Lecture 5

We've defined the Borel-Serre bordification $\overline{X}$ of $X = O(n)\backslash GL_n\mathbb{R}$ and shown $\Gamma = GL_n\mathbb{Z}$ acts properly and cocompactly. (end of proof (at least for $n=3$) is in the notes from Lecture 4)

Borel and Serre defined this for much more general G and Γ, specifically

G = semi-simple algebraic group defined over $\mathbb{Q}$
K = maximal compact subgroup
$X = K\backslash G$
Γ = arithmetic subgroup of G

They used this to determine the virtual cohomological dimension (VCD) of the groups Γ. The idea is to prove a certain kind of duality between H_* and H^* which shows both are zero outside an obvious range.

This duality generalizes Poincaré duality for closed manifolds to the setting of cohomology of groups.

Thm (Bieri–Eckman)

(see K. S. Brown's book *Cohomology of Groups* Thm 10.1, p. 220 and Prop 11.3, p 229.)

Let Γ be a group with sufficiently strong finiteness properties (including finitely generated, finitely presented; satisfied, eg if there is a compact Y with $\pi_1 Y = \Gamma$ and $\tilde{Y}$ contractible.)

Suppose there is $d > 0$ s.t.

$$H^k(\Gamma; \mathbb{Z}[\Gamma]) = \begin{cases} \text{torsion free} & k = d \\ 0 & k \neq d \end{cases}$$

Then $D = H^d(\Gamma; \mathbb{Z}[\Gamma])$ is a dualizing module for Γ, ie for any torsion free finite index subgroup $\Gamma' < \Gamma$ and any Γ'-module M there is a natural $\cong$

$$H^k(\Gamma'; M) \cong H_{d-k}(\Gamma'; M \otimes D).$$

In particular, $cd(\Gamma') = vcD(\Gamma) = d$.

To get at $H^*(\Gamma; \mathbb{Z}[\Gamma])$ they look at the action of Γ on $\overline{X}$, using

Thm: $\overline{X}$ contractible CW-cplex with proper, cocompact action by Γ then

$$H^i(\Gamma; \mathbb{Z}[\Gamma]) \cong H^i_c(\overline{X}; \mathbb{Z})$$

Proof The point is that elements of $\mathbb{Z}[\Gamma]$ are finite sums $\sum n_\gamma \gamma$.

Let $C(\overline{X})$ = chain complex for $\overline{X}$

Γ acts on $\overline{X} \Rightarrow C_k(\overline{X})$ is a $\mathbb{Z}[\Gamma]$-module for all k

$\overline{X}/\Gamma$ compact $\Rightarrow C_*(\overline{X})$ is finitely-generated as a $\mathbb{Z}[\Gamma]$-module.

To compute $H^*(\Gamma, \mathbb{Z}[\Gamma])$, use the cochain complex $\mathrm{Hom}_\Gamma(C_*(\overline{X}), \mathbb{Z}[\Gamma])$

If $F \in \mathrm{Hom}_\Gamma(C(\tilde{X}), \mathbb{Z}[\Gamma])$ then

$F(m) = \sum_\gamma f_\gamma(m)\gamma$ (with almost all $f_\gamma(m) = 0$)

F is Γ-module map $\Rightarrow f_\gamma(m) = f_1(\gamma^{-1}m)\ \forall \gamma$

Here $f_1 : C(\tilde{X}) \longrightarrow \mathbb{Z}$ is non-zero on only finitely many cells, ie is compactly supported

So $F \longmapsto f_1$ gives a map

$$\mathrm{Hom}_\Gamma(C(\tilde{X}), \mathbb{Z}[\Gamma]) \longrightarrow \mathrm{Hom}_c(C\tilde{X}, \mathbb{Z})$$

This map is an isomorphism: it has as inverse

$$f \longmapsto \left(m \mapsto \sum_{\gamma\in\Gamma} f(\gamma^{-1}m)\gamma\right)$$

//

Now: How do we compute $H^*_c(\tilde{X})$?
A: Poincaré-Lefschetz duality:

$\overline{X}$ is a non-compact manifold with ∂

$\dim \overline{X} = \frac{n(n+1)}{2} - 1 := m$

so Poincaré - Lefschetz duality gives an $\cong$ between H_* and H^* with compact supports:

$$H^k_c(\overline{X}) \cong H_{m-k}(\overline{X}, \partial\overline{X})$$

$$\cong \widetilde{H}_{m-k-1}(\partial\overline{X}) \quad (\text{since } \overline{X} \text{ is contractible})$$

$\partial\overline{X}$ is covered by the contractible sets $\overline{e(P)}$, for P parabolic

In fact the $\overline{e(P)}$ with P maximal parabolic cover $\partial\overline{X}$

$P = P_V = \mathrm{stab}(V)$ with V a subspace of $\mathbb{Q}^n \subseteq \mathbb{R}^n$.

