Outer space and Automorphisms of free groups

LECTURE 8

Last time we

1. Found a smaller chain complex which computes $H^*(\mathrm{Out}\,F_n)$

2. Defined a Lie algebra of derivations of the free Lie algebra

1. $C_k = \mathbb{Q}$-vector space generated by pairs (G, Φ)

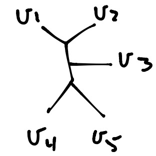
with $\begin{cases} G \text{ connected, trivalent, rank } n \\ \Phi \subset G \text{ a forest with } k \text{ (ordered) edges} \end{cases}$

modulo anti-symmetry and IHX relations

$\delta : C_k \longrightarrow C_{k+1}$ is $\delta = \delta_R + \delta_C$

$$(G, \Phi) \longmapsto \underbrace{\sum (G, \Phi \cup e)}_{\delta_R} + \underbrace{\sum (G^\alpha, \Phi^\alpha)}_{\delta_C}$$

(sign: new edge is last in ordering)

2. $\mathfrak{h}$ is generated by spiders 

= binary trees, leaves labelled by vectors in V = symplectic vector space $/$ AS, IHX

$$[S, S'] = \sum_{\substack{l \in S \\ l' \in S'}} \text{mate } S \text{ and } S' \text{ using } l \text{ and } l'$$

Claim. $H_*(\mathfrak{h})$ is closely related to $H^*(\mathrm{Out}\, F_n)$

How do you compute homology of a Lie algebra

(and why is it defined this way?)

Answer: Lie algebra = tang. space to id in a Lie group

= linear approximation to the Lie group.

If the Lie group is compact & simply-connected, it is determined by its Lie algebra, so you should be able to compute its cohomology from the Lie algebra, too

Lie algebra cohomology was defined to do this

H^* Lie group = de Rham cohomology

ie: Chains are differential forms $f\, dx_1 \wedge \cdots \wedge dx_k$

etc.

This motivates defn of Lie algebra (co)homology

$\mathfrak{h}_d$ = Lie algebra $\quad C_k = \Lambda^k \mathfrak{h}_d$

$$\partial : C_k \longrightarrow C_{k-1}$$

$$x_1 \wedge \cdots \wedge x_k \longmapsto \sum_{i<j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k .$$

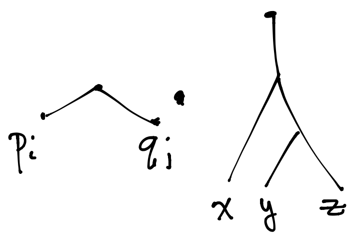
Doesn't look much like our cochain complex for Out F_n!

Trick: $\mathfrak{h}_d$ contains a copy of $\mathfrak{sp}_d$

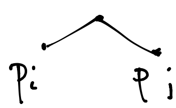
= 2-legged spiders

Recall $\mathfrak{sp}_{2d}$ = matrices A w/ ${}^tAJ + JA = 0$,

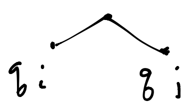
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$



· replaces q_i by q_j, p_j by $-p_i$ $\longleftrightarrow \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$



$\longmapsto \begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}$



$\longmapsto \begin{pmatrix} 0 & -E_{ij}-E_{ji} \\ 0 & 0 \end{pmatrix}$

and these generate sp_{2d}

<u>Next note</u>: 2-legged spiders act on h_d, (preserving # of legs. in fact):

$$s_0 \cdot s_1 \wedge \dots \wedge s_k = \sum_{i=1}^{k} [s_0, s_i] \wedge s_1 \wedge \dots \wedge \hat{s}_i \wedge \dots \wedge s_k$$

So $C_k = \Lambda^k h_d$ is an sp_{2d}-module

<u>Exercice</u> ∂ commutes with the sp-action, so (C_k^{sp}, ∂) is a sub-chain complex (C_*^{sp} = chains killed by sp-action)

Prop $H_* C_k = H_*(C_k^{sp})$

Proof (deferred) requires fact that sp_{2d} is simple.

