

THE COMPLEX OF FREE FACTORS OF A FREE GROUP

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1. Introduction

AN important tool in the study of the group $GL(n, \mathbb{Z})$ is provided by the geometric realization of the partially ordered set (poset) of proper direct summands of \mathbb{Z}^n . The natural inclusion $\mathbb{Z}^n \rightarrow \mathbb{Q}^n$ gives a one-to-one correspondence between proper direct summands of \mathbb{Z}^n and proper subspaces of \mathbb{Q}^n , so that this poset is isomorphic to the spherical building X_n for $GL(n, \mathbb{Q})$. The term “spherical” comes from the Solomon-Tits theorem [8], which says that X_n has the homotopy type of a bouquet of spheres:

SOLOMON-TITS THEOREM. *The geometric realization of the poset of proper subspaces of an n -dimensional vector space has the homotopy type of a bouquet of spheres of dimension $n - 2$.*

The building X_n encodes the structure of parabolic subgroups of $GL(n, \mathbb{Q})$: they are the stabilizers of simplices. X_n also parametrizes the Borel-Serre boundary of the homogeneous space for $GL(n, \mathbb{R})$. The top-dimensional homology $H_{n-2}(X_n)$ is the Steinberg module I_n for $GL(n, \mathbb{Q})$, and is a dualizing module for the homology of $GL(n, \mathbb{Z})$, i.e. for all coefficient modules M there are isomorphisms

$$H^i(GL(n, \mathbb{Z}); M) \rightarrow H_{d-i}(GL(n, \mathbb{Z}); M \otimes I_n),$$

where $d = n(n - 1)/2$ is the virtual cohomological dimension of $GL(n, \mathbb{Z})$.

If one replaces $GL(n, \mathbb{Z})$ by the group $Aut(F_n)$ of automorphisms of the free group of rank n , the natural analog FC_n of X_n is the geometric realization of the poset of proper free factors of F_n . The abelianization map $F_n \rightarrow \mathbb{Z}^n$ induces a map from FC_n to the poset of summands of \mathbb{Z}^n . In this paper we prove the analog of the Solomon-Tits theorem for FC_n :

THEOREM 1.1. *The geometric realization of the poset of proper free factors of F_n has the homotopy type of a bouquet of spheres of dimension $n - 2$.*

By analogy, we call the top homology $H_{n-2}(FC_n)$ the *Steinberg module* for $Aut(F_n)$. This leaves open some intriguing questions. It has recently been shown that $Aut(F_n)$ is a virtual duality group [1]; does the Steinberg module act as a

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dualizing module? There is an analog, called Autre space, of the homogeneous space for $GL(n, \mathbb{Z})$ and the Borel-Serre boundary; what is the relation between this and the “building” of free factors?

In [7], Quillen developed tools for studying the homotopy type of the geometric realization $|X|$ of a poset X . Given an order-preserving map $f : X \rightarrow Y$ (a “poset map”), there is a spectral sequence relating the homology of $|X|$, the homology of $|Y|$, and the homology of the “fibers” $|f/y|$, where

$$f/y = \{x \in X / f(x) \leq y\}$$

with the induced poset structure.

To understand FC_n then, one might try to apply Quillen’s theory using the poset map $FC_n \rightarrow X_n$. However, it seems to be difficult to understand the fibers of this map. Instead, we proceed by modelling the poset of free factors topologically, as the poset B_n of simplices of a certain subcomplex of the “sphere complex” $S(M)$ studied in [2]. There is a natural poset map from B_n to FC_n ; we compute the homotopy type of B_n and of the Quillen fibers of the poset map, and apply Quillen’s spectral sequence to obtain the result.

2. Sphere systems

Let M be the compact 3-manifold obtained by taking a connected sum of n copies of $S^1 \times S^2$ and removing a small open ball. A *sphere system* in M is a non-empty finite set of disjointly embedded 2-spheres in the interior of M , no two of which are isotopic, and none of which bounds a ball or is isotopic to the boundary sphere of M . The complex $S(M)$ of sphere systems in M is defined to be the simplicial complex whose k -simplices are isotopy class of sphere systems with $k + 1$ spheres.

