Last time: defined

$\mathcal{M}g_n$ = m.space of metric graphs (connected, $|v| \geq 3$ $\forall$ vertices v, $\chi = 1-n$). admissible

$= CV_n / Out(F_n)$

$\mathcal{M}g_n^* = CV_n^* / Out(F_n)$ ← ~~add~~ simplicial completion of CV_n

$\partial \mathcal{M}g_n^* = \mathcal{M}g_n^* \setminus \mathcal{M}g_n$

$\partial CV_n^* = \partial CV_n^* \setminus CV_n$

We observed $Cg_*^{(n)}$ = ~~odd~~ even commutative graph complex

$= C_*(\mathcal{M}g_n^*, \partial \mathcal{M}g_n^*)$

An admissible graph has $\leq 3n-3$ edges

so σ in $\mathcal{M}g_n^*$ has dim $\leq 3n-2$

$$\Rightarrow H_k(\mathcal{M}g_n^*, \partial \mathcal{M}g_n^*) = \begin{cases} 0 & \text{if } k > 3n-2 \end{cases}$$

$$\| \qquad H_k(Cg_*)$$

$C_k g = v - n + 1$ $\quad C_k g^{(n)}$: k = #vertices

C_1, C_0, C_{-1}, C_{-2}, C_{-3}

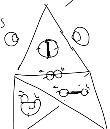
$$\begin{tikzcd}
 & & & & C_2 g^{(2)} \arrow[r] & C_1 g^{(2)} \\
 & & C_4 g^{(3)} \arrow[r] & C_3 g^{(3)} \arrow[r] & C_2 g^{(3)} \arrow[r] & C_1 g^{(3)} \\
C_6 g^{(4)} \arrow[r] & C_5 g^{(4)} \arrow[r] & C_4 g^{(4)} \arrow[r] & C_3 g^{(4)} \arrow[r] & C_2 g^{(4)} \arrow[r] & C_1 g^{(4)}
\end{tikzcd}$$

$\oplus$ $\oplus$ $\oplus$ $\oplus$ $\oplus$

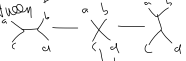
∂ operator collapses edges

2 vertices become one

one less edge



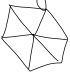
δ operator: splits a vertex, into two, adds an edge between them



This grading also gives chain & cochain complexes.

in C_0, in $C^{(n)}_{Ad}\mathcal{G}$

there is a cycle:



n-gon + central vertex = wheel W_n.

collapsing any edge gives a double edge

This is the grading used by Willwacher,

n even ⟹ $W_n = 0$

Thm: (Willwacher): There is a cocycle σ_{2n+1} for each n s.t. $\sigma_{2n+1}(W_n) \neq 0$

· W. identified $H^0(C\mathcal{G}_*)$ with $\mathfrak{grt}_1$ = "unipotent version of Grothendieck-Teichmüller Lie alg.

F. Brown proved $\mathfrak{grt}_1$ contains a free Lie algebra on odd generators σ_{2n+1}

Exercise: Compute $H_*(\mathcal{M}\mathcal{G}^*_n, \partial\mathcal{M}\mathcal{G}^*_n)$ for $n=2,3$ (and 4 - optional). (use $C\mathcal{G}^{(n)}_*$).

Back to K's theorem,

Back to K's thm for $\mathcal{O} = \mathrm{Lie}$:

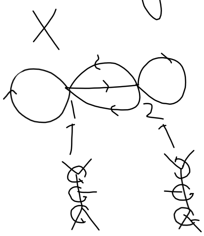
odd $\mathcal{O}$-graph complex CG_*

$$H_d(CG_*^{(n)}) = H^{2n-2-d}(\mathrm{Out}(F_n))$$

$$\| \quad H^{2n-2-d}(MG_n)$$

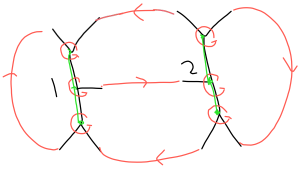
$$\| \quad H^{2n-2-d}(CV_n/\mathrm{Out}F_n)$$

Generator of $CG_*^{(n)}$:



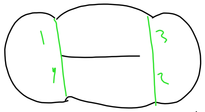
, vertices decorated by Lie spiders

$(X, or, \{T_v\}_{v \in X})$

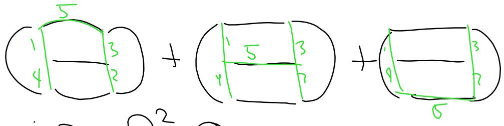
$G =$ 

In G, there's a natural forest Φ = interior edges of the T_v.

<u>Thm</u>: All of this orientation data is equivalent to just ordering the edges of Φ

∂ 

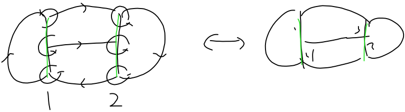
and the ∂ operator adds an edge to ϕ wherever possible, then sums

$=$ 

<u>Exercise</u>: $\partial^2 = 0$.

Linear

<u>Proof</u>: Linear algebra...