So we just need to figure out the invariants in $C_k = \Lambda^k \mathfrak{h}$

Actually, we don't need to figure them out, Weyl already did it!:

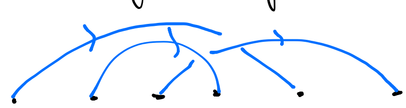
V = symplectic vector space

$(V \otimes V)^{sp}$ is 1-dimensional, gen. by $\sum_i p_i \otimes q_i - q_i \otimes p_i := \omega$

$(V^{\otimes k})^{sp} = 0$ unless k is even, $k = 2l$. Then $\underbrace{\omega \otimes \ldots \otimes \omega}_{l}$ is an invariant. Shorthand notation for this:

$\omega \otimes \omega \otimes \omega =$ 

Any such chord diagram gives another invariant, ed

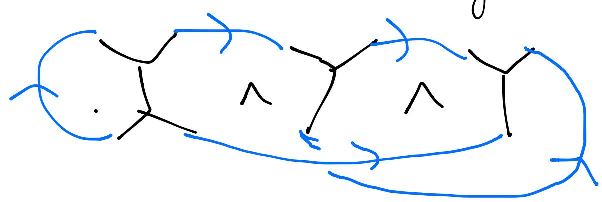


permutes all terms of $\omega \otimes \omega \otimes \omega$ by $\sigma = (2\,5\,3)$

Weyl's theorem says these spa the entire space of invariants.

This translates to our situation as follows:

Invariants of $\Lambda^k \mathfrak{h}$ $\longleftrightarrow$ wedges of spider whose legs are paired.



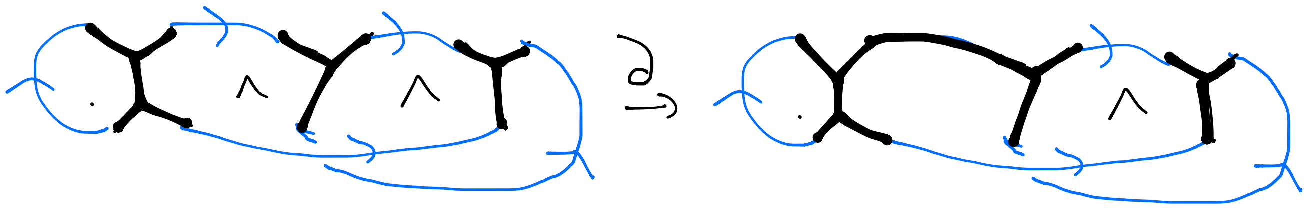
<u>The invariant in $\Lambda^k \mathfrak{h}$ is a (large!) sum:</u> a wedge of spiders is in the sum if the labels on the paired edges are a

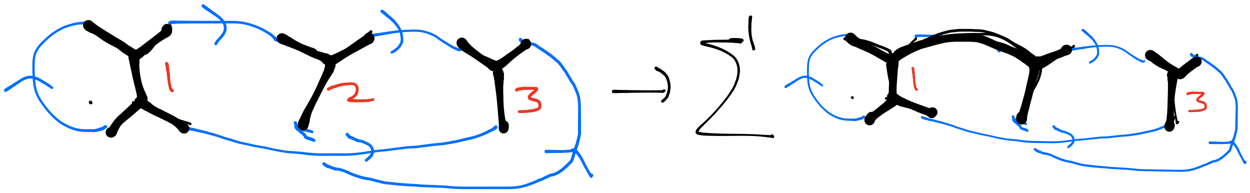
p_i-q_i pair: $\quad p_i \longrightarrow q_i \quad$ or $\quad q_i \longrightarrow p_i$

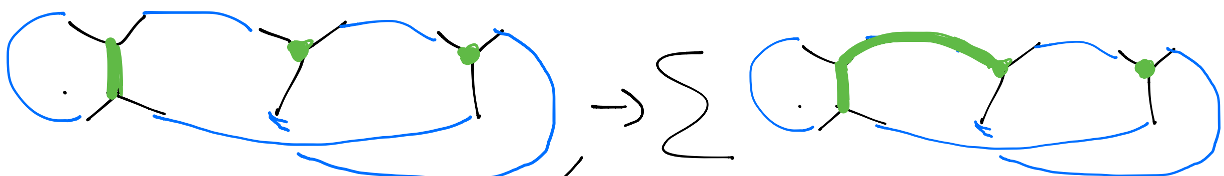
<u>Sign</u> of the term: each edge gets a sign $\quad p_i \xrightarrow{+} q_i$

sign of term $= \prod_e \text{sign}(e) \quad\quad q_i \xrightarrow{-} p_i$

Relationship with our old chain complex:





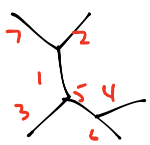


$$(G, \Phi) \xmapsto{\delta_R} \sum (G, \Phi \cup e)$$

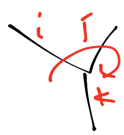
Prop The orientation on (G, Φ) given by ordering the trees in Φ, choosing a planar embedding of each tree in Φ and orienting the remaining edges is equivalent to that given by ordering the edges of Φ.

