ANNALES

DE

L'INSTITUT FOURIER

Martin R. BRIDSON & Karen VOGTMANN

**The ehn functions of $Out(F_n)$ and $Aut(F_n)$**

Tome 62, n° 5 (2012), p. 1811-1817.

<http://aif.cedram.org/item?id= IF_2012__62_5_1811_0>

nn. Inst. Fourier, Grenoble
**62**, 5 (2012) 1811-1817

# THE DEHN FUNCTIONS OF $Out\ F_n)$ ND $ut\ F_n)$

**by Martin R. BRIDSON & Karen VOGTM NN (*)**

BSTRACT. — For $n$ at least 3, the Dehn functions of $Out(F\ )$ and $ut(F\ )$ are exponential. Hatcher and Vogtmann proved that they are at most exponential, and the complementary lower bound in the case $n = 3$ was established by Bridson and Vogtmann. Handel and Mosher completed the proof by reducing the lower bound for $n$ bigger than 3 to the case $n = 3$. In this note we give a shorter, more direct proof of this last reduction.

RÉSUMÉ. — Pour $n$ au moins 3, les fonctions de Dehn de $Out(F\ )$ et $ut(F\ )$ sont exponentielles. Hatcher et Vogtmann ont montré qu'elles étaient au plus exponentielles, et la borne inférieure a été établie par Bridson et Vogtmann dans le cas $n = 3$. Handel et Mosher ont complété la démonstration en ramenant la preuve de la borne inférieure pour $n$ au moins 4 au cas $n = 3$. Dans cet article, nous donnons un argument plus direct permettant de passer du cas $n = 3$ au cas général.

Dehn functions provide upper bounds on the complexity of the word problem in finitely presented groups. They are examples of filling functions: if a group $G$ acts properly and cocompactly on a simplicial complex $X$, then the Dehn function of $G$ is asymptotically equivalent to the function that provides the optimal upper bound on the area of least-area discs in $X$, where the bound is expressed as a function of the length of the boundary of the disc. This article is concerned with the Dehn functions of automorphism groups of finitely-generated free groups.

Much of the contemporary study of $\mathrm{Out}(F_n)$ and $\mathrm{ut}(F_n)$ is based on the deep analogy between these groups, mapping class groups, and lattices in semisimple Lie groups, particularly $\mathrm{SL}(n,\mathbb{Z})$. The Dehn functions of mapping class groups are quadratic [9], as is the Dehn function of $\mathrm{SL}(n,\mathbb{Z})$ if $n \geqslant 5$ (see [10]). In contrast, Epstein *et l.* [6] proved that the Dehn function of $\mathrm{SL}(3,\mathbb{Z})$ is exponential. Building on their result, we proved

*Keywords:* utomorphism groups of free groups, Dehn functions.
*Math. classification:* 20F65, 20F28, 53C24, 57S25.
(*) Bridson is supported by an EPSRC Senior Fellowship. Vogtmann is supported by NSF grant DMS-0204185.

in [3] that ut($F_3$) and Out($F_3$) also have exponential Dehn functions. Hatcher and Vogtmann [8] established an exponential upper bound on the Dehn function of ut($F_n$) and Out($F_n$) for all $n \geqslant 3$. The comparison with $\mathrm{SL}(n,\mathbb{Z})$ might lead one to suspect that this last result is not optimal for large $n$, but recent work of Handel and Mosher [7] shows that in fact it is: they establish an exponential lower bound by using their general results on quasi-retractions to reduce to the case $n = 3$.

THEOREM. — *For $n \geqslant 3$, the Dehn functions of* ut($F_n$) *nd* Out($F_n$) *re exponenti l.*

This theorem answers Questions 35 and 37 of [4].

We learned the contents of [7] from Lee Mosher at Luminy in June 2010 and realized that one can also reduce the Theorem to the case $n = 3$ using a simple observation about natural maps between different-rank Outer spaces and uter spaces (Lemma 3). The purpose of this note is record this observation and the resulting proof of the Theorem.

## 1. Definitions

Let $A$ be a 1-connected simplicial complex. We consider simplicial loops $\ell\colon S \to A^{(1)}$, where $S$ is a simplicial subdivision of the circle. *simplici l filling* of $\ell$ is a simplicial map $L\colon D \to A^{(2)}$, where $D$ is a triangulation of the 2-disc and $L|_{\partial D} = \ell$. Such fillings always exist, by simplicial approximation. The filling area of $\ell$, denoted rea$_A(\ell)$, is the least number of triangles in the domain of any simplicial filling of $\ell$. The *Dehn function*[(1)] of $A$ is the least function $\delta_A\colon \to$ such that rea$_A(\ell) \leqslant \delta_A(n)$ for all loops of length $\leqslant n$ in $A^{(1)}$. The Dehn function of a finitely presented group $G$ is the Dehn function of any 1-connected 2-complex on which $G$ acts simplicially with finite stabilizers and compact quotient. This is well-defined up to the following equivalence relation: functions $f, g\colon \to$ are equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there is a constant $a > 1$ such that $f(n) \leqslant a\, g(an + a) + an + a$. The Dehn function can be interpreted as a measure of the complexity of the word problem for $G$ — see [2].

