

1. RUTH CHARNEY AND KAREN VOGTMANN: AUTOMORPHISM
GROUPS OF RIGHT-ANGLED ARTIN GROUPS

We propose some problems concerning outer automorphism groups of right-angled Artin groups (“RAAG’s”). Since F_n and \mathbb{Z}^n are examples of RAAG’s, it is tempting to view outer automorphism groups of general right-angled Artin groups as interpolating between $Out(F_n)$ and $GL(n, \mathbb{Z})$, and to ask to what extent properties common to both $Out(F_n)$ and $GL(n, \mathbb{Z})$ are shared by all of these groups.

Recall that a RAAG A_Γ based on a simplicial graph Γ is generated by the vertices of Γ , and the only relations are that v commutes with w if v and w are joined by an edge of Γ . Servatius [Ser89] and Laurence [Lau95] gave generators for $Aut(A_\Gamma)$, but not much else is known about this group except in the cases when Γ has no edges (so A_Γ is free) and Γ is the complete graph (so A_Γ is free abelian). Laurence’s generators are of four types: (1) inner automorphisms, (2) symmetries of Γ and inversions of the generators v , (3) *partial conjugations*, which conjugate everything in some connected component of $\Gamma - st(v)$ by v , and (4) *transvections*, which multiply v by w (on the right or left) if $lk(v) \subseteq st(w)$. Thus every element of $Out(A_\Gamma)$ lifts to an automorphism of the free group on the vertices of Γ , but the natural map from $Out(A_\Gamma)$ to $GL(n, \mathbb{Z})$ is not usually surjective.

There is a CAT(0) cube complex associated to any RAAG, whose 1-skeleton is the Cayley graph of the group, and which has a k -dimensional cube whenever the 1-skeleton of the cube appears (kind of a “cube-flag” complex, see [CD95]). The RAAG acts freely on this; the quotient has a loop for each generator and a k -torus for each complete subgraph on k vertices in Γ . This cube complex is 2-dimensional if and only if the graph Γ has no triangles. In this case, we have constructed an “outer space” for $Out(A_\Gamma)$, which is a contractible space on which $Out(A_\Gamma)$ acts with finite stabilizers [CCV]. Points in this outer space are morally actions of A_Γ on 2-dimensional CAT(0) complexes, though the actual description is in terms of products of trees. This outer space is finite-dimensional, and we obtain

Theorem 1.1. *If Γ is connected and triangle-free, then $Out(A_\Gamma)$ contains a torsion-free subgroup of finite index and it has finite virtual cohomological dimension.*

Although we obtain both upper and lower bounds on the virtual cohomological dimension, these bounds in general do not match, so we ask

Question 1.2. *What is the exact virtual cohomological dimension of the outer automorphism group of a 2-dimensional right-angled Artin group?*

The no-triangles condition on Γ was very convenient, but of course we would like to know what happens for any Γ :

Question 1.3. *Do the outer automorphism groups of all right-angled Artin groups have torsion-free subgroups of finite index?*

Question 1.4. *Calculate the virtual cohomological dimension of the automorphism group of any right-angled Artin group.*

REFERENCES

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