Proof is not transparent, recipe is complicated
basically takes orientation on the space of half-edges of G and groups the half-edges to produce canonical orientations of both types.

hint:



w/ cyclic ordering at each vertex



if $i < j < k$

Now we want to compute $H_*(h_d)$ ... BUT
Actually, this doesn't quite work: we need to take a limit $h_\infty = \lim_{d \to \infty} h_d$,
ie we allow ourselves an infinite supply of distinct vertex labels.

What is the issue?

We have a map

$$\Phi_d : \mathcal{G}_k \longrightarrow \Lambda^k \mathfrak{h}_d$$

$$(G, \Phi) \longmapsto \text{sp-invariant}$$

There is also a map

$$\Psi : \Lambda^k \mathfrak{h}_d \longrightarrow \mathcal{G}_k \ :$$

pair legs of $S_1 \wedge \cdots \wedge S_k$ in all possible ways,

multiply result by

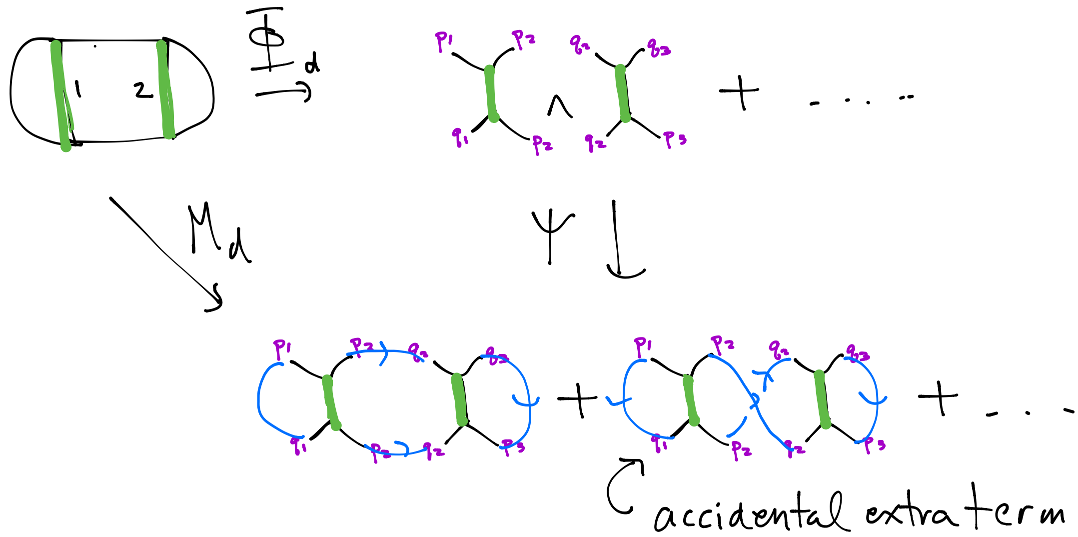
$$\prod_{l_i, l_j \text{ paired}} \langle v_i, v_j \rangle$$

v_i = label on leg l_i

Looks like an inverse to Φ_d

But $\Psi \circ \Phi_d = M_d$ is <u>not</u> (a multiple of) Id:

there are <u>accidental</u> pairings



Thm: d sufficiently large $\Rightarrow$ M_d is an isomorphism

$$\text{So}\quad \mathfrak{g}_d \hookrightarrow \Lambda^k \mathfrak{h}_d \twoheadrightarrow \mathfrak{g}_d$$

$$\uparrow$$

(Weyl) image $= (\Lambda^k \mathfrak{h}_d)^{sp}$

$$\text{So}\quad H_*(\mathfrak{g}_*) = H_*(\Lambda^* \mathfrak{h}_\infty)$$

$$\|$$

$$\bigoplus_n H_*(\mathrm{Out}\, F_n) \oplus H_*(sp_\infty)$$

Now: let's find some homology!

