finite number of stars of roses. Since K_n is locally finite, and the star of a rose is finite, there are only a finite number of roses

whose stars have nonempty intersection with B_k . We call the union of all these C_k , and define $N(k)$ to be the maximum of the norms of the roses in C_k .

To show connectivity at infinity, we show that for any k , any two roses ρ and ρ' which have norm at least $N(k)$ can be connected by path lying outside the ball of radius k ; to do this, we start with an arbitrary path between ρ and ρ' , and push it outside the ball of radius k by a homotopy.

A path $P = (\gamma_0, \gamma_1, \dots, \gamma_l)$ will be called *standard* if P has even length $l = 2k$, if γ_{2i} is a rose for all $0 \leq i \leq k$, and if $(\gamma_{2i}, \gamma_{2i+1}, \gamma_{2i+2})$ is an elementary Whitehead path for all $0 \leq i \leq k - 1$.

The fact that $Out(F_n)$ is generated by Whitehead automorphisms implies that there is a standard path $P = (\gamma_0, \gamma_1, \dots, \gamma_{2k})$ from ρ to ρ' .

In order to push P towards infinity, we start at a rose $\rho = \gamma_{2i}$ of minimal norm m along P . The preceding rose γ_{2i-2} is obtained from ρ by an elementary Whitehead move (β, f) , and the succeeding rose γ_{2i+2} is obtained from ρ by an elementary Whitehead move (α, e) . We may assume that ρ_e^α has norm strictly greater than m , i.e. the elementary Whitehead move (α, e) strictly increases norm ($|\alpha| > |e|$), and that (β, f) does not decrease norm ($|\beta| \geq |f|$).

We will replace the segment $(\gamma_{2i-2}, \dots, \gamma_{2i+2}) = (\rho_f^\beta, \rho^\beta, \rho, \rho^\alpha, \rho_e^\alpha)$ of the path P by a new standard path from ρ_f^β to ρ_e^α which passes only through roses of norm strictly greater than m . (The process is the opposite of the ‘‘peak reduction’’ process of Higgins and Lyndon.) Repeating this, we can eliminate all roses of minimal norm m from P , and eventually arrange that all roses on the path have norm at least $N(k)$. This guarantees that P lies outside B_k .

LEMMA 3.1. *Let α and β be compatible ideal edges of ρ , such that the elementary Whitehead move (α, e) strictly increases norm, and (β, f) does not decrease norm. Then there is a standard path from ρ_e^α to ρ_f^β with at most one rose in its interior; this rose has norm strictly greater than the norm of ρ .*

Proof. Let $\rho^{\alpha, \beta} = (g, \Gamma)$, and let a and b be the edges of Γ corresponding to α and β , respectively. Our standard path will lie in the link of (g, Γ) and we will denote its vertices by specifying the edges of Γ which must be collapsed to obtain the vertex; for example, $\rho_e^\alpha = \{b, e\}$. Our path will be of the form

$$(T_1, T_1 \cap T, T, T_2 \cap T, T_2)$$

where T_1, T_2 and T are maximal trees in Γ .

If $\{e, f\}$ is a tree in Γ , we may take $T_1 = \{b, e\}$, $T_2 = \{a, f\}$ and $T = \{e, f\}$. Since $|\alpha| > |e|$ and $|\beta| \geq |f|$, the rose obtained by collapsing T has norm strictly greater than the norm of ρ . If $e = f$, this degenerates to the elementary Whitehead path $(\rho_e^\alpha, \rho_e^{\alpha, \beta}, \rho_e^\beta)$.

Now suppose $\{e, f\}$ is not a tree in Γ . This happens, for example, if $\alpha = \beta$; then a, e and f all have the same endpoints. In this case we take $T_1 = \{e\}$, $T_2 = \{f\}$ and $T = \emptyset$, so that the path degenerates to $(\rho_e^\alpha, \rho^\alpha, \rho_f^\alpha)$. If $\alpha \neq \beta$, we set $T_1 = \{b, e\}$ and $T_2 = \{a, f\}$. Note that both $\{a, e\}$ and $\{b, f\}$ are trees. If $|e| > |f|$, we take $T = \{b, f\}$, and if $|f| \geq |e|$, we take $T = \{a, e\}$ to insure that the rose obtained by collapsing T has norm strictly greater than the norm of ρ . \square

The proof of Lemma 3.1 gives a simplicial map of a subdivided square into K_n which maps the center to $\Gamma = \rho^{\alpha, \beta}$ and the corners to $\rho, \rho_e^\alpha, \rho_T^{\alpha, \beta}$ and ρ_f^β . If $\alpha \neq \beta$ and $e \neq f$, then

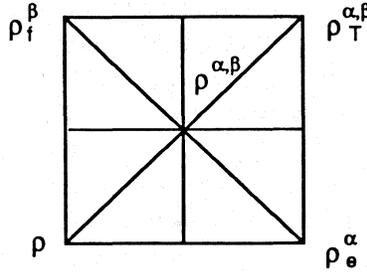


Fig. 1

$\rho_T^{\alpha,\beta}$ is a rose, which has norm strictly greater than the norm of ρ . This square gives a homotopy rel endpoints between the old path and the new one (Fig. 1).