$V \subset W \Rightarrow P_V \cap P_W = \mathrm{stab}(V \subset W)$ is also parabolic and $\overline{e(P_V)} \cap \overline{e(P_W)} = \overline{e(P_{V \subset W})}$

By definition, the nerve of the cover of $\partial\overline{X}$ by the $\overline{e(P_V)}$ has

* One vertex for each proper subspace V of $\mathbb{Q}^n$
* A k-simplex for each chain of k inclusions $V_0 \subset \cdots \subset V_k$

The intersections
$$\overline{e(P_{V_0})} \cap \cdots \cap \overline{e(P_{V_k})} = \overline{e(P_{V_0 \subset \cdots \subset V_k})}$$
are contractible, so the nerve is homotopy equivalent to the whole space

<u>Def</u> For any field k, the Tits building $T(k^n)$ is the simplicial complex with one vertex for each proper subspace V of k^n and one k-simplex for each chain $V_0 \subset \cdots \subset V_k$ of proper inclusions.

Theorem (Solomon-Tits) k any field.

Then $T(k^n) \simeq \bigvee S^{n-2}$

So we conclude

$$\tilde{H}^k(\partial \overline{X}) = \begin{cases} 0 & k \neq n-2 \\ \text{free abelian} & k = n-2 \end{cases}$$

Giving

$$H^k(\Gamma; \mathbb{Z}[\Gamma]) \cong \tilde{H}_{m-k-1}(\partial \overline{X})$$

$$= \begin{cases} \text{free abelian} & k = \left[\frac{n(n+1)}{2} - 1\right] - (n-2) - 1 \\ & \quad = \frac{n(n-1)}{2} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{vcd}(GL_n \mathbb{Z}) = \frac{n(n-1)}{2}$ ✓

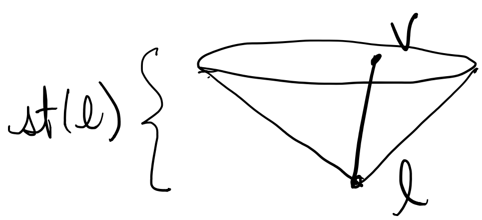
Proof of Solomon-Tits theorem:

By induction on n.

$n=2 \Rightarrow T(k^n)$ is discrete
$\Rightarrow \ = \bigvee S^0$ ✓

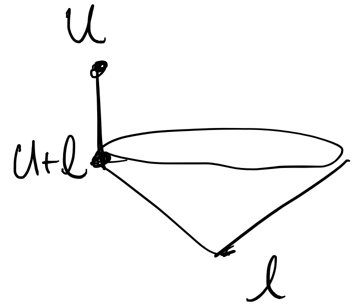
<u>$n>2$</u>. Fix a line $\ell \subseteq k^n$.

If $V \supset \ell$, then V is connected to ℓ, ie is contained in the star of ℓ:



$st(\ell)$ is contractible.

If $U \notin st(\ell)$ and $\dim U < n-1$, then $U+\ell$ is a proper subspace and is in $st(\ell)$:

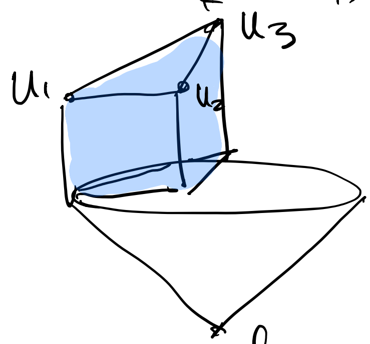


If $U_0 \subset \cdots \subset U_k$ is a k-simplex and $\dim U_k < n-1$, then

$U_0 + \ell \subset \cdots \subset U_k + \ell$ is a simplex

in $st(\ell)$

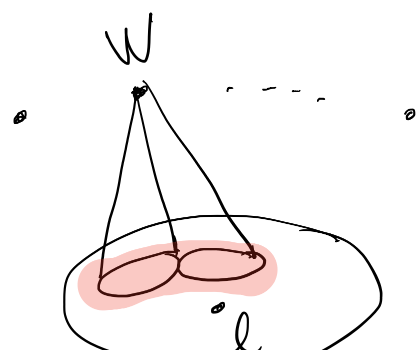
(possibly of $\dim = k-1$)