Fix a basepoint p on ∂M . The fundamental group $\pi_1(M, p)$ is isomorphic to F_n . Any automorphism of F_n can be realized by a homeomorphism of M fixing ∂M . A theorem of Laudenbach [4] implies that such a homeomorphism inducing the identity on $\pi_1(M, p)$ acts trivially on isotopy classes of sphere systems, so that in fact $Aut(F_n)$ acts on $S(M)$.

For H a subset of $\pi_1(M, p)$, define S_H to be the subcomplex of $S(M)$ consisting of isotopy classes of sphere systems S such that $\pi_1(M - S, p) \supseteq H$. Define Y_H to be the subcomplex of S_H consisting of isotopy classes of sphere systems S such that $M - S$ is connected. For H trivial, $S_H = S(M)$, and Y_H is the complex Y of [2].

The following theorems determine the homotopy types of S_H and Y_H . The proofs rely upon the proofs for $S(M)$ and Y given in [2].

THEOREM 2.1. S_H is contractible.

Proof. We claim that the contraction of $S(M)$ in [2] restricts to a contraction of the subcomplex S_H . To show this, the key point is the following:

LEMMA 2.2. *Any two simplices in S_H can be represented by sphere systems Σ and S such that every element of H is representable by a loop disjoint from both Σ and S .*

Proof. Enlarge Σ to a maximal sphere system Σ' , so the components of $M - \Sigma'$ are three-punctured spheres. By Proposition 1.1 of [2] we may isotope S to be in normal form with respect to Σ' ; this means that S intersects each component of $M - \Sigma'$ in a collection of surfaces, each having at most one boundary circle on each of the three punctures.

We can represent a given element of H by a loop γ_0 based at p , such that γ_0 is disjoint from S and transverse to Σ' . The points of intersection of γ_0 with Σ' divide γ_0 into a finite set of arcs, each entirely contained in one component of $M - \Sigma'$. Suppose one of these arcs α , in a component P of $M - \Sigma'$, has both endpoints on the same boundary sphere σ of P . Since the map $\pi_0(\sigma - (S \cap \sigma)) \rightarrow \pi_0(P - (S \cap P))$ is injective (an easy consequence of normal form), there is an arc α' in $\sigma - (S \cap \sigma)$ with $\partial\alpha' = \partial\alpha$. Since P is simply-connected, α is homotopic to α' fixing endpoints. This homotopy gives a homotopy of γ_0 eliminating the two points of $\partial\alpha$ from $\gamma_0 \cap \Sigma'$, without introducing any intersection points with S . After repeating this operation a finite number of times, we may assume there are no remaining arcs of $\gamma_0 - (\gamma_0 \cap \Sigma')$ of the specified sort.

Now consider a homotopy $F : I \times I \rightarrow M$ of γ_0 to a loop γ_1 disjoint from Σ . Make F transverse to Σ' and look at $F^{-1}(\Sigma')$. This consists of a collection of disjoint arcs and circles. These do not meet the left and right edges of $I \times I$ since these edges map to the basepoint p .

We claim that every arc component of $F^{-1}(\Sigma')$ with one endpoint on $I \times \{0\}$ must have its other endpoint on $I \times \{1\}$. If not, choose an “edgemoat” arc with both endpoints on $I \times \{0\}$, i.e. an arc such that the interval of $I \times \{0\}$ bounded by the endpoints contains no other point of $F^{-1}(\Sigma')$. Then γ_0 maps this interval to an arc α in $M - S$ which is entirely contained in one component P of $M - \Sigma'$ and has both endpoints on the same boundary sphere of P , contradicting our assumption that all such arcs have been eliminated.

Since the loop γ_1 is disjoint from Σ , it follows that γ_0 must be disjoint from Σ , and by construction γ_0 was disjoint from S .

Since elements of H are representable by loops disjoint from Σ and S , these loops remain disjoint from sphere systems obtained by surgering S along Σ , because such surgery produces spheres lying in a neighbourhood of $S \cup \Sigma$. This means that the contraction of $S(M)$ constructed in [2] restricts to a contraction of the subcomplex S_H . (Alternatively, we could use the simpler contraction technique of [3], which reverses the roles of S and Σ .)

Definition. A simplicial complex K is k -spherical if it is k -dimensional and $(k - 1)$ -connected. By convention, a (-1) -spherical complex is a single point. A complex is spherical if it is k -spherical for some k .

Now that we have S_H contractible, we use it to show that Y_H is spherical. In the course of the proof, we will need to consider complexes analogous to Y_H for manifolds with more than one boundary sphere. The next lemma shows that these have the same homotopy type as Y_H .

