Kontsevich: $H_*(h_\infty) \cong H_*(fG_*g)$

Recap: Defined a Lie algebra h_K (generated by spiders).

Needed
- cyclic operad $\mathcal{O}$
- symplectic vector space $V_k \cong \mathbb{R}^{2k}$

For $\mathcal{O}$ = Comm (cyclic version)

h_K = Lie algebra of polynomial functions on V_k (no linear or constant terms) with Poisson bracket: $\{f,g\} = \sum \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$

P. bracket with a fixed f is a derivation $D_f(xy) = D_f x\, y + x D_f y$

We observed: pictorial defn of Poisson algebra makes sense, with any $\mathcal{O}$ decorating the spiders

and saw spiders can be viewed as derivations

We also observed $h_k^{(2)}$ gen by 2-legged spiders, get a copy of sp_{2k}.

2-legged spiders act on k-legged spiders, preserving # legs.

Sketched proofs that

(1) $H_*(\Lambda^* h_k, \partial_{CE}) \cong H_*((\Lambda^* h_k)^{sp_k}, \partial_{CE})$

(2) $(\Lambda^* h_k)^{sp_k} \cong fG_*g$

2nd part: $H_*(h_\infty)$ is a Hopf algebra and the primitive elts (ie non-products)

$$PH_*(h_\infty) = PH_*(sp_\infty) \oplus H_*(C_* g)$$

for $\mathcal{O} = \text{Lie}$ $\quad H_d(C_* g) \cong \bigoplus_{n \geq 2} H^{2n-3-d}(\text{Out}(F_n))$

$\mathcal{O} = \text{Assoc}$ $\quad = \bigoplus_{s \geq 1} H^{\cdot}(\text{Mod}(S_{g,s}))$

Why is this good? Those H^* are hard to compute

this gives new tools.

eg $\mathcal{O}$-Lie h_K was studied by Morita

had $\text{Tr}: h_K \longrightarrow \mathfrak{a}$ = abelian Lie algebra ($[\,,\,]=0$)

$H^* h_K \leftarrow H^* \mathfrak{a} = \mathfrak{a}$ ← understand

image is "Morita cocycles"

eg K. used this ~~formalism~~ to define cycles and cocycles in $H^*(\text{Mod}(S_{g,s}))$ pairing non-trivially. ($\leftrightarrow$ stable $H^*(\text{Mod}\, S_{g,r})$)

In general, this provides connections with other areas of math.

How is $H_*(h_\infty)$ a Hopf algebra:

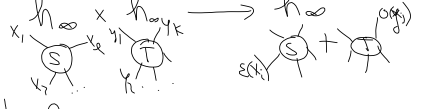
product $H_*(h_\infty)\otimes H_*(h_\infty)\to H_*(h_\infty)$?

Recall $B_k = \{p_1 \cdots p_k, q_1 \cdots q_k\}$ = basis for V_k.

Define $\mathcal{E}: B_\infty \to B_\infty$ $\quad O: B_\infty \to B_\infty$

$x_i \mapsto x_{2i}$ $\quad x_i \mapsto x_{2i+1}$.

$h_\infty \times h_\infty \longrightarrow h_\infty$



on homology

$H_*(h_\infty)\otimes H_*(h_\infty)\to H_*(h_\infty)$

on chain level:

$$\bigoplus_{k+l=n} \Lambda^k h_\infty \otimes \Lambda^l h_\infty \longrightarrow \Lambda^n h_\infty$$

$$S_1\wedge\cdots\wedge S_k \quad T_1\wedge\cdots\wedge T_l \mapsto \mathcal{E}(S_1)\wedge\cdots\wedge\mathcal{E}(S_k)\wedge O(T_1)\wedge\cdots\wedge O(T_l)$$

$\Psi \downarrow \qquad\qquad \downarrow \Psi_n$

$$\sum G^{ev} \sqcup G^{odd}$$

Upshot: primitive elts on RHS in fG_* are connected graphs.

$$PH_*(h_\infty) = H_*({}_{conn}C_* G).$$

may have vertices of valence 2.

$$h_\infty = h_\infty^{(1)} \oplus h_\infty^{+}$$

$$PH_*(h_\infty) = PH_*(h_\infty^{(1)}) \oplus PH_*(h_\infty^{+})$$

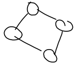
$$= PH_*(sp_\infty).$$

Exercise: Compute $PH_*(h_\infty^{(2)})$.

Hint: $H_*(h_\infty^{(2)}) = H_*(\mathrm{conn}\, C_*\mathcal{G}^{(2)})$

(What graphs can you make with 2-legged spiders? When are they 0 in the (odd) graph complex?

$$PH_*(h_\infty) = H_*(\mathrm{conn}\, C_*\mathcal{G}) \oplus \underline{PH_*(h_\infty^{(2)})}$$



Recall $fC_*\mathcal{G} = \bigoplus_{\chi = 1-r \le 0} fC_*\mathcal{G}^{(r)}$

$$\mathrm{conn}\, C_*\mathcal{G} = \underbrace{\mathrm{conn}\, C_*\mathcal{G}}_{\chi=0\,(r=1)} \oplus \underbrace{\bigoplus_{r\ge 2} \mathrm{conn}\, C_*\mathcal{G}^{(r)}}$$

G connected, no univalent vertices

$\chi = 1-r < 0$

$\operatorname{rank} \pi_1 G = r \geq 2$

Claim: $r \geq 2 \Rightarrow H_*(\mathrm{conn}\, C_*\mathcal{G}) \cong H_*(C_*\mathcal{G})$

no bivalent vertices

Pf is induction on # of bivalent vertices.