We will now prove several lemmas which find Whitehead moves which strictly increase the norm. Each uses the following lemma applied to a star graph.

LEMMA 3.2. *Let u, v and w be three vertices of a graph, and S the set of all other vertices. If $|\{u, v\}| \leq |v|$ and $|\{u, w\}| \leq |w|$, then $\{u\} \cdot \{v\} = \{u\} \cdot \{w\}$ and $\{u\} \cdot S = 0$.*

Proof. We have $|\{u, v\}| = \{u\} \cdot S + \{u\} \cdot \{w\} + \{v\} \cdot \{w\} + \{v\} \cdot S$ and $|v| = \{v\} \cdot S + \{v\} \cdot \{u\} + \{v\} \cdot \{w\}$. The inequality $|\{u, v\}| \leq |v|$ gives $\{u\} \cdot S + \{u\} \cdot \{w\} \leq \{u\} \cdot \{v\}$. Similarly, the inequality $|\{u, w\}| \leq |w|$ gives $\{u\} \cdot S + \{u\} \cdot \{v\} \leq \{u\} \cdot \{w\}$; together these imply $\{u\} \cdot S = 0$ and $\{u\} \cdot \{v\} = \{u\} \cdot \{w\}$. □

We define a *trio* of edges of a rose $\rho = (r, R)$ to be a set $\{e, f, g\}$ of three oriented edges of R which does not contain any pair $\{x, \bar{x}\}$, where \bar{x} denotes the image of x under the orientation involution. A trio has the property that each $(\{x, y\}, x)$, for $x, y \in \{e, f, g\}$, is an elementary Whitehead move.

LEMMA 3.3. *Let $\{e, f, g\}$ be a trio. Then at least one of the Whitehead moves $(\{x, y\}, x)$, for $x, y \in \{e, f, g\}$ strictly increases norm.*

Proof. Consider the portion of the star graph of W with vertices e, f and g . By applying Lemma 3.2 to e, f and g with e, f and g successively in the role of u , we find that there is an isolated component of the star graph with vertices e, f, g . But then the ideal edge $\{e, f, g\}$ intersects no edges of the star graph, i.e. has absolute value zero, contradicting Lemma 2.2. □

We define the *size* of an ideal edge to be the number of elements in the smaller of the two components of the ideal edge. The *size* of a Whitehead move is the size of its ideal edge.

PROPOSITION 3.4. *If $n \geq 4$, and α is an ideal edge, then there is a Whitehead move of size two compatible with α which strictly increases norm.*

Proof. If $n \geq 4$, then one of the components of α has at least 4 elements. This component must contain a trio, since otherwise it would not determine an ideal edge. Then we are done by Lemma 3.3. □

COROLLARY 3.5. *If $n \geq 4$, there is a standard path from ρ_e^α to ρ_f^β which contains only roses of norm greater than the norm of ρ in its interior.*

Proof. By Proposition 3.4, we can find strictly increasing Whitehead moves (α_1, e_1) compatible with α and (β_1, f_1) compatible with β , which have size two. By Lemma 3.1, we can connect ρ_e^α to $\rho_{e_1}^{\alpha_1}$ and $\rho_{f_1}^{\beta_1}$ to ρ_f^β by appropriate paths. If α_1 and β_1 are compatible, we are done by Lemma 3.1. If they cross, there are at least five edges which are not in the two-element components of α_1 and β_1 , and hence there is a trio compatible with both α_1 and β_1 . Thus we can find a strictly increasing elementary Whitehead move (γ, g) such that γ is compatible with both α_1 and β_1 . By Lemma 3.1, we can then connect $\rho_{e_1}^{\alpha_1}$ to ρ_g^γ and ρ_g^γ to $\rho_{f_1}^{\beta_1}$ by appropriate paths. The concatenation of all these paths gives a standard path from ρ_e^α to ρ_f^β with the required property. \square

This completes the essential step in the proof of connectivity for $n \geq 4$. We now consider the remaining case $n = 3$. We need slightly more subtle ways of finding strictly increasing Whitehead moves of size two. The following three lemmas accomplish this.

We define a *quartet* to be a set of ideal edges of the form $\{e, \bar{e}, f, \bar{f}\}$.

LEMMA 3.6. *Let $\{e, \bar{e}, f, \bar{f}\}$ be a quartet. Then at least one of the Whitehead moves involving $\{e, f\}$, $\{\bar{e}, \bar{f}\}$, $\{e, \bar{f}\}$ or $\{\bar{e}, f\}$ is strictly increasing.*

Proof. Assume all of the Whitehead moves in the statement of the proposition are reductive.