LEMMA 1. — *If $A$ nd $B$ re 1-connected simplici l complexes, $F\colon A \to B$ is simplici l m p, nd $\ell$ is loop in the 1-skeleton of $A$, then* rea$_A(\ell) \geqslant$ rea$_B(F \circ \ell)$.

(1) The standard definition of area and Dehn function are phrased in terms of singular discs, but this version is equivalent.

*Proof.* — If $L\colon D \to A$ is a simplicial filling of $\ell$, then $F \circ L$ is a simplicial filling of $F \circ \ell$, with the same number of triangles in the domain $D$.

COROLLARY. — *Let $A, B$ nd $C$ be 1-connected simplici l complexes with simplici l m ps $A \to B \to C$. Let $\ell_n$ be sequence of simplici l loops in $A$ whose length is bounded bove by line r function of $n$, let $\overline{\ell}_n$ be the im ge loops in $C$ nd let $(n) =$ rea$_C(\overline{\ell}_n)$. Then the Dehn function of $B$ s tisfies $\delta_B(n) \succeq (n)$.*

*Proof.* — This follows from Lemma 1 together with the observation that a simplicial map does not increase the length of any loop in the 1-skeleton.

## 2. Simplicial complexes associated to Out($F_n$) and ut($F_n$)

Let $K_n$ denote the spine of Outer space, as defined in [5], and $L_n$ the spine of uter space, as defined in [8]. These are contractible simplicial complexes with cocompact proper actions by Out($F_n$) and ut($F_n$) respectively, so we may use them to compute the Dehn functions for these groups.

Recall from [5] that a *m rked gr ph* is a finite metric graph $\Gamma$ together with a homotopy equivalence $g\colon R_n \to \Gamma$, where $R_n$ is a fixed graph with one vertex and $n$ loops. vertex of $K_n$ can be represented either as a marked graph $(g, \Gamma)$ with all vertices of valence at least three, or as a free minimal action of $F_n$ on a simplicial tree (namely the universal cover of $\Gamma$). vertex of $L_n$ has the same descriptions except that there is a chosen basepoint in the marked graph (respected by the marking) or in the simplicial tree. Note that we allow marked graphs to have separating edges. Both $K_n$ and $L_n$ are flag complexes, so to define them it suffices to describe what it means for vertices to be adjacent. In the marked-graph description, vertices of $K_n$ (or $L_n$) are adjacent if one can be obtained from the other by a forest collapse (*i.e.* collapsing each component of a forest to a point).

## 3. Three natural maps

There is a *forgetful m p* $\phi_n\colon L_n \to K_n$ which simply forgets the basepoint; this map is simplicial.

Let $m < n$. We fix an ordered basis for $F_n$, identify $F_m$ with the subgroup generated by the first $m$ elements of the basis, and identify ut($F_m$) with

the subgroup of ut$(F_n)$ that leaves $F_m < F_n$ invariant and fixes the last $n \quad m$ basis elements. We consider two maps associated to this choice of basis.

First, there is an equivariant *ugment tion m p* $\iota\colon L_m \to L_n$ which attaches a bouquet of $n \quad m$ circles to the basepoint of each marked graph and marks them with the last $n \quad m$ basis elements of $F_n$. This map is simplicial, since a forest collapse has no effect on the bouquet of circles at the basepoint.

Secondly, there is a *restriction m p* $\rho\colon K_n \to K_m$ which is easiest to describe using trees. point in $K_n$ is given by a minimal free simplicial action of $F_n$ on a tree $T$ with no vertices of valence 2. We define $\rho(T)$ to be the minimal invariant subtree for $F_m < F_n$; more explicitly, $\rho(T)$ is the union of the axes in $T$ of all elements of $F_m$. (Vertices of $T$ that have valence 2 in $\rho(T)$ are no longer considered to be vertices.)

