Outer space and Automorphisms of free groups

LECTURE 7

Change of time next week

<u>Last time</u>, showed how to compute

$H_*(\mathrm{Out}F_n; \mathbb{Q}) = H_*(K_n/\mathrm{Out}F_n; \mathbb{Q})$

using a chain complex derived from the cube complex structure of K_n

<u>Today</u>: connection of $H^*(K_n/\mathrm{Out}F_n; \mathbb{Q})$ with Kontsevich's graph homology.

<u>All</u> homology will be with trivial coefficients in a field of char. 0, so we will omit the coefficients in the notation

$H_*(K_n/\mathrm{Out}\,F_n)$ is the homology of the chain complex C_* with $C_k = \bigoplus_{(G,\Phi)} \mathbb{Q}$

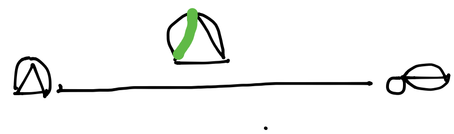
<u>where</u>
- G connected, $\pi_1 G \cong F_n$, all vertices have valence ≥ 3
- Φ is a forest in G with k edges $e_1, \ldots, e_k$
- (G, Φ) has no orientation-reversing automorphisms

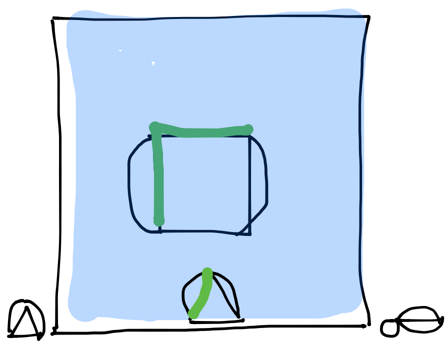
and
- $$\partial(G,\Phi) = \sum_{i=1}^{k} (-1)^{i+1} (G, \Phi - e_i) + (-1)^i (G_{e_i}, \Phi_{e_i})$$
$$= \partial_R(G,\Phi) + \partial_C(G,\Phi)$$

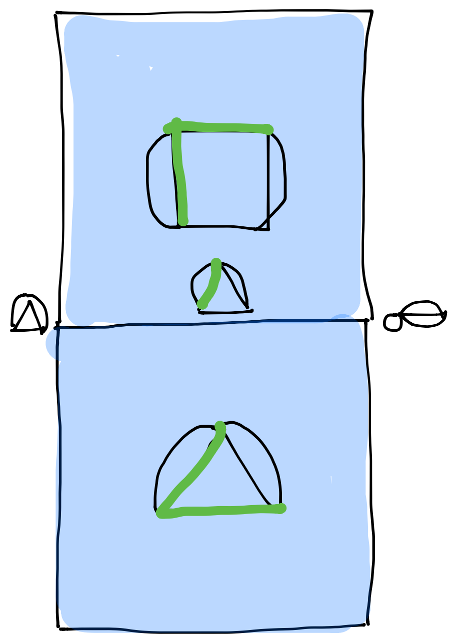
∂_R "removes edges from Φ"

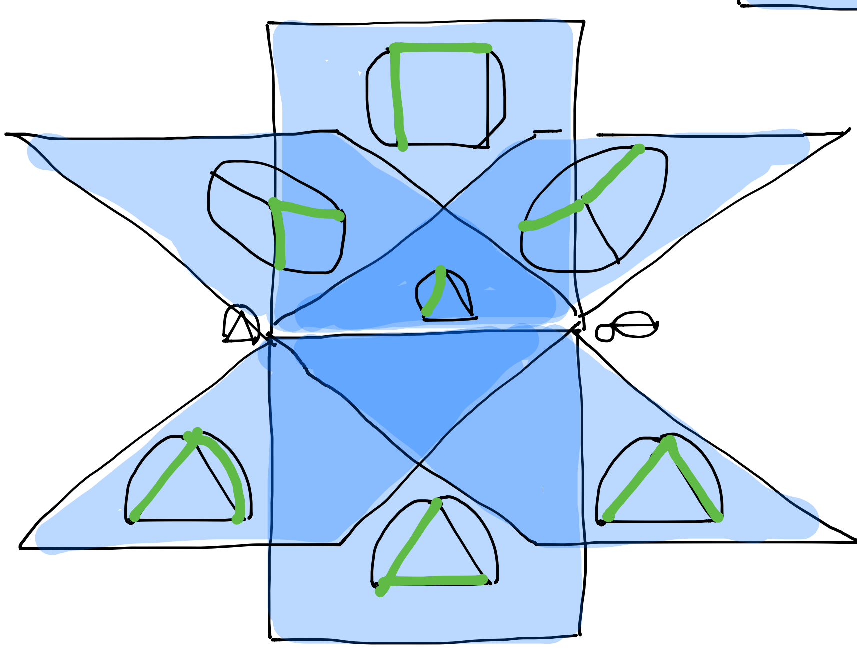
∂_C "collapses edges of Φ"