Apply Lemma 3.2 with $u = e, v = f, w = f$; with $u = \bar{e}, v = f, w = \bar{f}$; with $u = f, v = e, w = \bar{e}$; and with $u = \bar{f}, v = e, w = \bar{e}$. We find that the vertices e, f, \bar{e} and \bar{f} span an isolated component of the star graph, with $\{e\} \cdot \{\bar{e}\} = \{f\} \cdot \{\bar{f}\} = 0$ (Fig. 2).

Thus there is a subset $W_{e,f}$ of W of words which involve only e and f , and no other word in W involves either e or f . Since W contains a basis, there must be a basis for $F_2 = \langle e, f \rangle$ in $W_{e,f}$. But every word in $W_{e,f}$ must be of the form $e^{\epsilon_1} f^{\epsilon_2} \dots f^{\epsilon_{2k}}$ with $\epsilon_i = \pm 1$ since neither e and \bar{e} nor f and \bar{f} are connected in the star graph. If we abelianize and look mod 2, every word in $W_{e,f}$ is either $(0, 0)$ or $(1, 1)$; since these do not form a basis of \mathbf{Z}_2^2 , no subset of words in $W_{e,f}$ is a basis for F_2 . \square

LEMMA 3.7. *Let $n = 3$, and suppose the Whitehead move (α, e) increases norm. Then there is a strictly increasing Whitehead move of size two compatible with α .*

Proof. The ideal edge α is compatible with a trio, and hence with a strictly increasing Whitehead move, unless α is of the form $\{e, f, \bar{f}\}$.

Now suppose $\alpha = \{e, f, \bar{f}\}$, and assume that all size two Whitehead moves compatible with α are reductive. Let $S = \{\bar{e}, g, \bar{g}\}$ be the complement of $\{e, f, \bar{f}\}$. Applying Lemma 3.2 with $u = e, v = f$ and $w = \bar{f}$, we find that $\{e\} \cdot S = 0$ and $\{e\} \cdot \{f\} = \{e\} \cdot \{\bar{f}\}$. Since

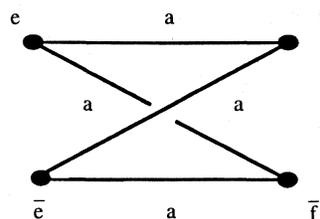


Fig. 2

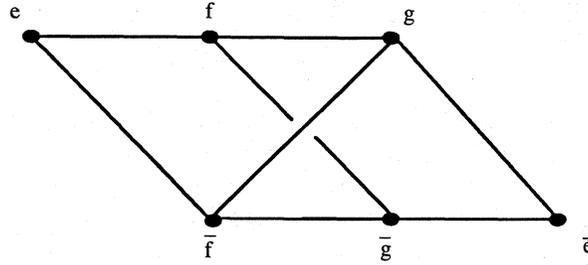


Fig. 3

$|f| = |\bar{f}|$, we then have $\{f\} \cdot S = \{\bar{f}\} \cdot S$. Since α is increasing, $|\{e, f, \bar{f}\}| \geq |e|$, which implies $\{f\} \cdot S \geq \{e\} \cdot \{f\}$. By assumption, $|\{e, f\}| \leq |e|$ and $|\{e, \bar{f}\}| \leq |e|$; together these imply $\{f\} \cdot \{\bar{f}\} = 0$. The same argument for $\{\bar{e}, g, \bar{g}\}$ shows $\{\bar{e}\} \cdot \{e, f, \bar{f}\} = 0$ and $\{g\} \cdot \{\bar{g}\} = 0$ (see Fig. 3).

Analysis of this graph shows that every cyclic word represented by the graph has to involve an equal number of f 's and g 's. If we abelianize and look mod 2, all words must be of the form $(0, 0, 0), (0, 1, 1), (1, 0, 0)$ or $(1, 1, 1)$. But no subset of these vectors forms a basis for \mathbb{Z}_2^3 , contradicting the fact that W contains a basis for F_3 . \square

PROPOSITION 3.8. *If $n = 3$, there is a standard path from ρ_e^α to ρ_f^β which contains only roses of norm greater than the norm of ρ in its interior.*

Proof. By applying Lemma 3.7, we may assume α and β have size two. If α and β are compatible, we are done by Lemma 3.1. If they cross and there is a trio compatible with both of them, we are done by Lemmas 3.3 and 3.1. The only case left to consider is if α is of the form $\{e, f\}$ and β is of the form $\{e, \bar{f}\}$. By Lemma 3.6, at least one Whitehead move involving $\{f, g\}, \{f, \bar{g}\}, \{\bar{f}, g\}$, or $\{\bar{f}, \bar{g}\}$ is strictly increasing; call it (γ, h) . Then γ is compatible with one of α or β , and there is a trio compatible with both γ and the other one of α or β , so we can connect ρ_e^α to ρ_h^γ and ρ_h^γ to ρ_e^β by paths whose concatenation gives a path containing only roses of norm greater than the norm of ρ . \square

We summarize the argument in the following theorem.

