Intro to graph homology.

= homology of graph complex.

What IS a graph complex?

Why study them?

Short answers

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(1) Chain complex $\to C_k \xrightarrow{d} C_{k-1} \to \cdots$

generators are isomorphism classes of finite graphs.

differential given by collapsing edges.

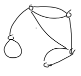
$\Rightarrow$ very combinatorial!

(2) finite graphs parameterize many mathematical objects,

$\Rightarrow$ many questions can be reduced to questions about graph complexes

Slightly longer answers

(1) Graph = finite 1-dimensional cell complex G



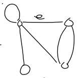
G^0 = vertices

G^1 = edges

(not necess connected)

C_k generated by graphs with k vertices

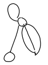
$d: C_k \to C_{k-1}$?



$G =$

$e \in G^1$, not a loop

$\rightsquigarrow G_e$ = shrink e to a point G_e =



Want to say $dG = \sum_{\substack{e \in G^1 \\ \text{not loops}}} G_e$

Problem don't have $d^2 = 0$.

Solution: orient graphs

Two ways to do this:

(1) order the edges of G.

say two orderings are equivalent if they differ by an even permutation.

Say $(G, or) = -(G, -or)$

oriented graphs generate C_k modulo

Another way to say this.

Convention for the whole course: fix a field $\mathbb{K}$ of characteristic 0 ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$)

$\bigwedge^{e(G)} \mathbb{K}^{e(G)}$ $\qquad e(G) = \#$ edges of G

$\cong \mathbb{K}$, gen by $e_1 \wedge \cdots \wedge e_{e(G)}$

or is a unit vector in $\bigwedge^{e(G)} \mathbb{K}^{e(G)} = \det \mathbb{K}^{e(G)}$

(± 1)

$(G, or) = -(G, -or)$.

$d(G, or) = \sum_i (-1)^i (G_{e_i}, \overline{or})$

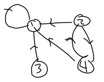
$G^1 = \{e_1, \ldots, e_q\}$

Collapsing e induces an ordering on G_e^1

Exercise: $d^2 = 0$. ("even" orientation)

2nd orientation:

order the vertices G^0 and orient the edges, ie order the half-edges $H(e)$ of ~~t~~ each edge e.



or is an $\sim$ of these data
Two are $\sim$ if differ by even no of vertex label transpositions and edge flips

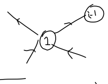
ie or is a unit vector in
$$\det \mathbb{k}^{v(G)} \otimes \bigotimes_{e \in G^1} \det \mathbb{k}^{H(e)}$$

$C_* = \text{gen by } (G, or) / (G,or) = -(G, \bar{or})$

$\overline{(G, or) \to (G_e, ?)}$

Choose a representative of or with

define or on G_e to be Image of e is labeled 1
reduce labels on other vertices by 1.



Lemma: $d(G) = \sum_{\substack{e \\ \text{not loops}}} (G_e, \bar{or})$

satisfies $d^2 = 0$.

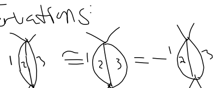
Exercise: For orientation (1) could do choose rep of or with $\overset{e=1}{\longleftarrow}$ then remaining edges have labels reduced by 1 after collapse $\overline{or}$ $d(G) = \sum_{\substack{e \text{ not a} \\ \text{loop}}} (G_e, \bar{or})$

Claim $d^2 = 0$.

"odd" orientation

Easy observations:

even or:



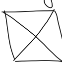
$\Rightarrow (G, or) = -(G, or) \Rightarrow (G, or) = 0$.

No gens for graphs w/ multiple edges.

odd or:

$\Rightarrow$ no gens for graphs with loops.

Exercise:



$= 0$ in both orientations

Easy observations:

* $G \to G_e$ $\Rightarrow$ If G has no leaves, vertices in G_e have valence $\geq$ vertices in G.
* G connected $\Rightarrow G_e$ is connected.
* $\chi(G) = v(G) - e(G) = \chi(G_e)$.

If G is connected $\chi = 1 - \operatorname{rank} \pi_1 G$
so $G \to G_e$ preserves $\operatorname{rank} \pi_1 G$

Define G admissible if all vertices have valence ≥ 3, and connected.

$A_* \subset C_*$ = subchain cplex generated by admissible graphs.

$$A_* = \bigoplus_n A_*^n$$

A_*^n gen by graphs w/ $\pi_1 \cong F_n$

Graph complexes introduced by
Kontsevich 1993
(1) Formal (non)-commutative symplectic geometry
(2) Feynman diagrams and low-dimensional topology 1994.

to study ~~deformation quantization~~;
Motivated by physics:
In classical mechanics, study the phase space of a system (coords for position and velocity of each point). M
M is a symplectic manifold. Want to study functions on M (eg energy - aka Hamiltonian)
$C^\infty(M)$ = commutative algebra.