To make the connection with Kontsevich's graph homology, need to simplify our chain complex. And, turns out it's more convenient to use the <u>cochain</u> complex.

Coboundary of $\sigma =$ 







<u>Cochain complex</u> A cube (G, Φ) is

a codimension 1 face of cubes

$(G, \Phi \cup e)$ and (G^α, Φ^α)

<u>where</u>: • e is any edge which can be added to Φ

without forming a cycle

• α blows up some vertex of G into 2

vertices, and Φ^e contains the new edge.

$$\delta(G, \Phi) = \delta_R(G, \Phi) + \delta_C(G, \Phi)$$

$$= \sum_e (G, \Phi \cup e) + \sum_\alpha (G^\alpha, \Phi^\alpha)$$

where $\Phi \cup e$ and Φ^α are oriented appropriately

In sphere system model, $G \longrightarrow S_G \subset M_n$

S_G a simple sphere system, spheres $\longleftrightarrow$ edges of G

Φ a forest $\longleftrightarrow$ sub-system $S_\phi \subset S_G$ (not simple)

$$(G, \Phi) \longleftrightarrow (S_G, S_\Phi)$$

Then

$$(G^\alpha, \Phi^\alpha) \longleftrightarrow (S_G \cup s_\alpha, S_\phi \cup s_\alpha)$$

so $\delta_c(G, \Phi) = \delta_c(S_G, S_\phi)$

$$= \sum_{\substack{s_\alpha \text{ compatible} \\ \text{with } S_G}}' (S_G \cup s_\alpha, S_\phi \cup s_\alpha)$$

We have

$$C_k = \bigoplus_{\substack{(G,\Phi) \\ e(\Phi)=k}} \mathbb{Q} = \bigoplus_{e(\Phi)=k} (G,\Phi) \qquad [\text{Notation}]$$

