Lecture 7

We have

$\mathcal{O}_n$ = (reduced) outerspace

$\sigma(g,G)$ = open simplex in $\mathcal{O}_n$

$$\mathcal{O}_n = \coprod \overline{\sigma(g,G)} \Big/ \text{face relations}$$

$\Sigma(g,G)$ = bordification of $\sigma(g,G)$
= closed cell with interior homeomorphic to $\sigma(g,G)$

$\overline{\mathcal{O}_n}$ = bordification of $\mathcal{O}_n$

$$= \coprod \Sigma(g,G) \Big/ \text{face relations}$$

Point in $\overline{\Theta}_n = (g, G = C_0 \supset \cdots \supset C_k)$,

with volume 1 metrics on each C_i,
st $C_i = \text{core}(\text{length } 0 \text{ edges of } C_{i-1})$

Vertex of $\overline{\Theta}_n = (g, G = C_0 \supset \cdots \supset C_n)$

where $C_i = \text{core}(C_{i-1} \setminus e_i)$
(longest possible chain)

T = edges in $G \setminus \{e_1, \dots, e_n\}$
form a maximal tree

so vertex $= (g, G, T, e_1, \dots, e_n)$

$e_1, \dots, e_n$ = edges of $G \setminus T$

$\text{Out}(F_n)$ acts by changing g

Thm: $\overline{\Theta}_n$ is contractible, action is cocompact.

Bieri-Eckmann tells us that $\Gamma = Out(F_n)$ is a virtual duality group of dimension d if

$$H^k(\Gamma; \mathbb{Z}[\Gamma]) = \begin{cases} \text{free abelian} & k = d \\ 0 & k \neq d \end{cases}$$

The relation with $\overline{\mathcal{O}}_n$ is that there is a natural identification

$$H^k(\Gamma; \mathbb{Z}[\Gamma]) \leftrightarrow H^k_c(\overline{\mathcal{O}}_n)$$

where H^*_c is cohomology with compact supports

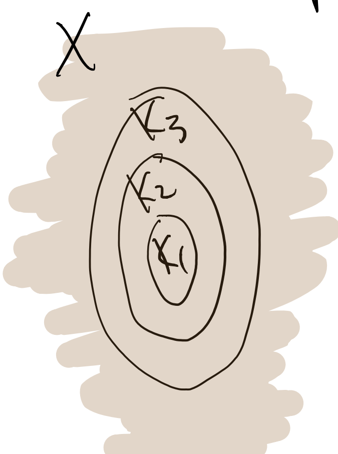
Unlike the case for GL_n, however, $\overline{\mathcal{O}}_n$ is <u>not</u> a manifold, so we <u>don't</u> have an identification

$$H^k_c(\overline{\mathcal{O}}_n) \cong H^{k-1}(\partial\overline{\mathcal{O}}_n)$$

So we have to compute H^k_c directly.

Cohomology with compact supports

Suppose $K_1 \subset K_2 \subset \cdots$ are compact subsets of a space X, $\bigcup K_i = X$.



$$X - K_1 \supset X - K_2 \supset \cdots$$

$C^k(X, X-K_i) =$ k-cochains on X that vanish on $X - K_i$

$=$ k-cochains supported on K_i

so $C^k(X, X-K_i) \hookrightarrow C^k(X, X-K_{i+1})$

and $\lim_{i\to\infty} C^k(X, X-K_i) = C^k_c(X)$

so $H^k_c(X) = \lim_{i\to\infty} H^k(X, X-K_i)$

For us $X = \overline{\Theta}_n$ is contractible, so the long exact sequence in cohomology gives