One can also describe $\rho$ in terms of marked graphs. The chosen embedding $F_m < F_n$ corresponds to choosing an $m$-petal subrose $R_m \subset R_n$. vertex in $K_n$ is given by a graph $\Gamma$ marked with a homotopy equivalence $g\colon R_n \to \Gamma$, and the restriction of $g$ to $R_m$ lifts to a homotopy equivalence $\widehat{g}\colon R_m \to \widehat{\Gamma}$, where $\widehat{\Gamma}$ is the covering space corresponding to $g_*(F_m)$. There is a canonical retraction $r$ of $\widehat{\Gamma}$ onto its *comp ct core*, i.e. the smallest connected subgraph containing all nontrivial embedded loops in $\Gamma$. Let $\widehat{\Gamma}_0$ be the graph obtained by erasing all vertices of valence 2 from the compact core and define $\rho(g,\Gamma) = (r \circ \widehat{g}, \widehat{\Gamma}_0)$.

LEMMA 2. — *For $m < n$, the restriction m p $\rho\colon K_n \to K_m$ is simplici l.*

*Proof.* — ny forest collapse in $\Gamma$ is covered by a forest collapse in $\widehat{\Gamma}$ that preserves the compact core, so $\rho$ preserves adjacency.

LEMMA 3. — *For $m < n$, the following di gr m of simplici l m ps commutes:*

$$\begin{array}{ccc} L_m & \to & L_n \\ \phi_m\downarrow & & \downarrow\phi_n \\ K_m & \overset{\rho}{\leftarrow} & K_n \end{array}$$

*Proof.* — Given a marked graph with basepoint $(g,\Gamma;v) \in L_n$, the marked graph $\iota(g,\Gamma;v)$ is obtained by attaching $n \quad m$ loops at $v$ labelled by the elements $a_{m+1},\dots,a_n$ of our fixed basis for $F_n$. Then $(g_n,\Gamma_n) := \phi_n \circ \iota(g,\Gamma;v)$ is obtained by forgetting the basepoint, and the cover of $(g_n,\Gamma_n)$ corresponding to $F_m < F_n$ is obtained from a copy of $(g,\Gamma)$ (with its labels) by attaching $2(n \quad m)$ trees. (These trees are obtained from the Cayley graph of $F_n$ as follows: one cuts at an edge labelled $a_i^{\varepsilon}$, with

$i \in \{m+1, \dots, n\}$ and $\varepsilon = \pm 1$, takes one component of the result, and then attaches the hanging edge to the basepoint $v$ of $\Gamma$.) The effect of $\rho$ is to delete these trees.

## 4. Proof of the Theorem

In the light of the Corollary and Lemma 3, it suffices to exhibit a sequence of loops $\ell_i$ in the 1-skeleton of $L_3$ whose lengths are bounded by a linear function of $i$ and whose filling area when projected to $K_3$ grows exponentially as a function of $i$. Such a sequence of loops is essentially described in [3]. What we actually described there were words in the generators of $\mathrm{ut}(F_3)$ rather than loops in $L_3$, but standard quasi-isometric arguments show that this is equivalent. More explicitly, the words we considered were $w_i = T^i A T^{\ i} B T^i A\ ^1 T^{\ i} B^{\ 1}$ where

$$T\colon \begin{cases} a_1 \mapsto a_1^2 a_2 \\ a_2 \mapsto a_1 a_2 \\ a_3 \mapsto a_3 \end{cases}, \qquad A\colon \begin{cases} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 \\ a_3 \mapsto a_1 a_3 \end{cases}, \qquad B\colon \begin{cases} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 \\ a_3 \mapsto a_3 a_2 \end{cases}.$$

To interpret these as loops in the 1-skeleton of $L_3$ (and $K_3$) we note that $A = \lambda_{31}$ and $B = \rho_{32}$ are elementary transvections and $T$ is the composition of two elementary transvections: $T = \lambda_{21} \circ \rho_{12}$. Thus $w_i$ is the product of $8i + 4$ elementary transvections. There is a (connected) subcomplex of the 1-skeleton of $L_3$ spanned by roses (graphs with a single vertex) and Nielsen graphs (which have $(n \quad 2)$ loops at the base vertex and a further trivalent vertex). We say roses are adjacent if they have distance 2 in this graph.

Let $I \in L_3$ be the rose marked by the identity map $R_3 \to R_3$. Each elementary transvection $\tau$ moves $I$ to an adjacent rose $\tau I$, which is connected to $I$ by a Nielsen graph $N_\tau$. composition $\tau_1 \dots \tau_k$ of elementary transvections gives a path through adjacent roses $I, \tau_1 I, \tau_1\tau_2 I, \dots, \tau_1\tau_2 \dots \tau_k I$; the Nielsen graph connecting $\sigma I$ to $\sigma\tau I$ is $\sigma N_\tau$. Thus the word $w_i$ corresponds to a loop $\ell_i$ of length $16i + 8$ in the 1-skeleton of $L_3$. Theorem of [3] provides an exponential lower bound on the filling area of $\phi \circ \ell_i$ in $K_3$.