We can decompose this further by # vertices of G

$$C_k = \bigoplus_{v>k} C_{v,k}$$

where $C_{v,k} = \bigoplus_{\substack{e\Phi=k \\ vG=v}} (G,\Phi)$

Top-dim'l cubes are

$$(G,T) = (\text{trivalent graph, maximal tree})$$

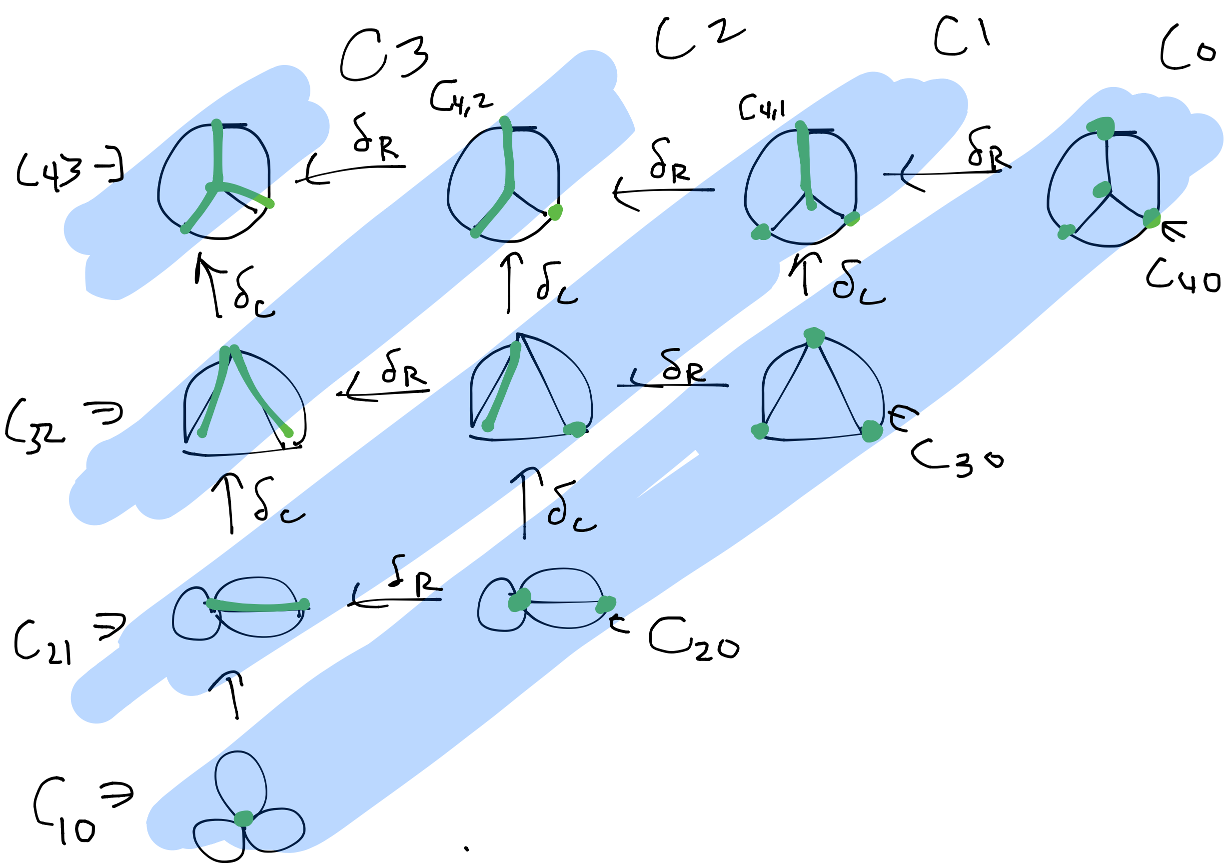
$$(vG = 2n-2 \ , \ eT = 2n-3)$$

next are $\quad v(G) = 2n-2 \ , \ e\Phi = 2n-4$

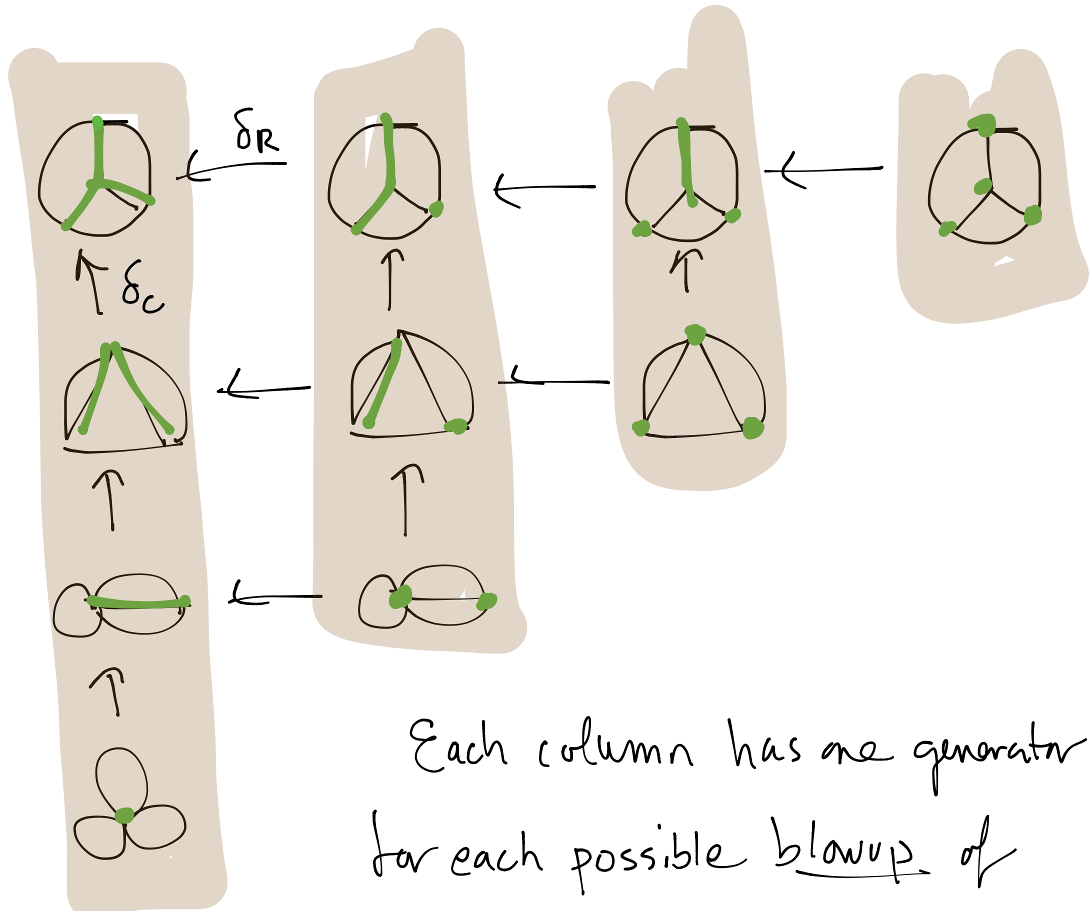
and $vG = 2n-3 \ , \ e\Phi = 2n-4$

Arrange the $C_{v,k}$ in a grid

Draw a sample (G,Φ) at each spot to keep this straight:



We can compute $H^*(C_*)$ by first computing the <u>vertical</u> homology, <u>then</u> the horizontal H_* (this is spectral sequences, but don't tell anybody)



Each column has one generator for each possible <u>blowup</u> of the graphs G at the bottom

Different blowups may result in isomorphic graphs, e.g. if G has an automorphism

If these were <u>marked</u> graphs, then the complex of blowups is $lk_{>G}$ in K_n, which we know is $\simeq \bigvee S^{2n-3-v(G)}$

