Last time: Defined 2 flavors of graph complex, generated by admissible graphs.

odd: orient a graph by ordering vertices & orienting edges.

even: order edges.

Differential given by edge collapse

Began describing Kontsevich's 3 Lie algebras
- commutative
- associative
- Lie version

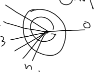
We will associate a Lie algebra to any cyclic operad:

operad = $\{ P(n) \}_{n \in \mathbb{N}}$ w/ action of Σ_n.

cyclic $O(n)$ if Σ_{n+1} acts on $P(n)$.

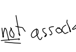
(any input slot could be the output.

Example: Ass $P(n) =$



Comm: associative plus no ordering of inputs

$P(n)$ has one object.



Assoc $O(n)_{\Sigma_n}$

Comm $O(n)_{\Sigma_n}$

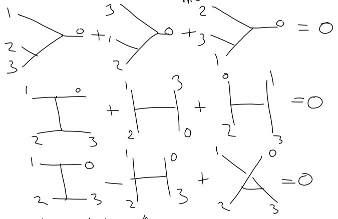
Lie operad: (not commutative, not associative

$[[1,[2,3]],4]$

$[x,y] = -[y,x]$ AS – antisymmetry

$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

Generator of $P(n)$ = rooted planar trivalent tree modulo AS and Jacobi.



"IHX-relation"

$P(n)$ has a cyclic version:
planar trivalent trees ~~on~~ with $n+1$ leaves modulo AS and IHX

Stasheff: (Notices 200?)
What is an operad?

From each of these (or any cyclic operad) and a symplectic vector space V construct a Lie algebra...

For comm this will be the algebra of polynomial functions on V with no constant or linear terms.

V ~~B~~ has basis $B=\{p_1,\dots,p_n,q_1,\dots,q_n\}$ with Poisson bracket $\{f,g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$

$V_k \hookrightarrow V_{k+1}$

$\langle p_1 \dots p_k, q_1 \dots q_k\rangle \subset \langle p_1 \dots p_{k+1}, q_1 \dots q_{k+1}\rangle$

will induce inclusion of Lie algebras.

What

What does this have to do with graph homology?

<u>Ref</u>: On a theorem of Kontsevich
J. Conant — K.V.

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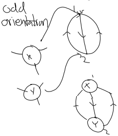
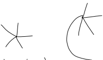
① There is a notion of (co)homology for a Lie algebra $\mathfrak{g}$ — defined by Chevalley–Eilenberg.
If G = simple, compact w/ Lie algebra $\mathfrak{g}$,
$$H_*(G) = H_*(\mathfrak{g}).$$

② Given a cyclic operad $\mathcal{O}$ and a graph G, can "decorate the vertices of G by generators of $\mathcal{O}$"



$|v|$ = valence of v
Stick elt of $\mathcal{O}((|v|))/\Sigma_{|v|}$ at the vertex
Get an "$\mathcal{O}$-graph".

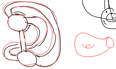
If $\mathcal{O}$ = Comm



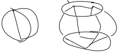
an $\mathcal{O}$-graph is just a graph.

If $\mathcal{O}$ = Assoc
$\mathcal{O}$-graph = graph with a cyclic ordering of edges entering each vertex.

surface =



$\mathcal{O}$ = <u>Lie</u>: more complicated

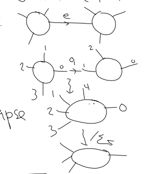


/AS, IHX.

$\mathcal{O}$-graphs $\rightsquigarrow$ Ch. complex generated by $\mathcal{O}$-graphs / $(G,-or)=(G,or)$.

<u>odd</u>:

<u>Differential</u>:



This gives an $\mathcal{O}$-graph, collapse edge

<u>Thm</u> (Kontsevich) $h_\infty = \mathcal{O}$-Lie algebra

$$PH_k^{CE}(h_\infty^+) = \text{homology of the } \mathcal{O}\text{-graph complex},$$

<u>Furthermore</u>:

- If $\mathcal{O}=$ Assoc, $= \bigoplus_{g,s\geq 1} H^{d-k}(\mathrm{Mod}(S_{g,s});\mathbb{R})$
- If $\mathcal{O}=$ Lie $= \bigoplus_{n\geq 2} H^{2n-3-k}(\mathrm{Out}(F_n))$
- If $\mathcal{O}=$ Comm $=$ "graph homology" contains invariants of odd-dimensional homology spheres.