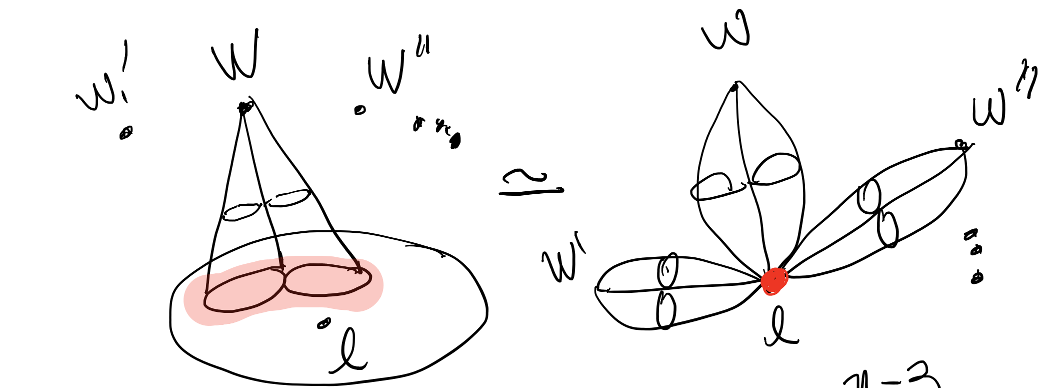
and you can push it down to $st(\ell)$

ie the map $U \mapsto U + \ell$ gives a deformation retraction into $st(\ell)$.

The only subspaces we haven't got in our (contractible) subcomplex yet are the $(n-1)$-dimensional subspaces W which don't contain ℓ.



The link of W is the complex spanned by all proper subspaces of W



By induction, $\mathrm{lk}(w) \simeq \bigvee S^{n-3}$

$$\therefore\ T(\mathbb{k}^n) \simeq \bigvee_{w^{n-1} \not\ni \ell} \mathrm{susp}(\mathrm{lk}\, w)$$

$$\simeq \bigvee_{w^{n-1} \not\ni \ell} \mathrm{susp}\left(\bigvee S^{n-3}\right)$$

$$\simeq \bigvee_{w^{n-1} \not\ni \ell} \bigvee S^{n-2} = \bigvee S^{n-2} \checkmark$$

We described the symmetric space $X = O(n)\backslash GL_n\mathbb{R}$ in several ways, including as space of marked lattices

$$\mathbb{Z}^n \xrightarrow{\Lambda} \mathbb{R}^n, \text{ modulo}$$

homothety and rotation.

We noted this could also be interpreted as the space of marked flat tori:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\Lambda} & \mathbb{R}^n \\ \cup & & \cup\! \mid \\ \mathbb{Z}^n & \longrightarrow & A\mathbb{Z}^n \end{array} \quad \rightsquigarrow \quad \mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\bar{\Lambda}} \mathbb{R}^n/A\mathbb{Z}^n$$

$T_n \equiv \mathbb{R}^n/\mathbb{Z}^n$ = "standard torus"

$T = \mathbb{R}^n/A\mathbb{Z}^n$ is homeo to T_n but has a different metric

So can describe a point of X_n as a flat torus together with a linear homeomorphism

$$T_n \xrightarrow{\bar{\Lambda}} T, \text{ called a } \underline{\text{marking}}$$

(then I don't need to say "modulo rotation")
(If I say "volume 1" I don't need to say "modulo homothety" either.)

<u>or</u> I could describe it as an <u>action</u> of $\pi_1 T^n = \mathbb{Z}^n$ on $\tilde{T}$ by isometries (without choosing a basis for $\tilde{T} \approx \mathbb{R}^n$)

We can make the same construction with other geometric objects in place of T_n:

<u>eg</u> $T_n \rightsquigarrow S_g$ = closed surface of genus g

S = surface w/ hyperbolic metric, genus g

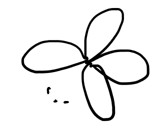
$S_g \xrightarrow{h} S$ a homeomorphism

$(h, S) \sim (h', S')$ if there's an isometry $S \xrightarrow{f} S'$ with $f \circ h \simeq h'$.