$$H^{k-1}(X) \to H^{k-1}(X-K_i) \to H^{k}(X, X-K_i) \to H^{k}(X)$$

so $k \geq 2$, get $H^k(X, X-K_i) \cong H^{k-1}(X-K_i)$

$k=1$, get $H^1(X, X-K_i) = H^0(X-K_i)/\mathbb{Z}$

$$\left(0 \to \mathbb{Z} \to H^0(X-K_i) \to H^1(X, X-K_i) \to 0 \right)$$

so
$$H^k_c(X) = \lim_{i \to \infty} H^{k-1}(X-K_i) \quad k>1$$

$$H^1_c(X) = \lim_{i \to \infty} H^0(X \setminus K_i)/\mathbb{Z}$$

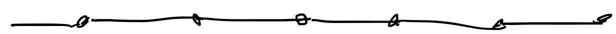
Ie there is a correspondence

$$\begin{array}{ccc} \bar{f} & \to & f \\ \delta \big\downarrow & & \big\downarrow \delta \\ \delta \bar{f} & & 0 \end{array}$$

$(k-1)$-cocycles supported on the <u>complements</u> of compact sets.

$\updownarrow$

k-cocycles supported on compact sets

eg $X = \mathbb{R}$ 

C^0X : an integer n_i at i $\quad f : i \mapsto n_i$

$f \in C^0 X$ is a <u>cocycle</u> iff $f(\partial e) = 0$ $\forall e$ iff f is constant.

$f \in C^0_c X$ is a cocycle iff $f \equiv 0$

so $H^0_c X = 0$

every $f \in C^1 X$ is a cocycle

But it's easy to construct $g \in C^0 X$ w $\delta g = g\partial(\cancel{e}) = f$ So $H^1(X) = 0$

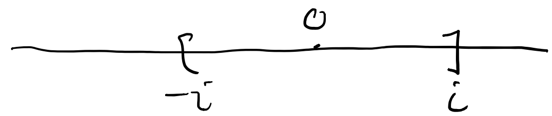
Similarly, every $f \in C^1_c X$ is a cocycle

But $f \longmapsto \sum_e f(e)$

is an isomorphism. $H^1_c X \cong \mathbb{Z}$:

$f - f' = \delta g$ iff $\sum_e f(e) = \sum_e f'(e)$

$K_i = [-i, i]$



$$H^1_c = \lim_{i \to \infty} H^0(X - K_i)/\mathbb{Z}$$

$$= \lim_{i \to \infty} \mathbb{Z}^2/\mathbb{Z}$$

$$= \mathbb{Z}$$

$$H^0_c = \lim_{i \to \infty} H^{-1}(X - K_i) = 0$$

$\overline{O}_n$ is a cell complex – <u>but</u> it's more convenient to replace $\overline{O}_n$ by a <u>simplicial</u> complex S_n

<u>vertices</u> = vertices of $\overline{O}_n$

$$= (g, G, T, e_1, \dots, e_n)$$

$$\sim (c_T \circ g, R = G/T, e_1, \dots, e_n)$$

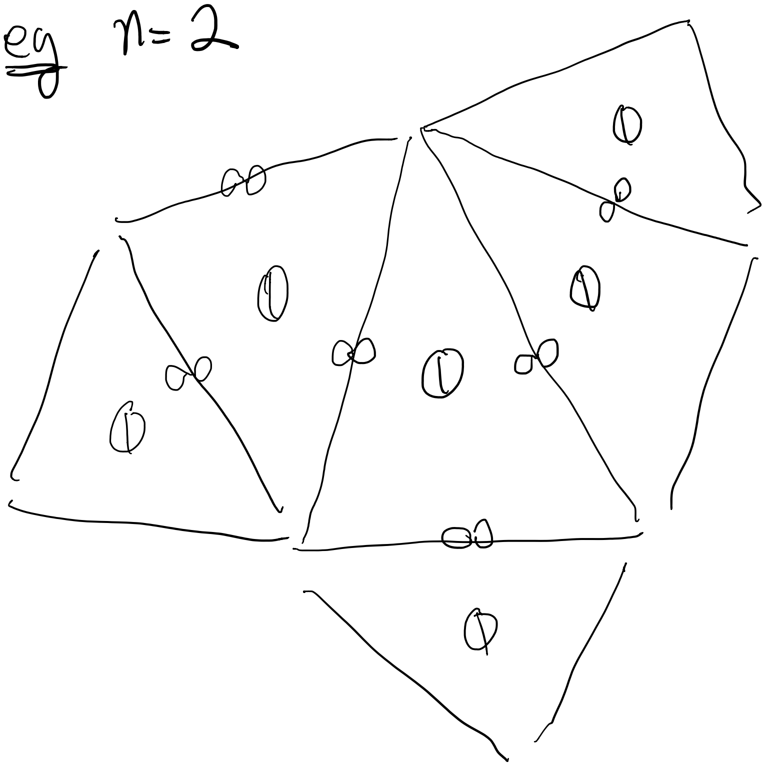
<u>simplices</u>: $v_0, \dots v_k$ form a