THEOREM 3.9. *If $n \geq 3$, K_n is connected at infinity.*

Proof. Fix k , and let $N = N(k)$. Let $P = (\gamma_0, \dots, \gamma_{2r})$ be a standard path joining roses γ_0 and γ_{2r} of norm at least N . Choose a rose γ_{2i} of minimal norm m on P , such that at least one of the roses γ_{2i-2} or γ_{2i+2} has norm strictly greater than m . By Corollary 3.5 and Proposition 3.8, we can find a standard path from γ_{2i-2} to γ_{2i+2} which contains only roses of norm strictly greater than m in its interior. Replace the segment $(\gamma_{2i-2}, \dots, \gamma_{2i+2})$ of P by this new path. Repeat until all roses of norm m have been eliminated from P , and then continue until all roses on the path have norm at least N ; the resulting path lies outside the ball of radius k , showing that K_n is connected at infinity. \square

4. SIMPLE CONNECTIVITY AT INFINITY

We show K_n is simply connected at infinity by showing that for any k , any loop which lies outside the ball of radius $N = N(k)$ bounds a disk which lies outside the ball of radius k . The following proposition shows that we may assume our loops are standard paths.

PROPOSITION 4.1. *Every path in $K_n - B_N$ is homotopic to a standard path in $K_n - B_N$ by a homotopy in $K_n - B_N$.*

Proof. Note that if γ is a marked graph which lies outside B_N , then every marked graph that can be obtained from γ by a forest collapse is also outside B_N . Fix a path $P = (\gamma_0, \dots, \gamma_k)$ in $K_n - B_N$. By collapsing maximal trees whenever possible, we may assume that γ_i is a rose for all even i or for all odd i . Let γ_{s-1} and γ_{s+1} be two successive roses in this path; they are obtained from the intervening vertex $\gamma_s = (g, \Gamma)$ by collapsing maximal trees F and F' of Γ , respectively.

CLAIM. *Let $\{e_1, \dots, e_k\}$ be the edges of F . There is a bijection between the edges of F and the edges of F' sending e_i to $e'_i \in F'$ so that the set $F_m = \{e'_1, \dots, e'_m, e_{m+1}, \dots, e_k\}$ is a maximal tree for all $0 \leq m \leq k$.*

Proof. Note that $F_0 = F$ is a maximal tree. We suppose that F_i is a maximal tree for all $i < m$, and show how to define e'_m so that F_m is also a maximal tree. Consider the geodesic P in F' which connects the endpoints v and w of e_m . Now $F_{m-1} - e_m$ has two components, F_v and F_w , which are connected by P . Define e'_m to be an edge of P with one vertex in F_v and one vertex in F_w . Then $F_m = F_v \cup F_w \cup e'_m$ is a maximal tree. □

Proof of Proposition 4.1 (continued). The path segment $(\gamma_{s-1}, \gamma_s, \gamma_{s+1})$ is homotopic to the standard path segment

$$(\gamma_{s-1} = \rho_0, \delta_1, \rho_1, \dots, \delta_k, \rho_k = \gamma_{s+1})$$

where ρ_i is obtained from γ_s by collapsing F_i , and δ_i is obtained from γ by collapsing $F_{i-1} \cap F_i$ □

We will contract our standard loop by a standard disk, which we now define.

Definition. A *standard 2-cell* is a simplicial map of a subdivided square or triangle (Fig. 4) into K_n , such that the corners are mapped to roses, the edges are elementary Whitehead paths, and the center is mapped to a marked graph (g, Γ) with at most three vertices. If Γ has only two vertices, the standard 2-cell is *degenerate*.

Let $(D, \{C\})$ be a triangulated disk which is the union of finitely many subdivided squares and triangles C with disjoint interiors. The *C-star* of a vertex v in D is the union of the squares and triangles C which contain v . If v is an interior vertex of D , the *C-link* of v is the boundary of the *C-star*. If v is on the boundary of D , the *C-link* of v is the closure of the boundary of the *C-star* minus the boundary of D . A simplicial map $p : D \rightarrow K_n$ is a *standard*

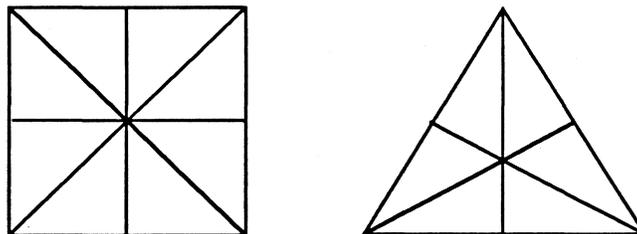


Fig. 4

disk if the restriction of p to each square or triangle C of D is a standard 2-cell. When there is no risk of ambiguity, we sometimes abuse notation by referring to standard disks by their domains. The proof of the Higgins–Lyndon lemma (Proposition 4.2.1 in [2]), shows that any standard loop in K_n bounds a standard disk.