Since they are <u>unmarked</u> graphs, we need to quotient $\mathcal{K}_{>G}$ by its stabilizer, which is $\cong \mathrm{Aut}(G)$

$\mathcal{K}_{>G}$ has no homology except in the top dimension

So same is true for $\mathcal{K}_{>G}/\mathrm{Aut}(G) \subset K_n/\mathrm{Out}(F_n)$

[pf uses equivariant H_* spectral sequence:

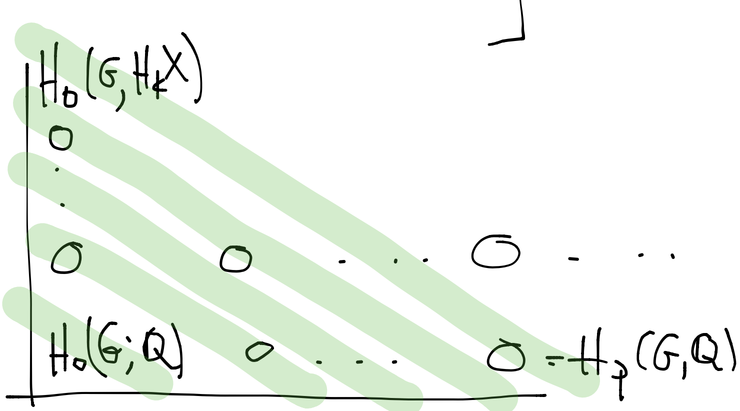
$X \simeq \vee S^k$, G finite acting on X cellularly

$$E^2_{p,q} = H_p(G; H_q X) \Rightarrow H_{p+q}(X/G)$$

Since $H_q X = 0$ unless $q = 0$ or k and

$H_p(G;\mathbb{Q}) = 0$ for $p>0$, Get $H_{p+q}(X/G) = 0$ for $0 < p+q < k$

(also $H_{p+q}(X/G) = 0$ for $p+q > k$)]



So after taking vertical H^*, co-chain complex becomes

$$0 \leftarrow \bigoplus_{e(T)=2n-2} (G,T) \big/ \mathrm{Im}\,\delta_c \xleftarrow{\delta_R} \bigoplus_{e(\Phi)=2n-3} (G,\Phi) \big/ \mathrm{Im}\,\delta_c \xleftarrow{\delta_R} \cdots \leftarrow \cdots \bigoplus_{e(\Phi)=0} (G,-)$$

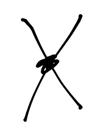
Where G is always trivalent.