Let M_k be the manifold obtained from the connected sum of n copies of $S^1 \times S^2$ by deleting k disjoint open balls rather than just a single ball. Choose the basepoint p on one of the spheres in ∂M . For $H \subseteq \pi_1(M_k, p)$, define $Y_H(M_k)$ to be the complex of isotopy classes of sphere systems S in M_k with connected complement such that $\pi_1(M - S, p) \supseteq H$.

LEMMA 2.3. *For $k \geq 1$, $Y_H(M_{k+1})$ deformation retracts onto a subcomplex isomorphic to $Y_H(M_k)$.*

Proof. Let ∂_0 and ∂_1 be two components of ∂M , with $p \in \partial_0$. Fix a sphere σ in the interior of M_{k+1} which separates M_{k+1} into two components, one of which is a three-punctured sphere P containing ∂_0 and ∂_1 , and the other of which is homeomorphic to M_k .

Let S be a sphere system representing a simplex of $Y_H(M_{k+1})$. We may assume that S is in normal form with respect to σ . This means that S intersects P in a collection of disjoint disks, each of which has boundary on σ and separates ∂_0 from ∂_1 . Perform a sequence of modifications of S by transferring, one at a time, each disk of $S \cap P$ to the other side of ∂_1 . This has no effect on $\pi_1(M - S, p)$, so the sphere systems which arise during this modification are still in $Y_H(M_{k+1})$. The final system resulting from transferring all the disks of $S \cap P$ to the other side of ∂_1 can be isotoped to be disjoint from σ .

This sequence of modifications can also be described as a sequence of surgeries on S using the disks in σ on the side away from ∂_0 . As explained on pp. 48–49 of [2], such a surgery process defines a piecewise linear flow on the sphere complex. In the present case this flow gives a deformation retraction of $Y_H(M_{k+1})$ onto the subcomplex of sphere systems in $M - \sigma$. This subcomplex can be identified with $Y_H(M_k)$ by choosing the basepoint for M_k to be in σ .

THEOREM 2.4. *Let H be a free factor of $F_n = \pi_1(M, p)$. Then Y_H is $(n - rk(H) - 1)$ -spherical.*

Proof. Let $i \leq n - rk(H) - 2$. Any map $g : S^i \rightarrow Y_H$ can be extended to a map $\hat{g} : D^{i+1} \rightarrow S_H$ since S_H is contractible. We can assume \hat{g} is a simplicial map with respect to some triangulation of D^{i+1} compatible with its standard piecewise linear structure. We will redefine \hat{g} on the stars of certain simplices in the interior of D^{i+1} to make the image of \hat{g} lie in Y_H .

To each sphere system S we associate a dual graph $\Gamma(S)$, with one vertex for each component of $M - S$ and one edge for each sphere in S . The endpoints of the edge corresponding to $s \in S$ are the vertices corresponding to the component or components adjacent to s . We say a sphere system S is *purely separating* if $\Gamma(S)$ has no edges which begin and end at the same vertex. Each sphere system S

has a *purely separating core*, consisting of those spheres in S which correspond to the core of $\Gamma(S)$, i.e. the subgraph spanned by edges with distinct vertices. The purely separating core of $S \in S_H$ is empty if and only if S is in Y_H .

Let σ be a simplex of D^{i+1} of maximal dimension among the simplices τ with $\hat{g}(\tau)$ purely separating. Note that all such simplices τ lie in the interior of D^{i+1} , since the boundary of D^{i+1} maps to Y_H . Let $S = \hat{g}(\sigma)$, and let N_0, \dots, N_r ($r \geq 1$) be the connected components of $M - S$, with $p \in N_0$. A simplex τ in the link $lk(\sigma)$ maps to a system T in the link of S , so that each $T_j = T \cap N_j$ is a sphere system in N_j and $H \leq \pi_1(N_0 - T_0, p)$. Furthermore $N_j - T_j$ must be connected for all j since otherwise the core of $\Gamma(S \cup T)$ would have more edges than $\Gamma(S)$, contradicting the maximality of σ . Thus \hat{g} maps $lk(\sigma)$ into a subcomplex of S_H which can be identified with $Y_H(N_0) * Y(N_1) * \dots * Y(N_r)$.

