Last time:

$\chi = 1-r$ Θ = cyclic operad $\rightsquigarrow$ "Θ-graph"

↓ and the odd (and even) graphcomplexes

$f\Theta_*^{(r)} \subset f(\Theta_*$ = gen by oriented graphs w/ no leaves

$C\Theta_*^{(r)} \subset C\Theta_*$ ← all vertices have valence ≥ 3 connected

Given $V_k \cong \mathbb{R}^{2k}$ $\mathcal{B} = \{p_1, \dots, p_k, q_1, \dots, q_k\}$ and Θ

defined h_∞ = Lie algebra generated by Θ-spiders.

$h_\infty^{(2)} \subset h_\infty$ ↔ spiders with 2 legs  $x, y \in \mathcal{B}$

$h_\infty^{(2)} \cong sp_\infty$ $\quad h_k^{(2)} \cong sp_k$

Thm (Kontsevich)

(1) $H_d^{CE}(h_\infty) \cong H_d(f(\Theta_*))$

This is a Hopf algebra.

The Primitive part is isomorphic to

$$PH_d^{CE}(sp_\infty) \oplus \bigoplus_r H_d(C\Theta_*^{(r)})$$

(2) For $h_\infty = l_\infty$ (Θ = Lie operad)

$$H_d(C\Theta_*^{(r)}) \cong H^{2r-2-d}(Out(F_r))$$

For $h_\infty = a_\infty$. (Θ = Assoc)

$$H_d(C\Theta_*^{(r)}) \cong \bigoplus_{\substack{\chi(S_{g,s}) = 1-r \\ s \geq 1}} H^{2r-2-d}(Mod\, S_{g,s})$$

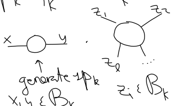
For $h_\infty = c_\infty$ (Θ = Comm)

$H_d(C\Theta_*^{(r)})$ contains invariants of odd-dimensional rational homology spheres

(M = manifold, $H^*(M, \mathbb{Q}) = H^*(S, \mathbb{Q})$)

Other variants of graph complexes (even orientations, other Θ, univalent vertices, directed edges) compute other invariants. We will see some in 2nd half of course.

Kontsevich's proof exploits the action of $sp_k = \mathfrak{h}_k^{(1)}$ on $\mathfrak{h}_k$.

Recall: 

$z_i \cdot y,\ z_i \mapsto x$
$z_i \cdot x,\ z_i \mapsto y$

$A \in sp$, S a spider

$$A \cdot S = \sum_{\text{legs } \lambda \text{ of } S} (\text{change } x_\lambda \text{ to } A \cdot x_\lambda)$$

Exercise: $A \cdot [S_1, S_2] = [A \cdot S_1, S_2] + [S_1, A \cdot S_2]$.

This extends to action of sp_k on $\Lambda^* \mathfrak{h}_k$

$$A \cdot (S_1 \wedge \dots \wedge S_n) = \sum_i S_1 \wedge \dots A \cdot S_i \wedge \dots \wedge S_n$$

The invariants of the action are

$$(\Lambda \mathfrak{h}_k)^{sp} = \{x : sp \cdot x = 0\}$$

Exercise: $A \cdot d^{CE}(x) = d^{CE} A \cdot x$.

So $(\Lambda^* \mathfrak{h}_k)^{sp} \xhookrightarrow{i} \Lambda^* \mathfrak{h}_k$ is a subcomplex.

Lemma: i induces an $\cong$ on homology.

$i : (\Lambda^* \mathfrak{h}_k)^{sp_k} \longrightarrow \Lambda^* \mathfrak{h}_k$

E finite-dim'l sp_k - module

sp_k reductive means $E = sp \cdot E \oplus E^{sp}$

Z_d = cycles in $\Lambda^d \mathfrak{h}_k$

$= sp \cdot Z_d \oplus Z_d^{sp}$

B_d = boundaries

$= sp \cdot B_d \oplus B_d^{sp}$

$$H_d = Z_d / B_d = \frac{sp \cdot Z_d}{sp \cdot B_d} \oplus \frac{Z_d^{sp}}{B_d^{sp}} = H_d(\Lambda \mathfrak{h}_k^{sp})$$

so just need to show $= 0$, ie

$sp \cdot Z_d \subseteq sp \cdot B_d$

follows if $sp \cdot Z_d \subset B_d \rightarrow$

sp_k is simple, not abelian $\rightarrow [sp, sp] \cdot Z_d \subset sp\, B_d$

$"sp\, Z_d \subset sp \cdot B_d"$

$\rightarrow \xi = {}^x \!\! - \!\! \bigcirc^y \in sp.$

Exercise $\xi a = d(a \wedge \xi) + da \wedge \xi$

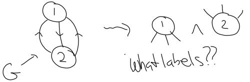
so if $da = 0$, $\xi a = d(a \wedge \xi)$ ✓

$(\Lambda^* \mathfrak{h}_k)^{sp} \longleftrightarrow \mathcal{O}g$

$\phi_k : \mathcal{O}g \longrightarrow \Lambda^* \mathfrak{h}_k$

$\psi_k : \Lambda^* \mathfrak{h}_k \longrightarrow \mathcal{O}g$

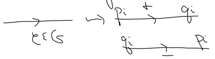
ϕ_k:

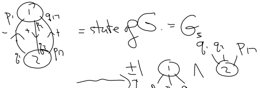


What labels??