Much smaller chain complex!

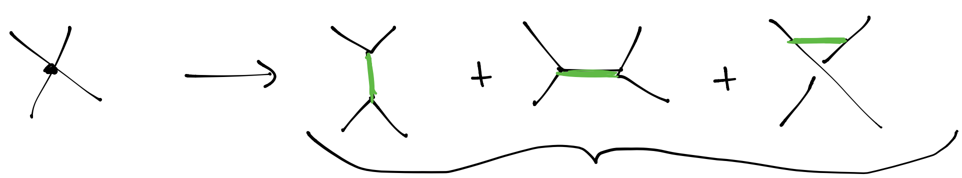
$$\delta_c : C_{2n-3,k-1} \longrightarrow C_{2n-2,k}$$

sends $(G,\Phi) \to \sum_\alpha (G^\alpha, \Phi^\alpha)$

G^α trivalent $\Longrightarrow$

G is trivalent except at one 4-valent vertex v 

There are 3 ways of expanding this into two trivalent vertices. The new edge becomes part of the forest.



This is called an IHX-vector.

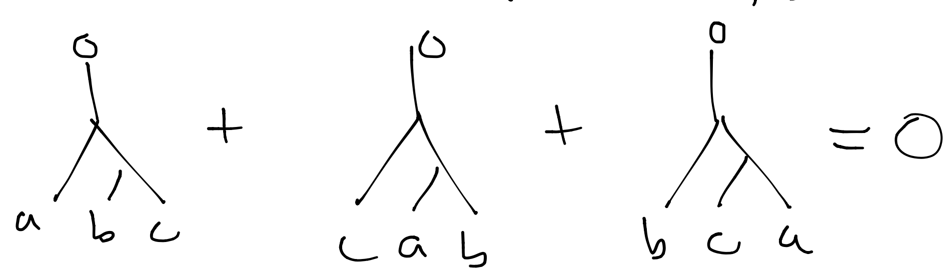
So our chain complex is

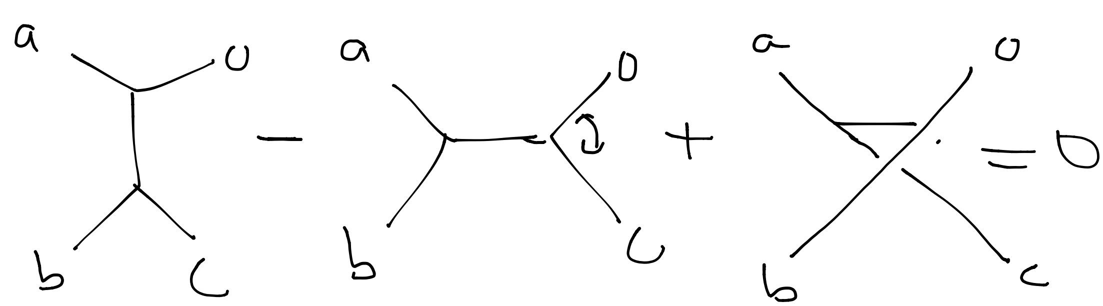
$$0 \leftarrow \bigoplus_{e(T)=2n-3} (G,T)\Big/\text{IHX} \longleftarrow \bigoplus_{e\Phi=2n-4} (G,T-e)\Big/\text{IHX} \longleftarrow \cdots$$

This is the first hint that $H^*(\mathrm{Out}F_n)$ is related to Lie algebras: (actually the Lie <u>operad</u>)

Modding out by IHX is an encoding of the <u>Jacobi identity</u>:

$$[a,[b,c]]+[c,[a,b]]+[b,[c,a]]=0$$





Kontsevich's theory

We now turn from something easy to understand (graphs) to something harder to understand (algebra!). Aim: convert algebras into graphs.

$V_d = \mathbb{R}^{2d}$ equipped with standard symplectic form $\langle v, w \rangle$, ie

$\mathbb{R}^d$ has basis $B = \{p_1, \ldots, p_d, q_1, \ldots q_d\}$

$\langle p_i, q_i \rangle = 1 = -\langle q_i, p_i \rangle$ all other $\langle x, y \rangle = 0$ for $x, y \in B$.

C_d = free commutative algebra on B
= polynomials in the p_i, q_i with real coeffs.