Since σ is a simplex in the interior of D^{i+1} , $lk(\sigma)$ is a sphere of dimension $i - \dim(\sigma)$. Each N_j has fundamental group of rank $n_j < n$, so by Lemma 2.3 and induction, $Y_H(N_0)$ is $(n_0 - rk(H) - 1)$ -spherical and, for $j \geq 1$, $Y(N_j)$ is $(n_j - 1)$ -spherical. Hence $Y_H(N_0) * Y(N_1) * \dots * Y(N_r)$ is spherical of dimension $(\sum_{j=0}^r n_j) - rk(H) - 1$.

Now $n = (\sum_j n_j) + rk(\pi_1(\Gamma(S))) = (\sum_j n_j) + m - r$ where m is the number of spheres in S , i.e., edges in $\Gamma(S)$. Since a simplicial map cannot increase dimension, we have $\dim(\sigma) \geq m - 1$. Since $i \leq n - rk(H) - 2$, we have

$$\begin{aligned} i - \dim(\sigma) &\leq n - rk(H) - 2 - \dim(\sigma) \\ &\leq n - rk(H) - m - 1 \\ &= \left(\sum_j n_j \right) - rk(H) - 1 - r \\ &< \left(\sum_j n_j \right) - rk(H) - 1. \end{aligned}$$

Hence the map $\hat{g} : lk(\sigma) \rightarrow Y_H(N_0) * Y(N_1) * \dots * Y(N_r)$ can be extended to a map of a disk D^k into $Y_H(N_0) * Y(N_1) * \dots * Y(N_r)$, where $k = i + 1 - \dim(\sigma)$. The system S is compatible with every system in the image of D^k , so this map can be extended to a map $\sigma * D^k \rightarrow S_H$. We replace the star of σ in D^{i+1} by the disk $\partial(\sigma) * D^k$, and define \hat{g} on $\partial(\sigma) * D^k$ using this map.

What have we improved? The new simplices in the disk $\partial(\sigma) * D^k$ are of the form $\sigma' * \tau$, where σ' is a face of σ and $\hat{g}(\tau) \subset Y_H(N_0) * Y(N_1) * \dots * Y(N_r)$. The image of such a simplex $\sigma' * \tau$ is a system $S' \cup T$ such that in $\Gamma(S' \cup T)$ the edges corresponding to T are all loops. Therefore any simplex in the disk $\partial(\sigma) * D^k$ with purely separating image must lie in the boundary of this disk, where we have not modified \hat{g} .

We continue this process, eliminating purely separating simplices until there is none in the image of \hat{g} . Since every system in $S_H - Y_H$ has a non-trivial purely separating core, in fact the whole disk maps into Y_H , and we are done.

3. Free factors

We now turn to the poset FC_n of proper free factors of the free group F_n , partially ordered by inclusion. A k -simplex in the geometric realization $|FC_n|$ is a flag $H_0 < H_1 < \dots < H_k$ of proper free factors of F_n , each properly included in the next. Each H_i is also a free factor of H_{i+1} (see [6, p. 117]), so that a maximal simplex of $|FC_n|$ has dimension $n - 2$.

We want to model free factors of F_n by sphere systems in $Y = Y(M)$, by taking the fundamental group of the (connected) complement. A sphere system with n spheres and connected complement, corresponding to an $(n - 1)$ -dimensional simplex of Y , in fact has simply-connected complement. But we want to consider proper free factors, so instead we consider the $(n - 2)$ -skeleton $Y^{(n-2)}$. Since Y is $(n - 2)$ -connected by Theorem 2.4, $Y^{(n-2)}$ is $(n - 2)$ -spherical.

In order to relate $Y^{(n-2)}$ to FC_n , we take the barycentric subdivision B_n of $Y^{(n-2)}$. Then B_n is the geometric realization of a poset of isotopy classes of sphere systems, partially ordered by inclusion. If $S \subseteq S'$ are sphere systems, we have $\pi_1(M - S, p) \geq \pi_1(M - S', p)$, reversing the partial ordering. Taking fundamental group of the complement thus gives a poset map $f : B_n \rightarrow (FC_n)^{op}$, where $(FC_n)^{op}$ denotes FC_n with the opposite partial ordering.

PROPOSITION 3.1. $f : B_n \rightarrow (FC_n)^{op}$ is surjective.