$h_k = c_k$ Comm, a_k Assoc, ℓ_k Lie

Generators: Θ-spiders

 $\in \Theta(n)/\Sigma_{n+1}$.

V_k sympl. vector space, basis $\mathcal{B} = \{p_i, q_i\}$

Stick a basis elt on each leg

 $= \underline{\underline{\Theta\text{-spider}}}$.

 $\longleftrightarrow p_1^2 q_2$ in free comm algebra on V.

 $\longleftrightarrow p_1^2 q_2$ / cyclic permutations.

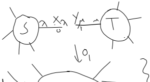
 $\longleftrightarrow$?

Θ-spiders generate $\mathfrak{h}$. (they have ≥ 2 legs).

$[S,T] = ?$

Given λ = leg of S, with label x_λ

μ = " " T, with label y_μ



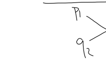
 $\Big\} = (ST)_{\lambda\mu}$

Define $[S,T] = \displaystyle\sum_{\substack{\lambda \text{ leg of } S \\ \mu \text{ leg of } T}} (ST)_{\lambda\mu}$

Exercise: (i) $[S,T] = -[T,S]$

(ii) Jacobi identity.

Exercise: $\mathcal{O} = \mathrm{Comm}$,

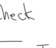


$\longleftrightarrow p_1 p_2 q_1^2 q_2$

Check $[S,T] = \{S,T\}$.

Claim: I can think of spiders as derivations of free $(\mathcal{O})$-algebras

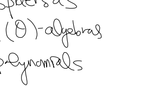
free comm algebra on B = polynomials

$P =$ 

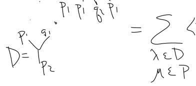
$\longleftrightarrow p_1^3 q_2$

$D =$ 

$$= \sum_{\substack{\lambda \in D \\ \mu \in P}} \langle x_\lambda, y_\mu \rangle$$



Derivation: $D(AB) = DA \cdot B + A \cdot DB$.



$D(AB) = DA \cdot B + A\, DB$

Free assoc, free Lie work the same way, ie

Spiders give derivations of the free objects.

D = spider 

$= D$

Derivations $D: A \to A$ A = algebra.

form a Lie algebra

$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Claim: The bracket I've defined on spiders $\leftrightarrow$ this bracket on derivations.

$h_k \hookrightarrow h_{k+1} \hookrightarrow \cdots$

legs decorated by elts of B_k

Define $h_\infty = \varinjlim h_k$.

CE homology of a Lie algebra h.

$C_k = \Lambda^k h = \{\sum r \cdot x_1 \wedge \dots \wedge x_k\}$.

$C_k \xrightarrow{d} C_{k-1}$

$x_1 \wedge \dots \wedge x_k \longmapsto \sum (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \hat{x}_i \dots \hat{x}_j \dots \wedge x_k$

Check $d^2 = 0$ $H_*^{CE}(h) = H_*$ of this chain complex.

To prove K's theorem make more graphs admissible:

1 Allow bivalent vertices

2 Allow disconnected graphs.

"full" graph complex fC_*

Allow 2-legged spiders.

x —○— y , $x, y \in B$.

Mating a 2-legged spider with a k-legged spider gives a (sum of) k-legged spiders.

In particular, 2-legged spiders form a sub-Lie algebra $h^{(2)}$.

<u>claim</u> $h^{(2)}_k \cong sp_k$ = Lie algebra of Sp_k.

$Sp_k = \{A \mid {}^tAJA = J\}$ $\quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$sp_k = \{A \mid {}^tAJ + JA = 0\}$

generated by $\begin{pmatrix} -E_{ij} & 0 \\ 0 & E_{ji} \end{pmatrix}$ $\quad$ E_{ij} = all 0 except 1 in ij-place

p_i —○— q_j

$\begin{pmatrix} 0 & 0 \\ -E_{ij}-E_{ji} & 0 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}$

q_i —○— q_j $\qquad$ p_i —○— p_j