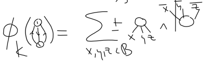
A state of G is a labelling of half-edges of G by elts of $\mathcal{B}_k$ and $\pm$ on each edge

rule:





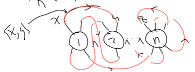
$\phi_k(G) = \sum_{\text{states } s}' G_s$ / cut open



Claim: $\phi_k(G)$ is an sp_k-invariant. (pf deferred)

Next: given $S_1 \wedge \dots \wedge S_n \in \Lambda^n h_k$, how to get a graph in $\mathcal{O}g$?

$\psi_k(S_1 \wedge \dots \wedge S_n)$



Let π = pairing of legs

this gives a graph $(S_1 \wedge \cdots \wedge S_n)^{\pi}$

$$\omega(\pi) = \prod_{\text{pairs } (x_\lambda, x_\mu)} \langle x_\lambda, x_\mu \rangle$$

Define $\Psi_k(S_1 \wedge \cdots \wedge S_n) = \sum_{\pi} \omega(\pi)\,(S_1 \wedge \cdots \wedge S_n)^{\pi}$

$$\Lambda^* h_k \xrightarrow{\Psi_k} \mathcal{O}g \xrightarrow{\phi_k} \Lambda^* h_k \xrightarrow{\Psi_k} \mathcal{O}g$$

$$\bigoplus \mathcal{O}g_{n,s} \xrightarrow{\phi_{n}} \bigoplus_{n,s} \Lambda_{n,s} \xrightarrow{\Psi} \mathcal{O}g_{n,s}$$

($\mathcal{O}g_{n,s}$: n vertices, s edges; $\Lambda_{n,s}$: gen. by spiders with in Λ^n with a total of 2s legs)

Lemma $\mathrm{Im}(\phi_k) = (\Lambda^* h_k)^{sp}$.

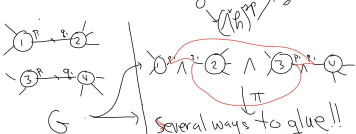
$$\mathcal{O}g \xrightarrow{\phi_k} (\Lambda^* h_k)^{sp} \xrightarrow{\Psi_k} \mathcal{O}g$$

(composite: id?)

ϕ_k is not a chain map. But Ψ_k is!

but $\Psi_k \circ \phi_k \neq id$!

Why is $\Psi_k \circ \phi_k \neq \mathrm{id}$? : $OG \longrightarrow OG$

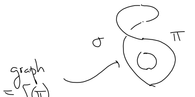


Several ways to glue!!
not all give back G

Claim: $\Psi_k \circ \phi_k = M_k : OG \longrightarrow OG$ is given by:

G : π = pairing of half-edges of G.

σ = "standard pairing"



$c(\pi)$ = # components of $\Gamma(\pi)$

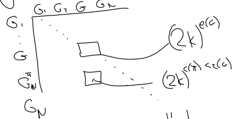
$$M_k(G) = \sum_{\pi}{}' (2k)^{c(\pi)} G^{\pi}$$

(sum over π = pairings of half-edges of G; G^{π} = result of gluing using π)

$c(\sigma)$ = # edges of G

$c(\pi) < c(\sigma)$ if $\pi \neq \sigma$

Look a matrix of M_k on $OG_{n,s}$



For $k \gg 0$ this matrix is invertible,
diagonal entries dominate
so matrix is invertible,

$$\begin{array}{ccc} \mathcal{O}g_{n,s} & \xrightarrow[\cong]{M_k} & \mathcal{O}g_{n,s} \\ & {\scriptstyle \phi_k} \searrow \quad \nearrow {\scriptstyle \psi_k} & \\ & \Lambda^*_{n,s} h_k & \end{array}$$

$$\begin{array}{ccc} \mathcal{O}g & \xrightarrow{M_\infty} & \mathcal{O}g \\ & {\scriptstyle \phi_\infty} \searrow \quad \nearrow {\scriptstyle \psi_\infty} & \\ & \Lambda^* h_\infty & \end{array}$$

$\operatorname{im} \phi_\infty = (\Lambda^* h_\infty)^{sp}$

$\psi_\infty : (\Lambda^* h_\infty)^{sp} \longrightarrow \mathcal{O}g$

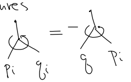
ψ_∞ is a chain map, so ψ_∞ induces an isomorphism on homology.

Why is $\operatorname{im} \phi_k \subset (\Lambda^* h_k)^{sp}$

Weyl: V_k = finite-dimensional symplectic vector space $B_k = \{p_1 \dots p_k\, q_1 \dots q_k\}$

then on $V \wedge V \ni w = \sum p_i \wedge q_i$ is invariant under the sp_k-action

in pictures



sp generated by x —— y, $x, y \in B$.

acts by mating as usual.

eg p_1 —— q_2

$q_1 \to q_2$
$p_2 \to p_1$