Proof. Every simplex of FC_n is contained in a simplex of dimension $n - 2$ so it suffices to show f maps onto all $(n - 2)$ -simplices. The group $Aut(F_n)$ acts transitively on $(n - 2)$ -simplices of FC_n , and all elements of $Aut(F_n)$ are realized by homeomorphisms of M , so f will be surjective if its image contains a single $(n - 2)$ -simplex, which it obviously does.

COROLLARY 3.2. FC_n is connected if $n \geq 3$.

Proof. Theorem 2.4 implies that B_n is connected for $n \geq 3$. So, given any two vertices of FC_n , lift them to vertices of B_n by Proposition 3.1, connect the lifted vertices by a path, then project the path back down to FC_n .

For any proper free factor H , let $B_{\geq H}$ denote the fiber f/H , consisting of isotopy classes of sphere systems S in B_n with $\pi_1(M - S, p) \geq H$.

PROPOSITION 3.3. Let H be a proper free factor of F_n . Then $B_{\geq H}$ is $(n - rk(H) - 1)$ -spherical. If $rk(H) = n - 1$ then $B_{\geq H}$ is a single point.

Proof. The fiber $B_{\geq H}$ is the barycentric subdivision of Y_H , so is $(n - rk(H) - 1)$ -spherical by Theorem 2.4.

Suppose $rk(H) = n - 1$, so that $\pi_1(M, p) = H * \langle x \rangle$ for some x . An element of $B_{\geq H}$ is represented by a sphere system containing exactly one sphere s , which is non-separating with $\pi_1(M - s, p) = H$. Suppose s and s' are two such spheres. Since s and s' are both non-separating, there is a homeomorphism h of M taking s

to s' . Since automorphisms of H can be realized by homeomorphisms of M fixing s , we may assume that the induced map on $\pi_1(M, p)$ is the identity on H .

Claim. The induced map $h_* : \pi_1(M, p) \rightarrow \pi_1(M, p)$ must send x to an element of the form $Ux^{\pm 1}V$, with $U, V \in H$.

Proof. Let $\{x_1, \dots, x_{n-1}\}$ be a basis for H , and let W be the reduced word representing $h_*(x)$ in the basis $\{x_1, \dots, x_{n-1}, x\}$ for $\pi_1(M, p)$. By looking at the map induced by h on homology, we see that the exponent sum of x in W must be ± 1 . Since h_* fixes H , $\{x_1, \dots, x_{n-1}, W\}$ is a basis for $\pi_1(M, p)$. If W contained both x and x^{-1} , we could apply Nielsen automorphisms to the set $\{x_1, \dots, x_{n-1}, W\}$ until W was of the form $x^{\pm 1}W_0x^{\pm 1}$. But $\{x_1, \dots, x_{n-1}, x^{\pm 1}W_0x^{\pm 1}\}$ is not a basis, since it is Nielsen reduced and not of the form $\{x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}, x^{\pm 1}\}$ (see [5], Prop. 2.8)).

The automorphism fixing H and sending $x \mapsto Ux^{\pm 1}V$ can be realized by a homeomorphism h' of M which takes s to itself (see [4], Lemma 4.3.1). The composition $h'h^{-1}$ sends s' to s and induces the identity on π_1 , hence acts trivially on the sphere complex. Thus s and s' are isotopic.

COROLLARY 3.4. *FC_n is simply connected for $n \geq 4$.*

Proof. Let e_0, e_1, \dots, e_k be the edges of an edge-path loop in FC_n , and choose lifts \tilde{e}_i of these edges to B_n . Let $e_{i-1}e_i$ or $e_k e_0$ be two adjacent edges of the path, meeting at the vertex H . The lifts \tilde{e}_{i-1} and \tilde{e}_i may not be connected, i.e. \tilde{e}_{i-1} may terminate at a sphere system S' and \tilde{e}_i may begin at a different sphere system S . However, both S and S' are in the fiber $B_{\geq H}$, which is connected by Proposition 3.3, so we may connect S and S' by a path in $B_{\geq H}$. Connecting the endpoints of each lifted edge in this way, we obtain a loop in B_n , which may be filled in by a disk if $n \geq 4$, by Proposition 3.3. The projection of this loop to FC_n is homotopic to the original loop, since each extra edge-path segment we added projects to a loop in the star of some vertex H , which is contractible. Therefore the projection of the disk kills our original loop in the fundamental group.