k-simplex if they are all vertices of $\Sigma(g, G)$ for some (g, G)

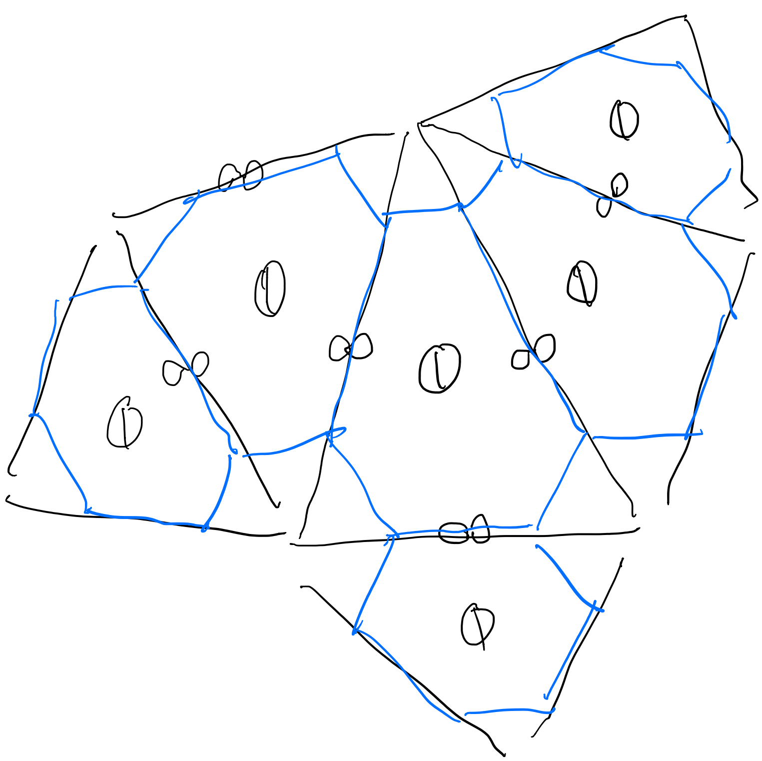
<u>special</u> <u>simplices</u>: spanned by <u>all</u> vertices of some $\Sigma(g, G)$