Let $p: D \rightarrow K_n$ be a standard disk, and let v be a vertex of D which maps to a rose ρ . The vertices on the (simplicial) link of v map to marked graphs which are obtained from ρ by blowing up ideal edges $\alpha_1, \dots, \alpha_k$ and ideal trees $\{\alpha_i, \alpha_{i+1}\}$. If v is on the boundary of D , the sequence of ideal edges $\alpha_1, \dots, \alpha_k$ is the *ideal link* of v . If v is in the interior of D , the ideal link is a cycle of ideal edges, and we call k the *circumference* of the ideal link. We define the *cell-star* of v , denoted $CST(v)$, to be the restriction of p to the C -star of v , and the *cell-link* of v , denoted $CLK(v)$ to be the restriction of p to the C -link of v . Note that the cell-link of v is a standard path.

We now fix a standard loop σ which lies outside the ball of radius $N(k)$, and choose a standard disk $p: D \rightarrow K_n$ with boundary σ as above. Let v be a vertex in the interior of D which maps to a rose ρ of minimal norm, say $\|\rho\| = m$. If $m > N = N(k)$ the image of the entire disk lies outside the ball of radius k and we are done. Otherwise, we will show that the cell-link $CLK(v)$ bounds a standard disk whose image contains only roses of norm strictly greater than m in its interior. We replace $CST(v)$ by this new standard disk; in this way, we eventually eliminate all roses of norm m from the image of the disk. We continue until all roses in the image of the standard disk have norm at least $N(k)$.

The following lemma is a three-dimensional analog of Lemma 3.1.

LEMMA 4.2. *Let $p: C \rightarrow K_n$ be a standard 2-cell in $CST(v)$, with three corners mapping to ρ, ρ_α^e and ρ_β^f . Let (γ, g) be a strictly increasing size two Whitehead move such that γ is compatible with both α and β . If (α, e) and (β, f) are both increasing, then there is a standard disk $q: D(C) \rightarrow K_n$ with boundary equal to the boundary of C such that*

- (i) *if w is an interior vertex of $D(C)$ with $q(w)$ a rose, then $q(w)$ has norm strictly greater than the norm of $p(v) = \rho$, and*
- (ii) *the ideal link of v in $D(C)$ is equal to α, γ, β .*

Proof. Let $(g, \Gamma) = \rho^{\alpha, \beta, \gamma}$, and suppose edges a, b and c of Γ correspond to α, β and γ , respectively. All of our standard 2-cells will map to the link of (g, Γ) in K_n , and each vertex w will be labelled by the edges of Γ which must be collapsed to obtain the marked graph $p(w)$. For instance, if $p(w) = \rho_g^y$, then w has the label $\{a, b, g\}$. Note that, since γ is size two, the edge c is a free edge in the tree $\{a, b, c\}$.

We first assume that C is a subdivided square, α, β and γ are distinct, $e \neq f$ and $\{e, f\}$ is a tree in $\rho^{\alpha, \beta}$. The roses on the boundary of C then correspond to maximal trees $\{a, b, c\}, \{a, f, c\}, \{e, b, c\}$ and $\{e, f, c\}$. In most cases of the proof, we will be able to find trees TA, TB and TX so that Fig. 5 determines a standard disk with the required properties.

The trees TA and TB are determined by Lemma 3.1: if $\{a, f, g\}$ is a tree in Γ , then $TA = \{a, f, g\}$. If $\{a, f, g\}$ is not a tree and $|f| > |g|$, we take $TA = \{a, c, g\}$. If $|g| \geq |f|$, we take $TA = \{a, b, f\}$. Similarly, $TB = \{a, b, g\}$ if $\{a, f, g\}$ is a tree; otherwise $TB = \{a, f, g\}$ when $|g| \geq |e|$ and $\{a, f, g\}$ when $|e| > |g|$.

We say two edges of Γ are *parallel* if they have the same endpoints. If g is parallel to c , then $TA = \{a, f, g\}, TB = \{e, b, g\}$ and we can take $TX = \{e, f, g\}$. If e is parallel to a , then $TB = \{e, b, g\}$. If $TA = \{a, x, y\}$, we can set $TX = \{e, x, y\}$. If f is parallel to b , then $TA = \{a, f, g\}$. If $TB = \{x, b, y\}$, we can set $TX = \{x, f, y\}$.