Remark. It is possible to describe the complex Y_H purely in terms of F_n . Suppose first that H is trivial. Define a simplicial complex Z to have vertices the rank $n - 1$ free factors of F_n , with a set of k such factors spanning a simplex in Z if there is an automorphism of F_n taking them to the k factors obtained by deleting the standard basis elements x_1, \dots, x_k of F_n one at a time. There is a simplicial map $f : Y \rightarrow Z$ sending a system of k spheres to the set of k fundamental groups of the complements of these spheres. These fundamental groups are equivalent to the standard set of k rank $n - 1$ factors under an automorphism of F_n since the homeomorphism group of M acts transitively on simplices of Y of a given dimension, and the standard k factors are the fundamental groups of the complements of the spheres in a standard system in Y . The last statement of Proposition 3.3 says that f is a bijection on vertices, so f embeds Y as a subcomplex of Z . The maximal simplices in Y and Z have dimension $n - 1$ and the groups $\text{Homeo}(M)$

and $Aut(F_n)$ act transitively on these simplices, so f must be surjective, hence an isomorphism. When H is nontrivial, f restricts to an isomorphism from Y_H onto the subcomplex Z_H spanned by the vertices which are free factors containing H .

We are now ready to apply Quillen’s spectral sequence to compute the homology of B_n and thus prove the main theorem.

THEOREM 3.5. *FC_n is $(n - 2)$ -spherical.*

Proof. We prove the theorem by induction on n . If $n \leq 4$, the theorem follows from Corollaries 3.2 and 3.4.

Quillen’s spectral sequence [7, 7.7] applied to $f : B_n \rightarrow (FC_n)^{op}$ becomes

$$E_{p,q}^2 = H_p(FC_n; H \mapsto H_q(B_{\geq H})) \Rightarrow H_{p+q}(B_n),$$

where the E^2 -term is computed using homology with coefficients in the functor $H \mapsto H_q(B_{\geq H})$.

For $q = 0$, Corollary 3.2 gives $H_0(B_{\geq H}) = \mathbb{Z}$ for all H , so $E_{p,0}^2 = H_p(FC_n, \mathbb{Z})$.

For $q > 0$, we have $E_{p,q}^2 = H_p(FC_n; H \mapsto \tilde{H}_q(B_{\geq H}))$, and we follow Quillen ([7], proof of Theorem 9.1) to compute this.

For a subset A of FC_n , let L_A denote the functor sending H to a fixed abelian group L if $H \in A$ and to 0 otherwise. Set $U = FC_{\leq H} = \{H' \in FC_n \mid H' \leq H\}$ and $V = FC_{< H} = \{H' \in FC_n \mid H' < H\}$. Then

$$H_i(FC_n; L_V) = H_i(V; L), \text{ and}$$

$$H_i(FC_n; L_U) = H_i(U; L) = \begin{cases} L & \text{if } i = 0 \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

since $|U|$ is contractible. The short exact sequence of functors

$$1 \rightarrow L_V \rightarrow L_U \rightarrow L_{\{H\}} \rightarrow 1$$

gives a long exact homology sequence, from which we compute

$$H_i(FC_n; L_{\{H\}}) = \tilde{H}_{i-1}(FC_{< H}; L).$$

Now, $H \mapsto \tilde{H}_q(B_{\geq H})$ is equal to the functor

$$\bigoplus_{rk(H)=n-q-1} \tilde{H}_q(B_{\geq H})_{\{H\}}$$

since $B_{\geq H}$ is $(n - rk(H) - 1)$ -spherical by Proposition 3.3. Thus

$$\begin{aligned} E_{p,q}^2 &= H_p(FC_n; H \mapsto \tilde{H}_q(B_{\geq H})) \\ &= \bigoplus_{rk(H)=n-q-1} H_p(FC_n; \tilde{H}_q(B_{\geq H})_{\{H\}}) \\ &= \bigoplus_{rk(H)=n-q-1} \tilde{H}_{p-1}(FC_{< H}; \tilde{H}_q(B_{\geq H})). \end{aligned}$$