$$= s(g, G)$$

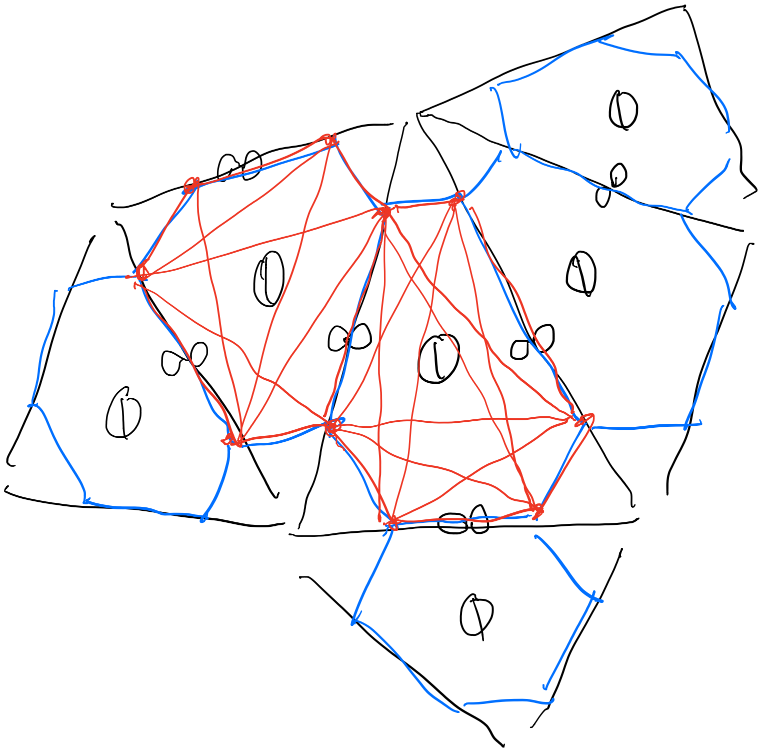
eg $n=2$



Θ_K



$\overline{\Theta}_K$



$\mathcal{S}_n$

The special simplices of $\mathcal{S}_2$ are

5-dimensional $s(g, \Phi)$

and 1-dimensional $s(g, \infty)$

$\mathcal{S}_n = \bigcup s(g, G) /$ face rel'ns

$\overline{\Theta}_n = \bigcup \Sigma(g, G) /$ face rel'ns

$\Theta_n = \bigcup \overline{\sigma(g, G)} /$ face rel'ns where $X(g', G')$ is a face of $X(g, G)$

if there is a forest collapse c

with
$$\begin{tikzcd} G \arrow[r, "c"] & G' \\ R_n \arrow[u, "g"] \arrow[ur, "c\circ g \simeq g'"'] & \end{tikzcd}$$

All 3 spaces are covered by $X(g, G)$ with G maximal,
intersections are $X(g', G')$'s
all $X(g, G)$'s are contractible
same nerve $K_n \Rightarrow$ all are homotopy equiv. to $K_n \simeq pt$ by CV theorem.

Want to exhaust $\mathcal{S}_n$ by compact subsets

idea: Define a Morse function on the vertices of $\mathcal{S}_n$

$$\mu : V(\mathcal{S}_n) \longrightarrow A = \text{ordered abelian gp}$$

which has a unique minimum at a vertex v_0 and totally orders the rest of the vertices

$$\mu(v_0)<\mu(v_1)<\mu(v_2)<\cdots$$

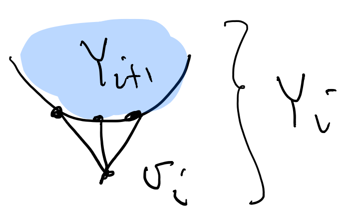
Let K_i = subcomplex of S_n spanned by $v_0,\dots,v_i$

$$\emptyset\subset K_0\subset K_1\subset K_2\subset\cdots \qquad \bigcup K_i = S_n$$

$$S_n\supset S_n-K_0\supset S_n-K_1\supset\cdots$$

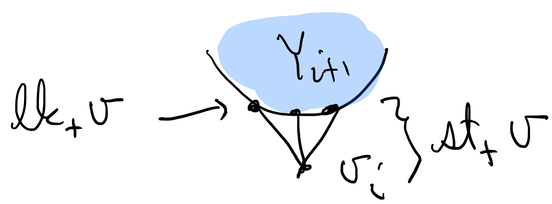
$$S_n\supset Y_0\supset Y_1\supset\cdots \qquad \bigcap Y_i=\emptyset$$

so $$H^k_c(S_n)=\lim_{i\to\infty} H^{k-1}(Y_i)$$



Define $\mathrm{lk}_+ v$ = subcomplex of $\mathrm{lk}\, v$ spanned by vertices u with $\mu(u) > \mu(v)$