(actually slightly more natural to look for cohomology ...)

Abelianize

$$\mathfrak{h} \longrightarrow \mathfrak{h}_{ab}$$

Lie algebra, $[\,,\,]=0$

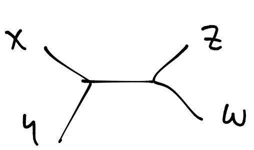
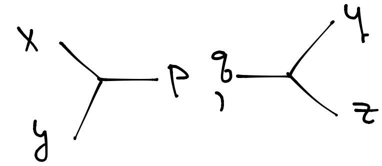
Get backwards map $H^*(\mathfrak{h}_{ab}) \longrightarrow H^*(\mathfrak{h})$

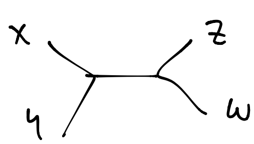
Since $[\,,\,]=0$ on $\mathfrak{h}_{ab}$, $\partial = 0$ ($\Rightarrow \delta = 0$)

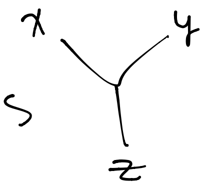
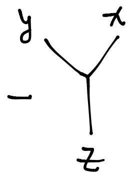
$$\Rightarrow H^k(\mathfrak{h}_{ab}) = \Lambda^k \mathfrak{h}_{ab} \longrightarrow H^*(\mathfrak{h}) = \bigoplus H_*(\mathrm{Out}\, F_n)$$

Now have to find elements of $\mathfrak{h}_{ab}$

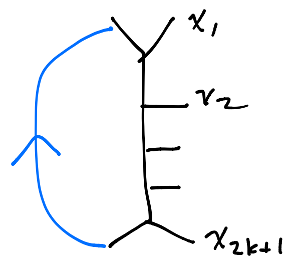
In $\mathfrak{h}_{ab}$, brackets are 0

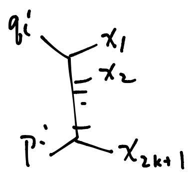
eg  $=$ $\Big[$  $\Big]$

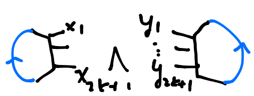
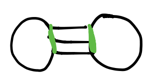
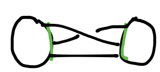
$\Rightarrow$  $= 0$ in $\mathfrak{h}_{ab}$

So the only tree in $\mathfrak{h}_{ab}$ is  $= -$  $= \cdots$

So $\mathfrak{h}_{ab}$ contains a copy of $\Lambda^3 V$

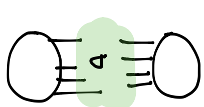
$\mathfrak{h}_{ab}$ also contains 

$= \sum_{i \geq 1}$ 

So $\Lambda^2 \mathfrak{h} \ni$  $\longrightarrow$  $+$ 

$+ \cdots$

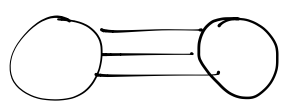
$\in C_{4k}(\mathrm{Out}F_{2k+2})$

$\sum_\sigma$  $= \mu_k = k^{\text{th}}$ Morita class

Conjecture: μ_k represents a non-trivial homology class (true for $k=1,2,3$)

Now that we know where to look, we can see μ_k directly in the quotient of Outer space by $Out(F_n)$, ie in the moduli space of metric graphs Q_n.

eg $\mu_1 \in Q_4$ is the top-dimensional class of a manifold $M_4 \subset Q_4$. M_4 is the image of a torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$

Each S^1 is a circle of metric graphs, obtained by moving the attaching point of a horizontal edge in  around one of the circles.

There are symmetries in this picture $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ so the torus is not embedded in Q_4, just the quotient $M_4 = T^4 / \mathbb{Z}/2 \times \mathbb{Z}/2$.

Note dim $\mathcal{O}_4 = 3\cdot 4 - 3 = 9$, so it is possible that M_4 bounds, ie is trivial in homology

In fact M_4 is <u>not</u> a boundary, but we don't have a general proof which works for all M_k.