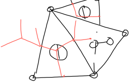
Graph complexes

$\leftrightarrow$ spaces of graphs.

i.p. Outer space, moduli space of graphs CV_n $\mathcal{M}g_n = CV_n / \mathrm{Out}F_n$

We defined the spine $K_n \subset CV_n$:

$=$ s.complex, k-simplex is a chain of k forest collapses, geom. realization of the poset of simplices $\sigma(G,g)$.

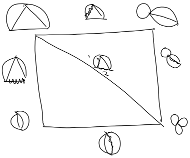


Fact: K_n is a def. retract of CV_n invariant under $\mathrm{Out}(F_n)$

$\Rightarrow H_*(K_n/\mathrm{Out}F_n) \cong H_*(\mathrm{Out}F_n)$.

Claim: K_n is a cube complex.

e.g. $n=3$:

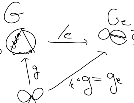


Cube $\leftrightarrow (G, g, \Phi)$ Φ = forest in G.

Orientation? $\leftrightarrow$ ordering the edges of Φ.

What is the ∂ map in this cube complex?

$$\partial(G,g,\Phi) = \sum_{e\in\Phi}{}' \, (G,g,\Phi-e) - (G_e, g_e, \Phi_e)$$



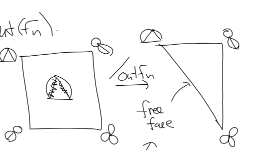
$= \partial_r - \partial_c$

remove edges (∂_r), collapse edges (∂_c).

Take the quotient by $\mathrm{Out}(F_n)$: the marking disappears.

The square I drew embeds in $K_n/\mathrm{Out}(F_n)$.

However



can push in along the free face, eliminate this (quotient-cube).

So, in chain complex for $K_n/\mathrm{Out}F_n$, don't need this graph.

Prop: (G,Φ) contributes a generator to the rational chains for $K_n/\mathrm{Out}F_n$ iff (G,Φ) has no automorphisms $G\to G$, $\Phi\to\Phi$ that give an odd permutation of $E(\Phi)$.

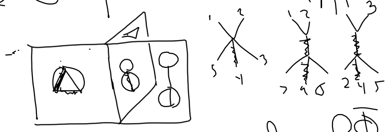
Lemma: The quotient of a sphere by a finite group is a rational homology sphere or disk, depending on whether the group has an orientation-reversing element.

Pf (equivariant homology spectral sequence).

Heuristics:

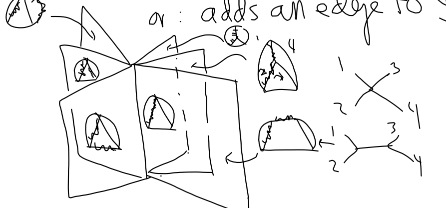
The homology of a finite group with trivial rational coefficients is 0. $\rightarrow$ equiv. H_* spec seq.

What is the δ operator in K_n?



∂ operator: collapses an edge of Φ
. adds an edge to Φ

δ operator: splits a vertex into two
or: adds an edge to Φ



$$\delta(G,g,\Phi) = \delta_a + \delta_s$$

add an edge to Φ (a)

split a vertex of G.

$$= \sum_{\substack{e \in G - \Phi \\ \text{st. } \Phi \cup e \text{ is a forest}}} (G, \Phi \cup e, g) + \sum_{v \in G} (G^\pi, \Phi^\pi, g^\pi)$$

π = partition of $e(v)$ into 2 pieces

(G^π, Φ^π, g^π)

This almost the forested graph complex ($\leftrightarrow$ Lie graph complex)

<u>except</u>: ① We have non-trivalent graphs

② In forested graph complex, have IHX relations.

③ The δ_s-term isn't there. (you can't split a trivalent vertex.

CG_* = Lie graph cplex ⟷ forested graph cplex.

CG_k = gen by pairs (G,Φ)

G trivalent, Φ is a forest with k trees, edges of Φ ordered

modulo $(G,\Phi,or) = -(G,\Phi,-or)$

and IHX on edges of Φ

$$\partial(G,\Phi,or) = \sum_{\substack{e\in G-\Phi \\ \text{st } \Phi\cup e \text{ is a forest}}} (G,\Phi\cup e, or)$$

C_* = chain complex for $K_n/\mathrm{Out}F_n$

C_k gen by ~~graphs~~ pairs (G,Φ) w/ no odd symmetries

$$\delta(G,\Phi) = \sum (G,\Phi\cup e) - \sum (G^{\pi},\Phi^{\pi})$$

Claim: Both chain complexes have the same homology.