$$\mathrm{st}_+ v = v * \mathrm{lk}_+ v$$



Then $Y_i = Y_{i+1} \cup_{\mathrm{lk}_+ v_i} \mathrm{st}_+ v_i$

M-V sequence is

$$\leftarrow \tilde H^k(\mathrm{lk}_+ v_i) \leftarrow \tilde H^k(Y_{i+1}) \oplus \overset{0}{\cancel{\tilde H^k(\mathrm{st}_+ v)}} \leftarrow \tilde H^k(Y_i) \leftarrow$$

$$\tilde H^{k-1}(\mathrm{lk}_+ v_i) \leftarrow \tilde H^{k-1}(Y_{i+1}) \oplus \overset{0}{\cancel{\tilde H^{k-1}(\mathrm{st}_+ v)}} \leftarrow \tilde H^{k-1}(Y_i) \leftarrow$$

$$\boxed{\text{Suppose } \mathrm{lk}_+ v_i \simeq \bigvee S^d \text{ for all } i}$$

Then the Mayer-Vietoris sequence says

$$\tilde{H}^k(Y_{i+1}) \cong H^k(Y_i) \cong \dots H^k(Y_0) = 0$$

if $k \neq d, d+1$

and

$$\begin{tikzcd}
0 & & & & \\
H^{d+1}(Y_{i+1}) \arrow[u] & H^{d+1}(Y_i) \arrow[l] & H^d(\mathrm{lk}_+ v_i) \arrow[l] & H^d(Y_{i+1}) \arrow[l] & H^d(Y_i) \arrow[l] \\
& & & & 0 \arrow[u]
\end{tikzcd}$$

$H^{d+1}(Y_i) = 0$ by induction, which $\Rightarrow$

$H^{d+1}(Y_{i+1}) = 0$

and we are left with:

$$\begin{tikzcd}
0 & \mathbb{Z}^k \arrow[l] \arrow[r, bend right] & H^d(Y_{i+1}) \arrow[l] & H^d(Y_i) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

splits b/c $\mathbb{Z}^k$ is free

$H^d(Y_i)$ free abelian by induction

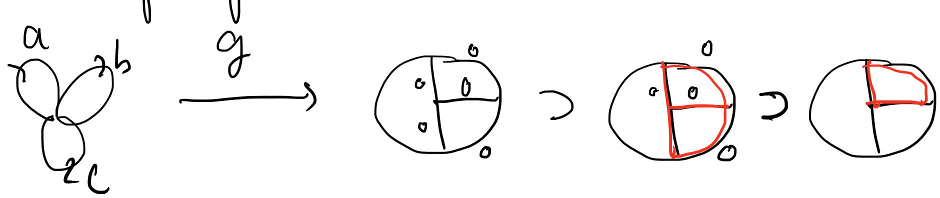
$\Rightarrow$ $H^d(Y_{i+1})$ is free abelian

So, we need a Morse function on the vertices of $\mathcal{S}_n$ = vertices of $\overline{\mathcal{O}_n}$ such that $lk_+ v \simeq \bigvee S^{2n-4}$ for all vertices v.

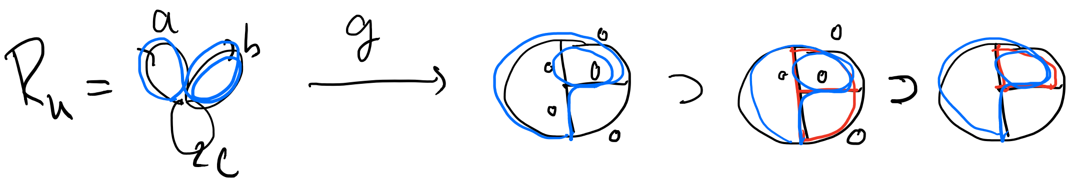
B-F define a Morse function on all points of $\mathcal{O}_n$

point $x = (g, G = C_0 \supset C_1 \supset \cdots \supset C_k)$
w/ volume 1 metric l_i on each C_i
C_i = core (length 0 edges of C_{i-1})