$M \longrightarrow \mathbb{R}$

$C^\infty(M)$ has a poisson bracket

$M \cong \mathbb{R}^{2n}$ w/ coords $(\underbrace{q_1, \ldots, q_n}_{\text{posn}}, \underbrace{p_1, \ldots, p_n}_{\text{velocity}})$

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

Lie algebra
$\{f,g\} = -\{g,f\}$ anti-symmetry
$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$
Jacobi identity.

$D_f : C^\infty M \longrightarrow C^\infty M$ is a derivation
$g \longmapsto \{f,g\}$

$D_f(gh) = D_f g \cdot h + g \cdot D_f h$
$\{f, gh\} = \{f,g\}h + g\{f,h\}$.

In quantum mechanics replace M by a Hilbert space H
replace $C^\infty(M)$ by operators on H.
the algebra of operators is not commutative

Want "Poisson bracket" on this non-commutative algebra.
Want: A = comm. algebra, w bracket $\{\,,\}$
$A((\hbar))$ = algebra of formal power series.
define a new operation $f \star g$ on $A((\hbar))$
st. $f \star g = fg + O(\hbar)$, associative, unital
and $[f,g] = f\star g - g \star f = \{f,g\} + O(\hbar^2)$
$(A((\hbar)), \star)$ is "algebra of fcns on a non-comm. mfld"

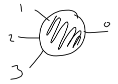
K. used graph complexes and Feynman integrals to define $\star$ with all the properties he wanted.

Want to decorate graph complexes with (non)-commutative objects, get Lie algebras

cyclic operads.

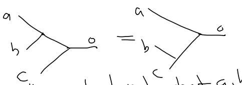
"commutative" operad Comm.
"associative" operad Ass
"Lie" operad Lie.

Operads: set of black boxes that take ordered inputs and spit out an output of the same type

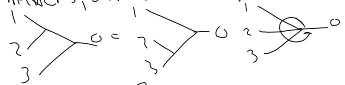


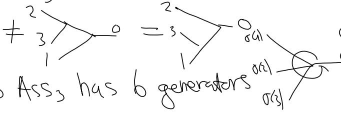
Operad = smile w/o the Cheshire cat.

<u>Ass:</u> $(a \cdot b) \cdot c = a \cdot (b \cdot c)$



order matters, but not what a, b, c, o are.

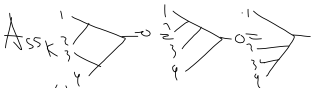




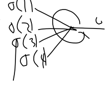
$\sigma \in \Sigma_3$

So Ass_3 has 6 generators

for $k > 3$ Ass_k

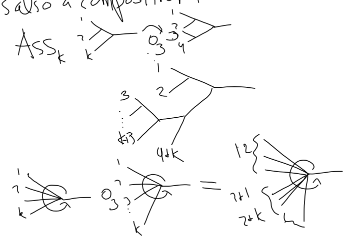


Gen of Ass_4 =



There's also a composition

Ass_k



Axioms for an operad:

<u>Sets P_k</u>, action of Σ_k on P_k.

<u>composition</u> | <u>an element 1 in P_1</u>

$\theta \in P_k,\ \theta_1 \in P_{i_1}, \dots, \theta_k \in P_{i_k}$

$(\theta, \theta_1, \dots, \theta_k) \longrightarrow \theta \circ (\theta_1, \dots, \theta_k)$

$\in (P_k\ P_{i_1} \dots P_{i_k}) \qquad \in (P_{i_1 + \dots + i_k})$

with a unit $1 \in P_1$

<u>Axioms:</u> ~~unit~~ $\theta \circ (1, \dots, 1) = \theta = 1 \circ \theta$

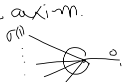
<u>associativity</u> $\theta \circ (\theta_1 \circ (\theta_{1,1} \dots \theta_{1,k_1}), \theta_2 \circ (\theta_{2,1} \dots \theta_{2,k_2}) \dots$

$= (\theta \circ (\theta_1, \dots, \theta_k)) \circ (\theta_{1,1} \dots \theta_{1,k_1}, \theta_{2,1} \dots)$

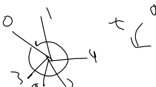
<u>equivariance:</u> ... acting by the Σ_{k_i}'s

<u>natural</u> axiom.

<u>Cyclic</u> operad?



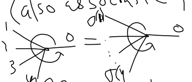
The action of Σ_k extends to an action of σ_{k+1}



Action of Σ_k on P_k extends to an action by Σ_{k+1}

"Any input could also be the output"

<u>Comm:</u> (also associative)



so $P_k = \{$  $\}$

Σ_k action is trivial.