A_d = free associative algebra on B
= polynomials in non-commuting variables
p_i, q_i

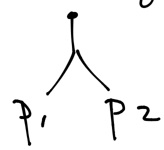
L_d = free Lie algebra on B: generated by
bracket expressions in the p_i, q_i
(eg $[p_1, p_2]$ or $[[p_1,[q_2,p_1]],[q_2,q_4]]$)

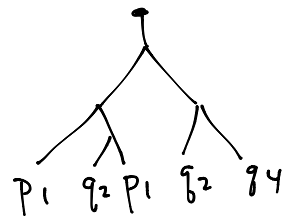
<u>modulo</u> antisymmetry $[X,Y] = -[Y,X]$
Jacobi identity
$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

Now turn these into algebras of labelled, rooted graphs: Start w/ L_d

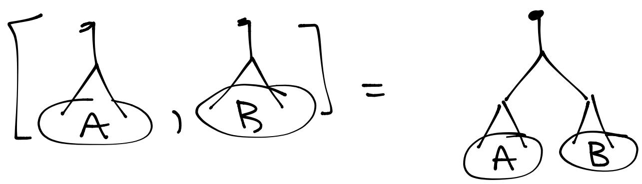
Ld:

bracket expression $\longleftrightarrow$ rooted planar binary tree

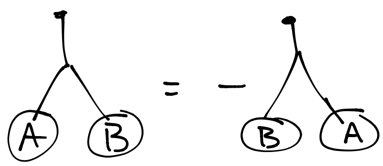
$[p_1, p_2] \longleftrightarrow$ 

$[[p_1,[q_2,p_1]],[q_2,q_4]] \longleftrightarrow$ 

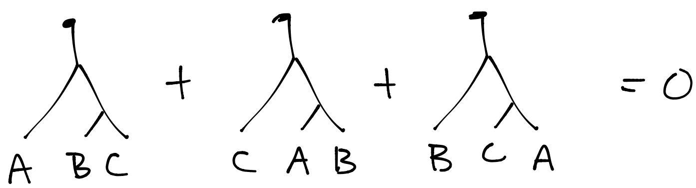
Bracket:

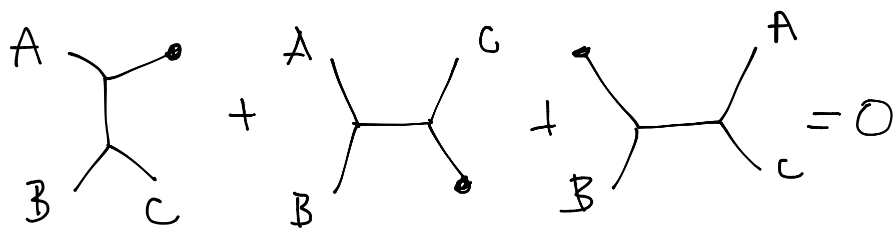


Anti-symmetry

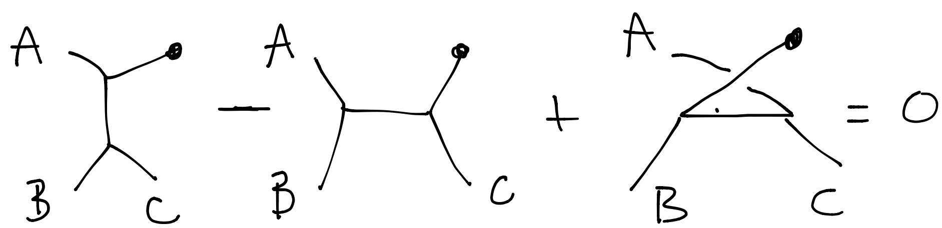


Jacobi:

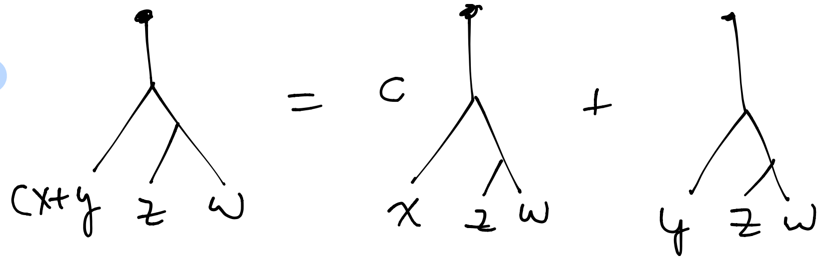




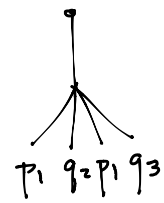
IHX:



multi Linearity

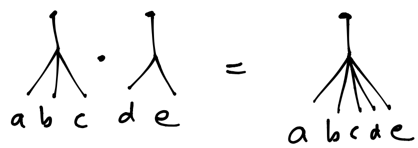


A_d generated by planar rooted labelled star-graphs:



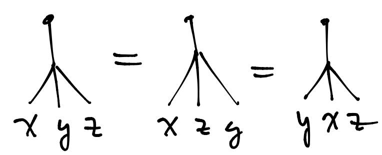
$\longleftrightarrow$ monomial $p_1 q_2 p_1 q_3$

Product :



C_d: generated by rooted trees (no planar embedding)

(so



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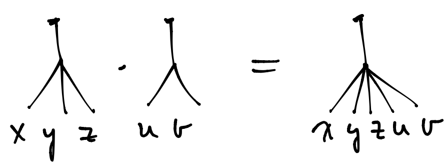
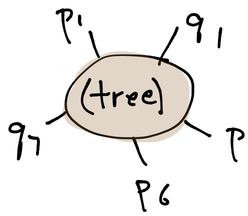
Product:



Let $H_d = C_d, A_d$ or L_d

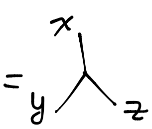
Define h_d = Lie algebra based on H_d generated by (appropriate) trees with <u>all</u> leaf vertices labelled (so there is no root)



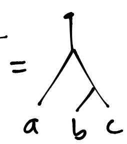
:= (Lie, commutative or associative) <u>Spider</u>

<u>Claim</u>: A spider corresponds to a <u>derivation</u> $D: H_d \to H_d$, ie $D(A * B) = DA * B + A * DB$, where $*$ is product ($H_d = A_d$ or C_d) or bracket ($H_d = L_d$)

<u>eg</u> $H_d = L_d$, $D =$

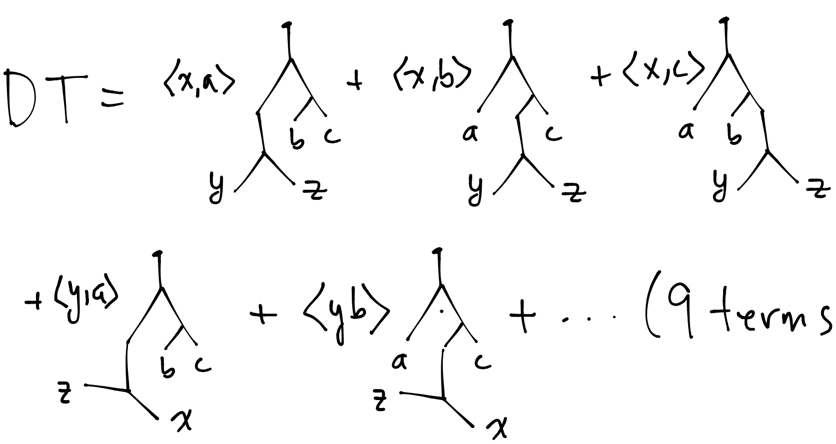


, $T =$



$\varepsilon\ L_d$

$DT =$

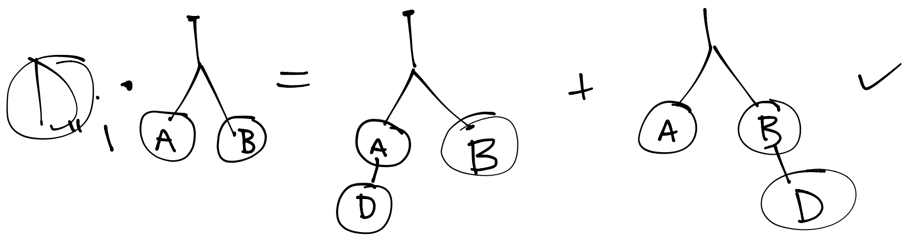


+ … (9 terms in all)