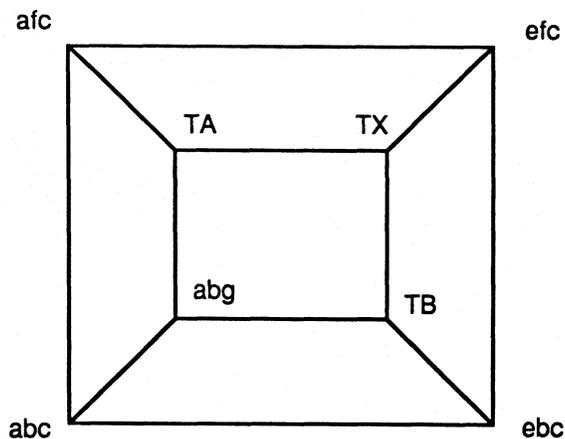


Fig. 5

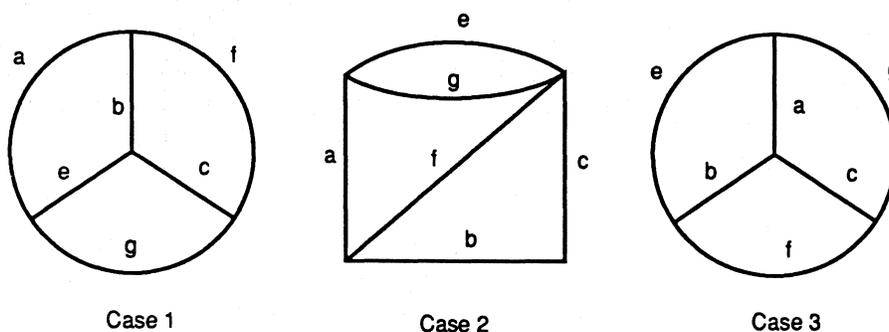


Fig. 6

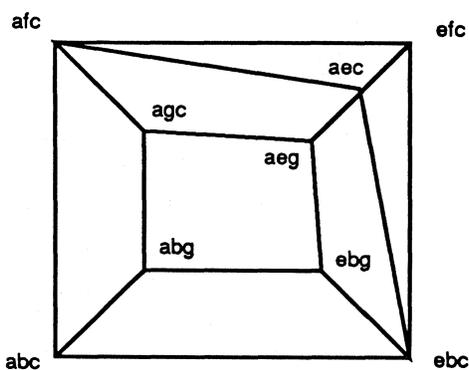


Fig. 7

If e is parallel to g and $TA = \{a, f, g\}$ (i.e. $\{a, f, g\}$ is a tree), then for $TB = \{x, b, y\}$ we can set $TX = \{x, f, y\}$. If f is parallel to g and $TB = \{e, b, g\}$, then for $TA = \{a, x, y\}$ we can set $TX = \{e, x, y\}$.

This takes care of all but three cases, up to the symmetry $a \leftrightarrow b, e \leftrightarrow f$. The graphs Γ and edges a, b, c, e, f, g in these cases are shown in Fig. 6.

In Case 1, we have $TB = \{e, b, g\}$; if $|g| \geq |f|$ we have $TA = \{a, b, f\}$ and we can take $TX = \{e, b, f\}$; if $|f| > |g|$ and $|e| > |g|$, we have $TA = \{a, g, c\}$ and we can take $TX = \{g, f, c\}$. If $|f| > |g| \geq |e|$, see Fig. 7.