α = a set of cyclic words in F_n

$R_n =$ 

Represent $w \in \alpha$ by an immersed loop in R_n



$w = ab^2$

measure total length of $g(w)$ (tightened) in $C_0, C_1, \ldots, C_k$

Add up lengths for all $w \in \alpha$

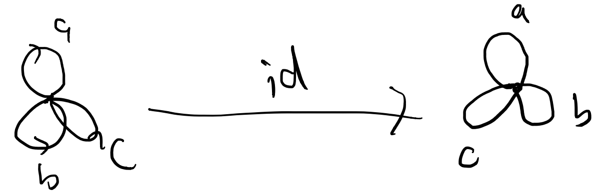
get k-tuple of numbers

$$(l_0(\alpha), l_1(\alpha), \ldots l_k(\alpha)) \in \mathbb{R}_+^k$$

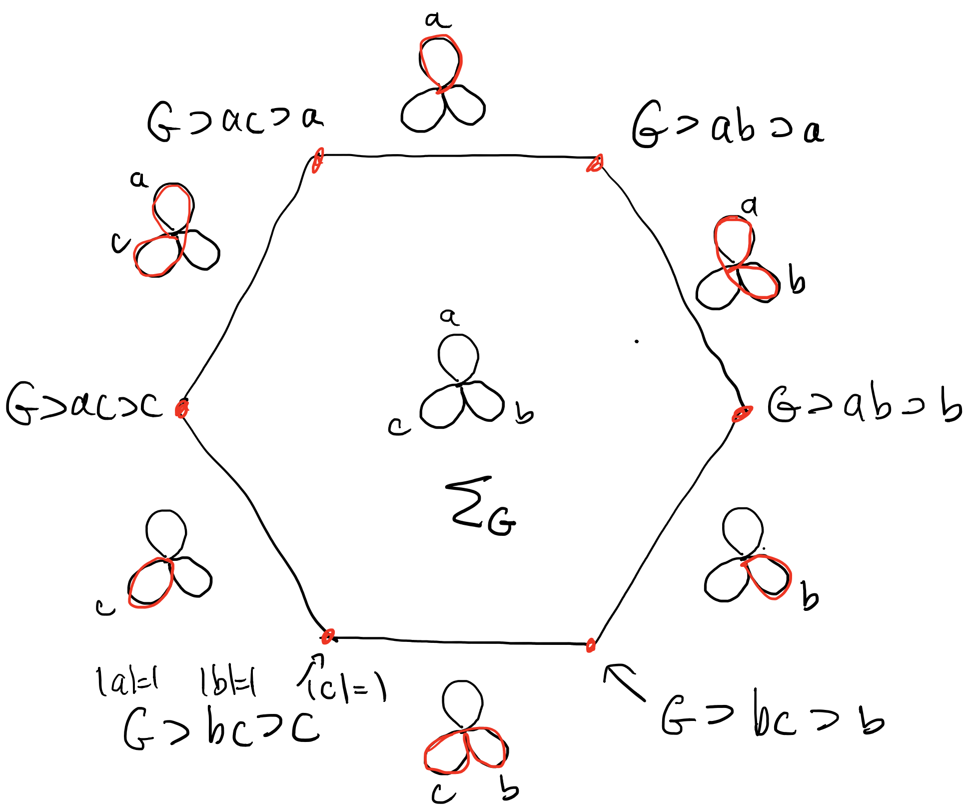
$$(\mathbb{R}_+ = [0, \infty))$$

Note length of flag is $\leq n$, embed $\mathbb{R}_+^k \hookrightarrow \mathbb{R}_+^n$ as first k coords, get

$$L^\alpha(x) = (l_0(\alpha), \ldots l_k(\alpha), 0, \ldots, 0) \in \mathbb{R}_+^n$$