Free factors of F_n contained in H are also free factors of H . Since H has rank $< n$, $FC_{<H}$ is $(rk(H) - 2)$ -spherical by induction. Therefore $E_{p,q}^2 = 0$ unless $p - 1 = (n - q - 1) - 2$, i.e. $p + q = n - 2$. Since all terms in the E^2 -term of the spectral sequence are zero except the bottom row for $p \leq n - 2$ and the diagonal $p + q = n - 2$, all differentials are zero and we have $E^2 = E^\infty$ as in the following diagram:

$$\begin{array}{cccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & E_{0,n-2}^2 & & 0 & & 0 & & 0 \\
 & 0 & & E_{1,n-3}^2 & & 0 & & 0 \\
 & 0 & & 0 & & E_{2,n-4}^2 & & 0 \\
 & 0 & & 0 & & 0 & & E_{3,n-5}^2 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 H_0(FC_n) & H_1(FC_n) & H_2(FC_n) & H_3(FC_n) & \dots & H_{n-2}(FC_n) & 0
 \end{array}$$

Since FC_n is connected and the spectral sequence converges to $H_*(B_n)$, which is $(n - 3)$ -connected, we must have $\tilde{H}_i(FC_n) = 0$ for $i \neq n - 2$. Since FC_n is simply-connected by Corollary 3.4, this implies that FC_n is $(n - 3)$ -connected by the Hurewicz theorem. The theorem follows since FC_n is $(n - 2)$ -dimensional.

4. The Cohen-Macaulay Property

In a PL triangulation of an n -dimensional sphere, the link of every k -simplex is an $(n - k - 1)$ -sphere. A poset is said to be *Cohen-Macaulay* of dimension n if its geometric realization is n -spherical and the link of every k -simplex is $(n - k - 1)$ -spherical (see [7]). Spherical buildings are Cohen-Macaulay, and we remark that FC_n also has this nice local property.

To see this, let $\sigma = H_0 < H_1 < \dots < H_k$ be a k -simplex of FC_n . The link of σ is the join of subcomplexes $FC_{H_i, H_{i+1}}$ of FC_n spanned by free factors H with $H_i < H < H_{i+1}$ ($-1 \leq i \leq k$, with the conventions $H_{-1} = 1$ and $H_{k+1} = F_n$). Counting dimensions, we see that it suffices to show that for each r and s with $0 \leq s < r$, the poset $FC_{r,s}$ of proper free factors H of F_r which properly contain F_s is $(r - s - 2)$ -spherical. The proof of this is identical to the proof that FC_n is $(n - 2)$ -spherical, after setting $n = r$ and replacing the complex Y by Y_{F_s} .

5. The map to the building

As mentioned in the introduction, the abelianization map $F_n \rightarrow \mathbb{Z}^n$ induces a map from the free factor complex FC_n to the building X_n , since summands of \mathbb{Z}^n correspond to subspaces of \mathbb{Q}^n . Since the map $Aut(F_n) \rightarrow GL(n, \mathbb{Z})$ is surjective, every basis for \mathbb{Z}^n lifts to a basis for F_n , and hence every flag of summands of \mathbb{Z}^n lifts to a flag of free factors of F_n , i.e. the map $FC_n \rightarrow X_n$ is surjective.

Given a basis $\{v_1, \dots, v_n\}$ for \mathbb{Q}^n , consider the subcomplex of X_n consisting of

all flags of subspaces of the form $\langle v_{i_1} \rangle \subset \langle v_{i_1}, v_{i_2} \rangle \subset \dots \subset \langle v_{i_1}, \dots, v_{i_{n-1}} \rangle$. This subcomplex can be identified with the barycentric subdivision of the boundary of an $(n - 1)$ -dimensional simplex, so forms an $(n - 2)$ -dimensional sphere in X_n , called an *apartment*.

The construction above applied to a basis of F_n instead of \mathbb{Q}^n yields an $(n - 2)$ -dimensional sphere in FC_n . In particular, $H_{n-2}(FC_n)$ is non-trivial. This sphere maps to an apartment in X_n showing that the induced map $H_{n-2}(FC_n) \rightarrow H_{n-2}(X_n)$ is also non-trivial.

The property of buildings which is missing in FC_n is that given any two maximal simplices there is an apartment which contains both of them. For example, for $n = 3$ and F_3 free on $\{x, y, z\}$ there is no “apartment” which contains both the one-cells corresponding to $\langle x \rangle \subset \langle x, y \rangle$ and $\langle yxy^{-1} \rangle \subset \langle x, y \rangle$, since x and yxy^{-1} do not form part of a basis of F_3 .

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