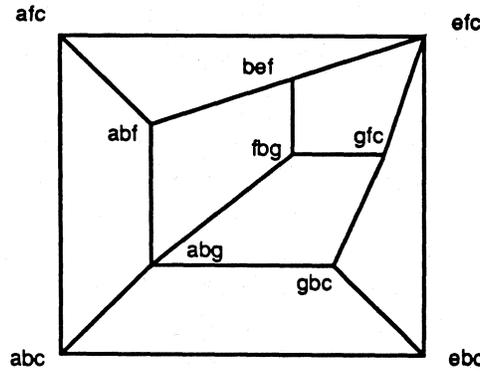


Fig. 8

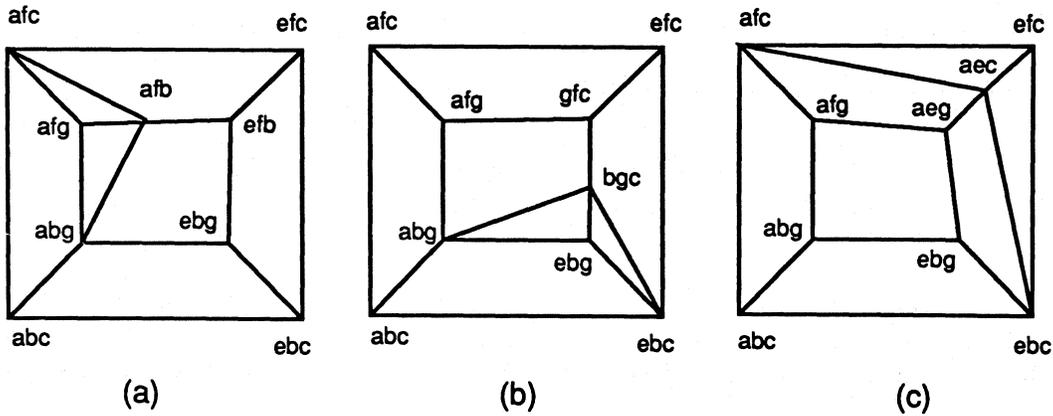


Fig. 9

In Case 2, if $|e|, |f| > |g|$, we have $TA = \{a, c, g\}$, $TB = \{g, b, c\}$ and we can set $TX = \{g, f, c\}$. If $|g| \geq |e|, |f|$ we have $TA = \{a, b, f\}$, $TB = \{a, b, e\}$ and we can take $TX = \{e, b, f\}$. If $|f| > |g| \geq |e|$, we have $TA = \{a, c, g\}$, $TB = \{a, b, e\}$ and we can set $TX = \{a, e, c\}$. If $|e| > |g| \geq |f|$, see Fig. 8.

In Case 3, we have $TA = \{a, f, g\}$ and $TB = \{e, b, g\}$. If $|g| \geq |f|$ we can complete the disk by adding two roses, $TX = \{e, f, b\}$ and $TA' = \{a, f, b\}$ as in Fig. 9(a); if $|e| > |g|$ we add roses, $TX = \{g, f, c\}$ and $TB' = \{g, b, c\}$ as in Fig. 9(b); in the remaining case $|e| \leq |g| < |f|$ we add roses $TX' = \{a, e, c\}$ and $TX = \{a, e, g\}$ as in Fig. 9(c).

The other possibilities for C are treated in a similar manner, but are easier than the above and are left to the dedicated reader. □

LEMMA 4.3. *Let $p: C \rightarrow K_n$ and $p: C' \rightarrow K_n$ be adjacent standard cells in $CST(v)$, and let α, β, α' be the ideal link of v in $C \cup C'$. Let (γ, g) be a strictly increasing size two Whitehead move, such that γ is compatible with α, β and α' . Then there is a standard disk $q: D(C, C') \rightarrow K_n$ with boundary equal to the boundary of $C \cup C'$ such that*

- (i) *if w is an interior vertex of $D(C, C')$ with $q(w)$ a rose, then the norm of $q(w)$ is strictly greater than the norm of $p(v) = \rho$, and*
- (ii) *the ideal link of v in $D(C, C')$ is equal to α, γ, α' .*

Proof. By Lemma 4.2, we can replace each of the cells C and C' by standard disks $q: D(C) \rightarrow K_n$ and $q': D(C') \rightarrow K_n$ satisfying condition (i), so that the ideal link of v in $D(C) \cup D(C')$ is equal to $\alpha, \gamma, \beta, \gamma, \alpha'$.

Here $D(C)$ consists of standard 2-cells A ($q(A)$ contains ρ^β), B ($q(B)$ contains ρ^α) and a standard disk $X(C)$ filling in the rest; similarly, q' consists of standard 2-cells A' and B' and a standard disk $X(C')$. If $A = A'$, the standard disk $B \cup X(C) \cup X(C') \cup B'$ satisfies both of our conditions. If $A \neq A'$, this means the corners ρ_f^β of C and $\rho_{f'}^\beta$ of C' are not equal i.e. $f \neq f'$. In this case we use $D(C \cup C') = B \cup X(C) \cup X(C') \cup B' \cup Y(C, C')$, where $Y(C, C')$ is a standard disk with boundary roses $\rho_g^\gamma, \rho_{TA}^{\beta, \gamma}, \rho_f^\beta, \rho_{f'}^\beta$ and $\rho_{TA'}^{\beta, \gamma}$. This standard disk $Y(C, C')$ lies in the link of $\Gamma = \rho^{\beta, \gamma}$, and has no interior roses unless TA is obtained from Γ by collapsing $\{a, f\}$ and TA' by collapsing $\{f', g\}$; in this case one can check that the rose obtained from Γ by collapsing $\{f, f'\}$ has norm strictly greater than the norm of ρ , and a standard disk $Y(C, C')$ can be formed by using $\{f, f'\}$ as the single interior vertex. \square

We now show that we may assume the ideal edges in the ideal link of v are all of size two.

PROPOSITION 4.4. *If $n > 4$, $CLK(v)$ bounds a standard disk $q: D(v) \rightarrow K_n$ containing v such that*

- (i) *each ideal edge in the ideal link of v in $D(v)$ has size two, and*
- (ii) *if $w \neq v$ is an interior vertex of $D(v)$ which maps to a rose $q(w)$, then $q(w)$ has norm greater than m .*

Proof. A collection of ideal edges partitions the edges of ρ . If any piece of the partition contains 4 or more vertices, we can find a strictly increasing Whitehead move of size two which is compatible with all ideal edges in the collection, by applying Lemmas 3.3 and 3.6.