Eg $x =$  $\in \overline{\Theta}_n$

$w = a^2 b^3 c^5$, $\alpha = \{w\}$



$l_i(\alpha)$ = length of w in C_i

$L^\alpha(x) = (l_1(\alpha), l_2(\alpha), l_3(\alpha))$

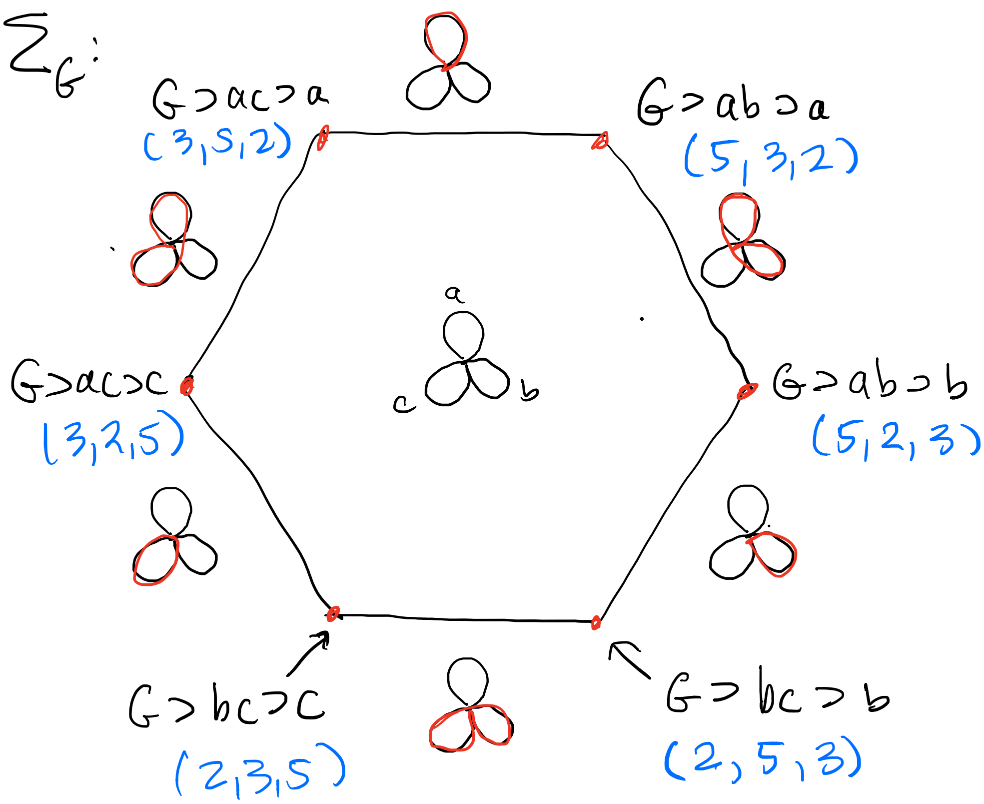
at a vertex, in each C_i only <u>one</u> edge has non-zero length, which is $= 1$.

eg at $v = G \supset bc \supset c$

$$\begin{array}{ccc} G & \supset bc & \supset c \\ \uparrow & \uparrow & \uparrow \\ |a|=1 & |b|=1 & |c|=1 \end{array}$$

so $L^{\alpha}(v) = (2,3,5)$

On Σ_G:



(Note the rose simplex in $\mathcal{O}_n$ is compactified to a permutohedron in $\overline{\mathcal{O}_n}$.)

Can also compute L^α on interior points, eg at the central point, $L^\alpha = (\frac{10}{3}, 0, 0)$

Unfortunately, can't distinguish all vertices with a single α.

$\mathcal{A} = (\alpha_1, \alpha_2, \ldots)$ a <u>sequence</u> of sets of cyclic words

Define

$$L^{\mathcal{A}}(p) = (l_0(\alpha_1), l_0(\alpha_2), \ldots, l_1(\alpha_1), l_1(\alpha_1), \ldots l_n(\alpha_1), l_n(\alpha_2), \ldots)$$

$$\in (\mathbb{R}_+^{\mathcal{A}})^n$$