The square of maps in Lemma 3 ought to have many uses beyond the one in this note (*cf.* [7]). We mention just one, for illustrative purposes. This is a special case of the fact that every infinite cyclic subgroup of $\mathrm{Out}(F_n)$ is quasi-isometrically embedded [1].

PROPOSITION. — *The cyclic subgroup of* $\mathrm{Out}(F_n)$ *gener ted by ny Nielsen tr nsform tion (element ry tr nsvection) is qu si-isometric lly embedded.*

*Proof.* — Each Nielsen transformation is in the image of the map

$$\Phi\colon \quad \mathrm{ut}(F_2) \to \quad \mathrm{ut}(F_n) \to \mathrm{Out}(F_n)$$

given by the inclusion of a free factor $F_2 < F_n$. Thus it suffices to prove that if a cyclic subgroup $C = \langle c \rangle < \quad \mathrm{ut}(F_2)$ has infinite image in $\mathrm{Out}(F_2)$, then $t \mapsto \Phi(c^t)$ is a quasi-geodesic. This is equivalent to the assertion that some (hence any) $C$-orbit in $K_n$ is quasi-isometrically embedded, where $C$ acts on $K_n$ as $\Phi(C)$ and $K_n$ is given the piecewise Euclidean metric where all edges have length 1.

$K_2$ is a tree and $C$ acts on $K_2$ as a hyperbolic isometry, so the $C$-orbits in $K_2$ are quasi-isometrically embedded. For each $x \in L_2$, the $C$-orbit of $\phi_2(x)$ is the image of the quasi-geodesic $t \mapsto c^t.\phi_2(x) = \phi_2(c^t.x)$. We factor $\phi_2$ as a composition of $C$-equivariant simplicial maps $L_2 \to K_n \xrightarrow{\phi} K_2$, as in Lemma 3, to deduce that the $C$-orbit of $\phi_n\iota(x)$ in $K_n$ is quasi-isometrically embedded.

slight variation on the above argument shows that if one lifts a free group of finite index $\Lambda < \mathrm{Out}(F_2)$ to $\quad \mathrm{ut}(F_2)$ and then maps it to $\mathrm{Out}(F_n)$ by choosing a free factor $F_2 < F_n$, then the inclusion $\Lambda \hookrightarrow \mathrm{Out}(F_n)$ will be a quasi-isometric embedding.

## BIBLIOGR PHY


[1] E. LIBEGOVIC, "Translation lengths in Out($F$ )", *Geom. Dedicata* **92** (2002), p. 87-93.

[2] M. R. BRIDSON, "The geometry of the word problem", in *Invitations to geometry and topology*, Oxf. Grad. Texts Math., vol. 7, Oxford Univ. Press, Oxford, 2002, p. 29-91.

[3] M. R. BRIDSON & K. VOGTMANN, "On the geometry of the automorphism group of a free group", *Bull. Math Londres. Soc.* **27** (1995), p. 544-552.

[4] ———, " utomorphism groups of free groups, surface groups and free abelian groups", in *Problems on mapping class groups and related topics*, Proc. Sympos. Pure Math., vol. 74, mer. Math. Soc., Providence, RI, 2006, p. 301-316.

[5] M. CULLER & K. VOGTMANN, "Moduli of graphs and automorphisms of free groups", *Invent. Math.* **4** (1986), no. 1, p. 91-119.

[6] D. B. . EPSTEIN, J. W. CANNON, D. F. HOLT, S. V. F. LEVY, M. S. PATERSON & W. P. THURSTON, *Word processing in groups*, Jones and Bartlett Publishers, Boston, M , 1992, xii+330 pages.

[7] M. HANDEL & L. MOSHER, "Lipschitz retraction and distortion for subgroups of Out($F$ )", arXiv:1009.5018, 2010.

[8] . HATCHER & K. VOGTMANN, "Isoperimetric inequalities for automorphism groups of free groups", *Pacific J. Math.* **173** (1996), no. 2, p. 425-441.

[9] L. Mosher, "Mapping class groups are automatic", *nn. of Math. (2)* **142** (1995), no. 2, p. 303-384.

[10] R. Young, "The Dehn function of SL(n;ℤ)", arXiv:0912.2697v1, 2009.

Manuscrit reçu le 7 décembre 2010,
accepté le 8 février 2011.

Martin R. BRIDSON
Mathematical Institute
24-29 St Giles'
Oxford OX1 3LB (U.K.)
bridson@maths.ox.ac.uk

Karen VOGTM NN
Cornell University
Department of Mathematics
Ithaca NY 14853 (US )
vogtmann@math.cornell.edu