Since α_i is compatible with α_{i+1} and $n > 4$, we can find a size two ideal edge β_i compatible with both, such that one of the Whitehead moves using β is strictly increasing. Replace the link $\alpha_1, \dots, \alpha_k$ by $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_k, \beta_k$ by applying Lemma 4.2.

If β_{i-1} is compatible with β_i , we can then eliminate α_i from the link by applying Lemma 4.3 with $\gamma = \beta_{i-1}$. Now suppose β_{i-1} crosses β_i . Using the fact that α_i is compatible with β_{i-1} and β_i and that $n > 4$, we can find a size two strictly increasing Whitehead move (γ_i, g_i) compatible with β_{i-1}, α_i , and β_i . We can then replace the section $(\beta_{i-1}, \alpha_i, \beta_i)$ of the ideal link by $(\beta_{i-1}, \gamma_i, \beta_i)$ by Lemma 4.3. \square

LEMMA 4.5. *Let k be the link circumference of v , and let α and β be two ideal edges in the ideal link of v which are compatible. If α and β are not adjacent, then there is a standard disk with boundary $CLK(v)$ such that two interior vertices map to ρ , each one with link circumference strictly smaller than k . If one of α or β is strictly increasing then any other rose in the image of the interior of the disk has norm strictly greater than m .*

Proof. Since α and β are in the ideal link of v , there are edges e and f of ρ so that the path $P = (\rho_e^\alpha, \rho^\alpha, \rho, \rho^\beta, \rho_f^\beta)$ divides $CST(v)$ into two subdisks L and R . The idea is to “slit” P and insert two cancelling disks. By Lemma 3.1, there is a standard 2-cell C whose boundary contains P from ρ_e^α to ρ_f^β . Then $L \cup C$ and $C \cup R$ are two standard disks, each with center mapping to ρ with strictly smaller link circumference; their union along $\partial C - P$ is a standard disk with boundary $CLK(\rho)$. \square

PROPOSITION 4.6. *Suppose $n \geq 5$, and let $\alpha_0, \dots, \alpha_{k-1}$ be distinct size two ideal edges. Suppose that α_i is compatible with α_j if and only if $i - j \equiv \pm 1 \pmod{k}$. Then $k \leq 6$.*

Proof. Suppose there are seven such ideal edges $\alpha_1, \dots, \alpha_7$; denote their order two components by A_1, \dots, A_7 . For $i \neq j$, α_i is compatible with α_j if and only if $A_i \cap A_j = \emptyset$. The conditions on the α_i imply that there are distinct oriented edges e, f, g, h , and k with $A_1 = \{e, f\}$, $A_2 = \{g, h\}$, $A_3 = \{e, k\}$, $A_4 = \{f, g\}$ and $A_5 = \{e, h\}$. Since A_6 crosses A_1 , it must contain f ; (it cannot contain e since it is compatible with A_5); since it crosses A_3 , it must contain h ; but then $A_6 = \{f, h\}$ does not cross A_4 . \square

THEOREM 4.7. K_n is simply connected at infinity for $n \geq 5$.

Proof. Let σ be a loop outside B_N ; by Proposition 4.1, σ is homotopic outside B_N to a standard loop. By the proof of the Higgins–Lyndon lemma (Proposition 4.2.1 in [2]), there is a standard disk $p: D \rightarrow K_n$ with boundary σ . Let $\rho = (r, R)$ be a rose of minimal norm m in the image of D , and v a vertex of D mapping to ρ . By Lemma 4.4, we may assume that the ideal link of v consists of strictly increasing ideal edges of size two. By Lemma 4.5, we may break up $CST(v)$ into a union of cell-stars $CST(v_i)$, where v_i maps to ρ and ideal edges in the ideal links of the v_i are size two and are not compatible unless they are adjacent. By Proposition 4.6, this implies that the circumference of each $CLK(v_i)$ is less than or equal to six; furthermore, the proof of Proposition 4.6 shows that these six size two ideal edges involve at most five oriented edges of R . If $n \geq 5$, there is a trio compatible with all ideal edges in the ideal link of v_i , i.e. there is a strictly increasing ideal edge γ_i compatible with all of them. By Lemma 4.3, each $CST(v_i)$ can be replaced by a standard disk with the property that interior vertices which map to roses map to roses of norm strictly greater than m . We continue in this fashion until the image of our disk contains only roses of norm strictly greater than m , and then until all roses in the image of our disk have norm at least N . For $N = N(k)$, the entire disk lies outside the ball of radius k , and we are done. \square

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