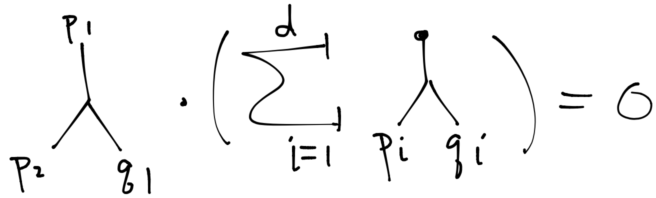
ie: sum $\langle x, a \rangle \cdot$ Result of "hanging D off T"

over all ways of pairing a leg x of D with a leg a of T

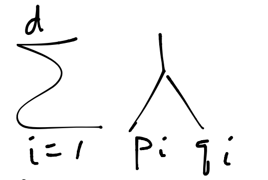
It's clear that D is a derivation:



Exercise:



More ambitious exercise: $w =$



then $Dw = 0$ for any $D \in \mathfrak{h}_d$.

Exercise: What does "hanging D onto T" mean for $H_d = A_d$ or C_d?

Another definition of $\mathfrak{h}_d$

= derivations of H_d killing ω,

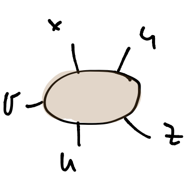
where $\omega = \sum [p_i, q_i]$ if $H_d = L_d$

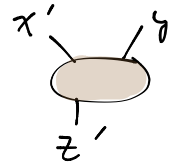
$\omega = \sum p_i q_i - q_i p_i$ if $H_d = A_d$

$\omega = \sum dp_i \wedge dq_i$ if $H_d = C_d$

Derivations of an algebra form a Lie algebra by defining $[D, D'] = D \circ D' - D' \circ D$

We can describe the bracket using graphs as follows:

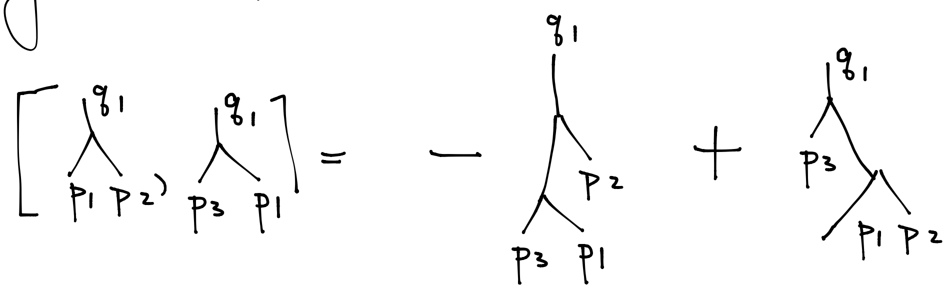
Let $S_i =$ 

, $S_i' =$ 

be spiders in H_d. Given a leg $x \in S_i$ and $x' \in S_i'$, can <u>mate</u> S_i and S_i' by gluing x_i to x_i', contracting the new edge if $H = A$ or C, and multiplying by $\langle x_i, x_i' \rangle$

Then $[\delta_i, \delta_i']$ is the sum of all possible matings.

<u>eg in L_d</u>:



(all other terms have coefficient 0)

$[A, B] = -[B, A]$ since each coefficient

$\langle x, x' \rangle$ becomes $\langle x', x \rangle = -\langle x, x' \rangle$

Jacobi identity is also true, but requires more effort to verify

The natural embedding $V_d \hookrightarrow V_{d+1}$ induces inclusions $\mathfrak{h}_d \hookrightarrow \mathfrak{h}_{d+1}$ (we're just allowing more labels). Define $\mathfrak{h}_\infty = \varinjlim \mathfrak{h}_d$

Kontsevich's Theorem:

$H_d = L_d$. Then

"$H_* \mathfrak{l}_\infty$ computes $H^*(\mathrm{Out}\, F_n)$ for all n"

More precisely:

$$P H_* \mathfrak{l}_\infty \cong H^*(\mathfrak{sp}_\infty) \oplus \bigoplus_{n \geq 2} H^*(\mathrm{Out}\, F_n)$$

$H_* \mathfrak{l}_\infty$ is a Hopf algebra, this is the "primitive part"

$\mathfrak{l}_\infty$ contains a copy of $\mathfrak{sp}_\infty$, corresp. to 2-legged spiders