The space of <u>marked</u> hyperbolic surfaces (area 1) is <u>Teichmüller space</u> $\mathcal{T}_g$

Note For a torus or a surface,
any homotopy equivalence is homotopic to a homeomorphism, (linear for a torus) so I could have made the markings homotopy equivalences instead of homeomorphisms

Why did I bring that up? Because I want to consider objects which are not manifolds, namely graphs:

Fix a model graph $R_n =$ 

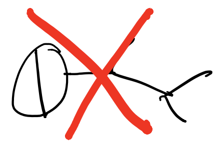
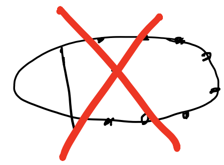
consider the set of marked graphs
$$R_n \xrightarrow{g} G, \quad g \text{ a h-equiv,}$$
modulo homothety and homotopy

We need a metric structure on graphs in order to define a space of marked graphs; varying the metric moves you in the space.

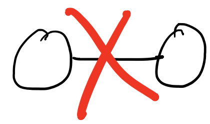
- Put positive real lengths on edges

To make the space finite-dimensional:

- Don't allow univalent or bivalent vertices

Also convenient at times to not allow any <u>separating</u> edges



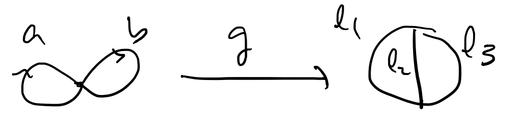
Say $(g, G) \sim (g', G')$ if there is an <u>isometry</u> $G \to G'$ making the diagram

$$\begin{tikzcd} G \arrow[rr] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$

commute up to homotopy

This space is known as Outer space $\mathcal{O}_n$, of rank n.

<u>$n=2$</u>: points look like



or

Just as $GL_n\mathbb{Z}$ acts on marked lattices by changing the marking:

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "\Lambda"] & \mathbb{R}^n \\ \mathbb{Z}^n \arrow[u, "A"] \arrow[ur, "\Lambda\circ A"'] & \end{tikzcd}$$

The group of homotopy equivalences of R_n acts on Outer space

$$\begin{tikzcd} R^n \arrow[r, "g"] & G \\ R^n \arrow[u, "h"] \arrow[ur, "g\circ h"'] & \end{tikzcd}$$

A homotopy equivalence $h: R_n \longrightarrow R_n$ (which may move the basepoint)

induces $h_*: \pi_1(R_n, b) \longrightarrow \pi_1(R_n, h(b))$

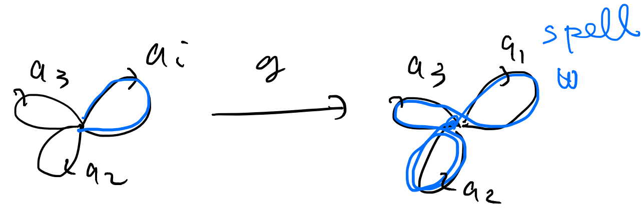
Only well-defined on $\pi_1(R_n, b)$ up to choosing a path from $h(b)$ to b, ie only well-defined up to inner automorphism.

$$\pi_0 HE(R_n) \longrightarrow Out(\pi_1 R_n) = Out(F_n)$$

Prop The group $\pi_0(HE(R_n))$ of homotopy equivalences (mod homotopy) of R_n is isomorphic to $Out(F_n)$

Proof: Can realize any automorphism by a homotopy equivalence

$\alpha : a_j \mapsto w_j$



$\Rightarrow$ surjective.

Injective: a homotopy equivalence inducing id on π_1 is homotopic to id.

So $Out(F_n)$ acts on $\mathcal{O}_n$.

We want to use $\mathcal{O}_n$ as a replacement for the symmetric space X_n used for studying $GL_n\mathbb{Z}$

Theorem (Culler–V 86) $\mathcal{O}_n$ is contractible

(recall this was easy for X_n. Not so easy for $\mathcal{O}_n$)

Exercise: The stabilizer of (g, G) is isomorphic to $\mathrm{Isom}(G)$.

in particular, it is finite.

In lecture 2, we briefly saw $\mathcal{O}_2 \approx \mathbb{H}^2$.
using a Jacobian map $\mathcal{O}_2 \longrightarrow \mathbb{H}^2$
which assigned a quadratic form to a marked metric graph.