<u>Recall</u>: An orientation determined by an ordering of $S=\{x_1,\dots,x_n\}$ is a choice of unit vector in $\Lambda^n \mathbb{R}S$, where $\mathbb{R}S$ = v. space with basis S

$$\left(x_1\wedge\cdots\wedge x_n \in \Lambda^n \mathbb{R}S \right) \longleftarrow \det \mathbb{R}S$$

So an orientation

$$(X, \mathrm{or}, \{T_v\})$$

half-edges of e

is a unit vector in

$$\det \mathbb{R}V(X) \otimes \bigotimes_{e\in E(X)} \det \mathbb{R}H(e) \otimes \bigotimes_{v\in X} \left(\bigotimes_{u\in V(T_v)} \det(\mathbb{R}H(u)) \right)$$

We claim all this ~~is a to~~ determines a unit vector in $\det \mathbb{R}(E(\Phi))$

Proof uses two elementary Lemmas.

<u>Lemma 1</u>: $S = \coprod S_i$ partition.

then $\bigotimes_i \det(\mathbb{R}S_i) \otimes \det\left(\bigoplus_{|S_i| \text{ odd}} \mathbb{R}\right) \overset{\text{canonical}}{\cong} \det \mathbb{R}S$

Pf: ordering each S_i, then ordering the set of S_i's gives me an ordering of S.

$$\underbrace{(x_1 \wedge \dots \wedge x_k)}_{S_1} \wedge \underbrace{(x_{k+1} \wedge \dots \wedge x_\ell)}_{S_2} \wedge \dots$$

If $|S_i|$ is even, I can move it past any other S_j without changing the orientation.

$$(x_1 \wedge x_2) \wedge (x_3 \wedge x_4 \wedge x_5) = (x_3 \wedge x_4 \wedge x_5) \wedge (x_1 \wedge x_2)$$

Sign changes if you ~~the~~ interchange two odd S_i. ✓

The map is well-defined.

Lemma 2: $0 \to U \xrightarrow{f} V \overset{s}{\underset{g}{\rightleftarrows}} W \to 0$

a short exact sequence of finite-dimensional vector spaces

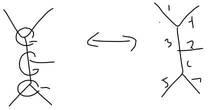
$$\det(U) \otimes \det(W) \leftrightarrow \det(V)$$
$$u \otimes w \longmapsto f(u) \wedge s(w).$$

(independent of s).

Exercise: $0 \to U \to V \to W \to Z \to 0$ exact $\Rightarrow \det U \otimes \det W \leftrightarrow \det V \otimes \det Z$

How to use these lemmas:

Claim



(Warning: the "obvious" equivalence doesn't work)

T_v

Pf: Look at reduced chain complex for T

$0 \to C_1 \to C_0 \to \mathbb{R} \to 0$ is exact since $\tilde{H}_*(T) = 0$

Lemma 2 $\det C_0 = \det C_1 \otimes \cancel{\det \mathbb{R}}$

$\det C_0 = \det C_1$

$\det \mathbb{R}V = \det \mathbb{R}E \otimes \bigotimes_{e \in E} \mathbb{R}H(e)$

Lemma 1 $= \det \mathbb{R}E \otimes \det \mathbb{R}H$

Lemma 1 $\det \mathbb{R}V = \underbrace{\det \mathbb{R}E}_{RH} \otimes \underbrace{\bigotimes_v \det H(v) \otimes \det \mathbb{R}V}_{LH \text{ orientation}}$

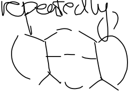
Now tensor both sides w/ $\det \mathbb{R}E$

$\det \mathbb{R}E \otimes \det \mathbb{R}V = \bigotimes_v \det H(v) \otimes \det \mathbb{R}V$

Now tensor with $\det \mathbb{R}V$

$\det \mathbb{R}E = \bigotimes_v \det H(v)$ ✓

Rest of proof uses these lemmas repeatedly, together with observation that $H(X) =$ leaves of Trees.



Also have to check $\partial(X, or_X, \{T_v\}) \leftrightarrow \partial(G, \Phi, or_\Phi)$

Back to Outer space!

$$H_*(CG_*^{(n)}) \leftrightarrow H^*(\mathrm{Out}F_n) = H^*(CV_n/\mathrm{Out}F_n).$$

$CV_n = \coprod$ open simplices $\sigma(G,g)$

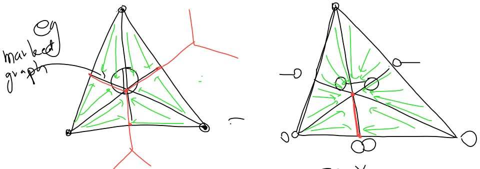
$\cap$

CV_n^* = simplicial completion.

$(CV_n^*)'$ = barycentric subdivision
(vertex = simplex in CV_n^*)

$\cup$

K_n = subcomplex with no faces in $CV_n^* \setminus CV_n$.



All maximal simplices in CV_n^* have at least one vertex at ∞ (in ∂CV_n^*) — eg vertex of ∂CV_n^* and one vertex inside CV_n ($\leftrightarrow$ trivalent graph)

So, 

$\subset CV_n$ can linearly project onto K_n.