Exercise (optional):  graph complex of lines has homology only in deg 0.

Graph homology and spaces of graphs.

$\mathcal{O}$ = Lie, Comm, Assoc   connected, $|v| \geq 3$.

<u>Moduli spaces of graphs:</u>

$\mathcal{MG}_r$ = space of ~~such~~ graphs w/ $\chi = 1-r$
<u>metric</u>:

$l_e \geq 0 \in \mathbb{R}$ for every $e \in E(G)$

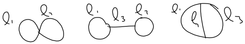
edge isometric to $(0, l_e) \subseteq \mathbb{R}$.

G gets <u>path metric</u>:

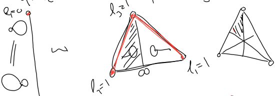
Move around by varying the l_e

Normalize so $\sum_{e \in G} l_e = 1$

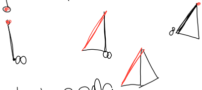
<u>$\mathcal{MG}_2$</u>:



$l_1 + l_2 = 1$ $l_1 + l_2 + l_3 = 1$



$\mathcal{MG}_2 \supset$



Better way to think of $\mathcal{MG}_2$

As a quotient space of Outer space

by $Out(F_n)$.

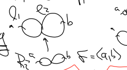
Add <u>markings</u> to metric graphs

$R_n =$  $g : R_n \xrightarrow{\simeq} G$ h.equiv.

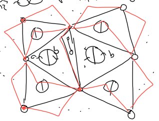
$\pi_1 R_n \cong F_n$ $g_* \; F_n \xrightarrow{\cong} \pi_1 G$ identifies $\pi_1 G$ with F_n.

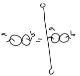
Point in OS_n is a <u>marked</u> metric graph.

<u>eg</u>



those are now <u>different</u>





$Out(F_n)$ acts on OS_n:

$\varphi : F_n \to F_n$ $a \mapsto abc$

Model φ by $f : R_n \to R_n$

$(G,g)\varphi = (G, g\circ f)$

$$\begin{tikzcd} R_n \arrow[r, "g"] & G \\ R_n \arrow[u, "f"] \arrow[ur, "g\circ f"'] & \end{tikzcd}$$

OS_n is <u>not</u> a simplicial complex, but <u>is</u> a union of open simplices

$\sigma(G,g)$: pt. lengths l_e w $\sum_e l_e = 1$ on G.

$OS_n^* =$ simplicial completion
$=$ union of $\underline{\text{closed}}$ simplices $\sigma(G, g)$

Action of $Out(F_n)$ extends to OS_n^*

$\mathcal{MG}_n = OS_n / Out F_n$

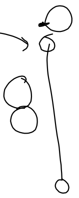
$\mathcal{MG}_n^* = OS_n^* / Out F_n$
$=$ "moduli space of tropical curves" of genus n.

<u>Thm</u> (Culler-V 1986):
OS_n's contractible

$Out(F_n)$ acts properly
$\operatorname{stab}(G, g) \cong \operatorname{Isom}(G)$

<u>Hurewicz</u>: $H_*(\mathcal{MG}_n) \cong H_*(Out F_n)$
Hitchcock: CV_n^* still contractible (trivial $\mathbb{Q}$ coeffs)

$CV_n^* \supseteq \partial CV_n^* = CV_n^* \setminus CV_n$



$\partial CV_n^* \to$

subcomplex:

$\leftrightarrow$ marked graphs with some cycle collapsed.

$H_*(CV_n^*, \partial CV_n^*)$

$C_k(CV_n^*, \partial CV_n^*) = C_k(CV_n^*) / C_k(\partial CV_n^*)$

$\cong$ one generator for every marked graph with $\pi_1 \cong F_n$

these have $\pi_1 \cong F_\ell$, $\ell < r$

How is a simplex of CV_n^* oriented?

order the edges!

rel chain cplex has one generator for every marked graph G: connected, $|v| \geq 3$, oriented by ordering edges.

If G has a loop,

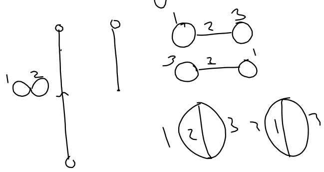
collapsing e changes π_1, lands in $C_*(\partial CV_n^*)$.

so only collapse non-loop edges.

$C_*(CV_n^*/\mathrm{Out}F_n, \partial CV_n^*/\mathrm{Out}F_n)$ are almost $= C_*\mathcal{G}$.

$C_*(\mathcal{MG}_n^*, \partial\mathcal{MG}_n^*)$

If G has an odd symmetry, then quotient $\cancel{\sigma}(G,g)/\mathrm{Isom}\,G$ doesn't contribute to chain complex



Exercise

Find a graph with no odd symmetries describe $\sigma(G,g)/\mathrm{Isom}(G)$.

$H_*(\mathcal{MG}_n^*, \partial\mathcal{MG}_n^*) \cong H_*(C_*\mathcal{G})^{\mathcal{O}=\text{comm},\ \text{or}=\text{even}}$