Also, $\mathrm{Out}(F_2)$ is isomorphic to $GL_2\mathbb{Z}$, and the map is equivariant, so you can see $\mathcal{O}_2 / \mathrm{Out}(F_2)$ is not compact

We want to bordify $\mathcal{O}_n$: add some stuff to make $\overline{\mathcal{O}_n}$, extend the action so the action is still proper and the.

quotient $\overline{\mathcal{O}_n} / \mathrm{Out}\, F_n$ is compact.

For $\mathcal{O}_n$, this is both easier and harder

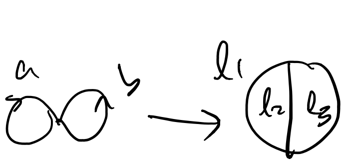
Easier because Θ_n has natural combinatorial structure which helps to define the bordification

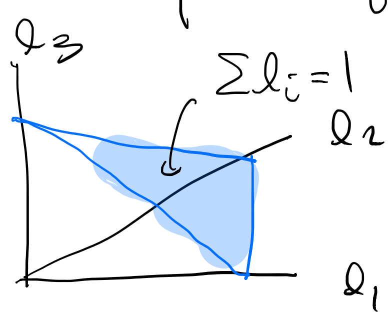
Harder because Θ_n is not a manifold if $n>2$, so can't use Poincaré-Lefschetz duality.

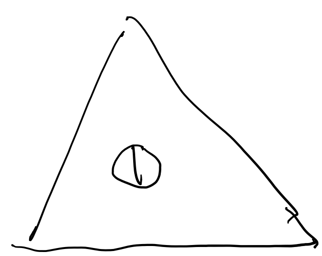
Combinatorial structure of Θ_n:

$(g,G) \in \Theta_n$. If G has k edges,

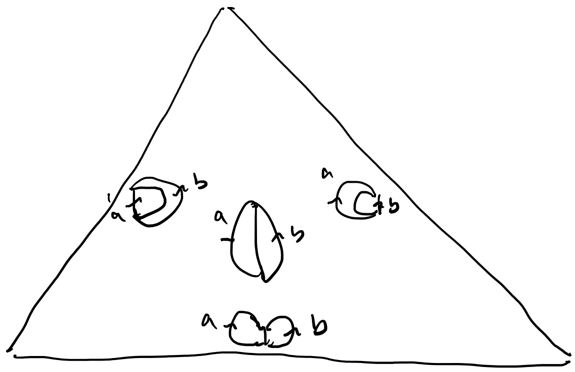
$\sum$ lengths $=1$, then varying the lengths fills out a subspace in Θ_n homeomorphic to an open $(k-1)$-simplex $\sigma(g,G)$

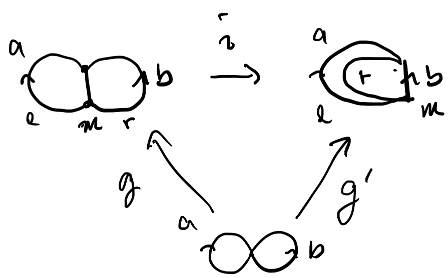






different points in $\sigma(g,G)$ may be isometric, but the isometry is not $\simeq$ id.





$i \circ g(b) = -b, \neq id$

so $\sigma(g, G) \hookrightarrow \mathcal{O}_n$

Since I can do this for *every* point (g, G)
I get
$$\mathcal{O}_n = \coprod \sigma(g, G),$$

a decomposition of $\mathcal{O}_n$ as a disjoint union of open simplizes

Faces of $\sigma(g, G) \leftrightarrow$ setting some edge lengths to 0

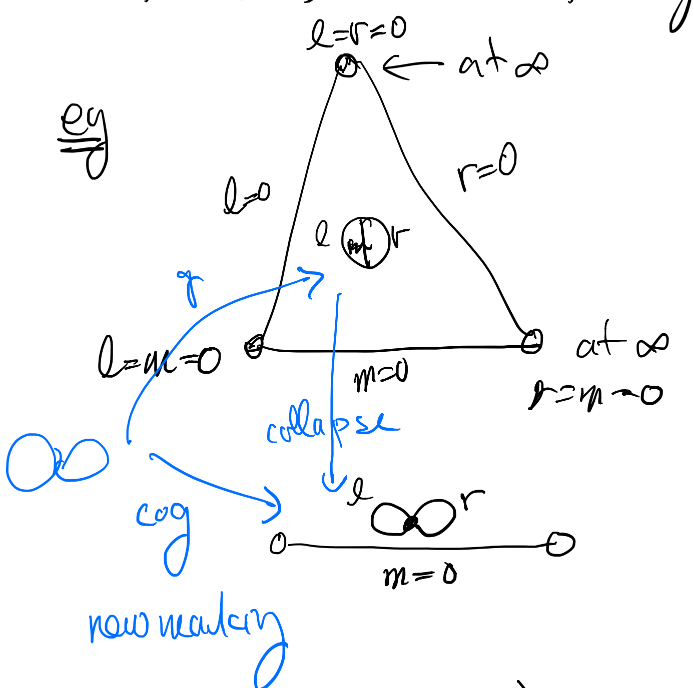
Some faces correspond to marked graphs

in O_n, some do not.