Proof: Decompose C_k = chains for K_n/Out:

$C_{p,q}$ = gen by graphs with p vertices and q trees in Φ.

(convention: Φ contains all vertices)

So in $C_{p,p}$: each tree in Φ is a single vertex.

 $\ni C_{3,3}$

Set $C_0 = \bigoplus_{p\geq 1} C_{p,p}$ ← no edges in Φ

$C_1 = \bigoplus_{p\geq 2} C_{p,p-1}$ ← 1 edge in Φ

$$C_k = \bigoplus_{p-q=k} C_{p,q} \leftarrow \begin{array}{l} q \text{ trees, } p \text{ vertices} \\ \Rightarrow k = p-q \text{ edges in } \Phi. \end{array}$$

Make a double complex

$$\begin{tikzcd}
C_{p+1,q-1} \arrow[d, leftarrow, "\delta_s"'] & C_{p+1,q} \arrow[l, "\delta_a"'] \arrow[d, leftarrow, "\delta_s"] \\
C_{p,q-1} & C_{p,q} \arrow[l, "\delta_a"']
\end{tikzcd}$$

$p-q=k+2$ $p-q=k+1$ $p-q=k$ C_k

This double complex is just another way to write $C_* =$ chain complex for K_n/Out.

Let's compute the vertical homology first: rank $G = n$

$$C_{p,p} \dashrightarrow C_{p,q} \xrightarrow{\delta_s} C_{p+1,q} \to \cdots C_{2n-2,1}$$

$$G \longrightarrow \text{split a vertex} \longrightarrow \text{split another}$$

$C_{p,p}$: p vertices, p trees, no edges

$C_{2n-2,1}$: trivalent graphs.

<u>Prop</u>: This chain complex has no homology except in top dim.

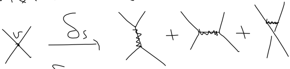
Assuming this, complex becomes

$$C_{2n-3,1}/\mathrm{im}\,\delta_s \leftarrow \quad \leftarrow C_{2n-3,q-1}/\mathrm{im}\,\delta_s \xleftarrow{\delta_a} C_{2n-2,q}/\mathrm{im}\,\delta_s$$

$$C_{2n-3,q} \xrightarrow{\delta_s} C_{2n-2,q}$$

$C_{2n-3,q}$: trivalent <u>except</u> 1 4-valent vertex v

$C_{2n-2,q}$: trivalent graphs



$\mathrm{im}\,\delta_s = IHX.$

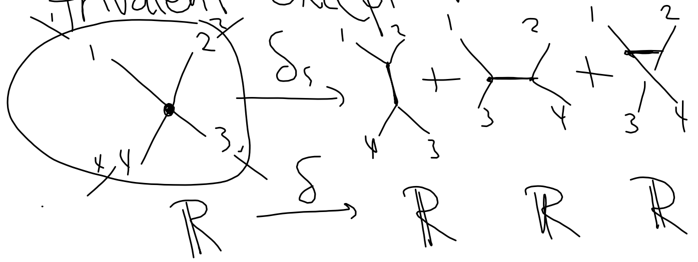
So $C_{2n-2,q}/\mathrm{im}\,\delta_s = \mathcal{CG}_q$ ✓.

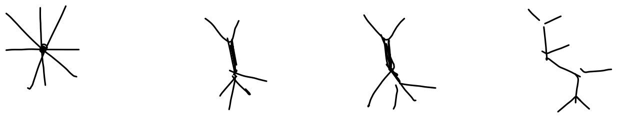
The vertical differential.

$C_{gg} \to \hat{C}_{p,q}$ has one generator for each (G, g, Φ).

Suffices to compute chain complex starting from each generator $(G, g, V(G))$.

For simplicity, assume all vertices are trivalent except one: v.





<u>Claim</u>: My vertical chain complex is the chain complex of a simplicial complex T_n whose vertices are 1-edge trees on n labeled leaves, edges are 2-edge trees

etc

<u>Prop</u> $T_n \simeq \bigvee S^{n-4}$.