<u>In</u> O_n if the collapsing edges form a <u>forest</u> (ie contain no closed loops)

<u>Not in</u> O_n if there's a loop – say the face is "at infinity"

<u>eg</u>

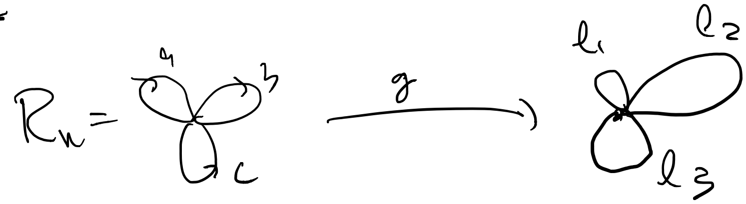


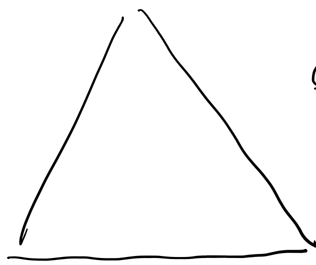
<u>Rule</u>: $\sigma(g', G')$ is a face of $\sigma(g, G)$

if $\exists$ forest collapse $G \xrightarrow{c} G'$ making diagram commute:

$$\begin{tikzcd} G \arrow[rr, "c"] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$

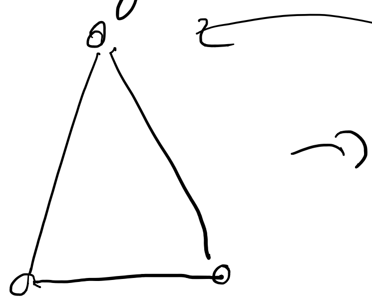
eg





all faces are at ∞!

To bordify $\mathcal{O}_n$, we will bordify each $\sigma(g, G)$ separately, replacing the faces at ∞ by cells

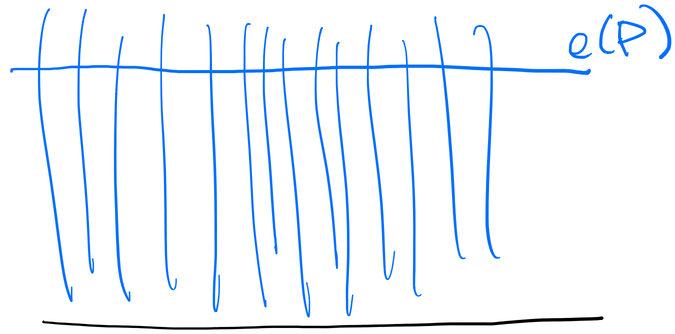


replace ~~vertex~~ by an interval

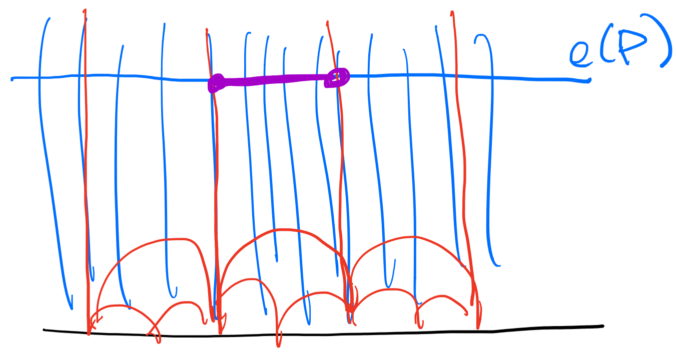
Then say how to glue the pieces together!

Picture for $n=2$:

$\mathbb{H}^2$ as upper half-plane: We replaced ∞ by a line $e(P)$



If we put on the Farey triangulation



Then e(P) is a union of line segments, one for each triangle with a vertex at ∞.