Lecture 8

$\mathcal{O}_n$ = (reduced) outerspace $= \bigcup \sigma(g,G)/\sim$

$\overline{\mathcal{O}_n}$ = bordification of $\mathcal{O}_n = \bigcup \Sigma(g,G)/\sim$

$\mathcal{S}_n$ = s.cplex $= \bigcup s(g,G)/\sim$

$s(g,G)$ = simplex on vertices of $\Sigma(g,G)$

point of $\Sigma(g,G)$ $= (g, G = C_0 \supset C_1 \supset \cdots \supset C_k)$
C_i = core of the length 0 subgraph of C_{i-1}

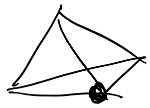
$$\Sigma(G) \subset \overline{\sigma}(G) \times \prod_{C \subsetneq G} \overline{\sigma}(C)$$

Compatibility: If the C-metric on $C' \subset C$ is not zero, then the C'-metric is the C-metric rescaled

So a point is a vertex of $\Sigma_G \Rightarrow$ it's a vertex of each $\overline{\sigma}(C)$
<u>ie</u> in each C only one edge has non-0 length (which must be 1)

Suppose e_0 is the non-zero edge in G

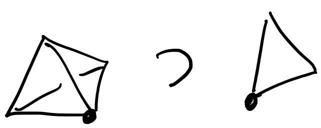
Compatibility $\Rightarrow$ if $C \subset G$ contains e_0, then the metric on C = a vertex of $\bar\sigma(C)$



$e_0 = 1$

But $C_1 \subseteq G - e_0$ can have <u>any</u> volume 1 metric

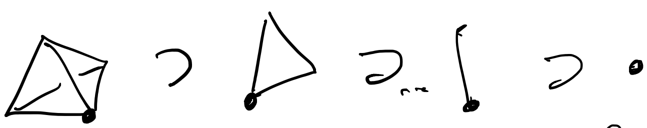
$G = C_0 > C_1$ at a vertex $\Rightarrow \exists!\ e_1 \in C_1$, of length 1



$e_0 = 1$ $e_1 = 1$

$\bar\sigma(G) \supset \bar\sigma(C_1)$

<u>etc</u>: have a chain of n core graphs ($n-1$ inclusions)



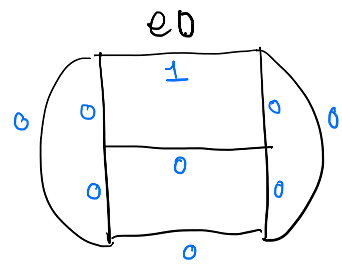
$e_0 = 1$ $e_1 = 1$ $e_{n-1} = 1$

$\bar\sigma(G) \supset \bar\sigma(C_1) \supset \bar\sigma(C_2) \supset \bar\sigma(C_{n-1})$

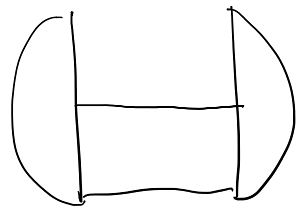
- $C_{i+1} = \text{core}(C_i - e_i)$
- metric on each C_i is 0 except at e_i

eg

G

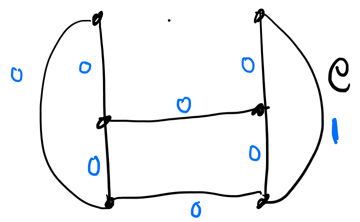


$G - e_0$

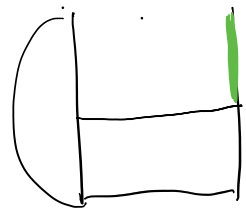


$\cup$

C_1

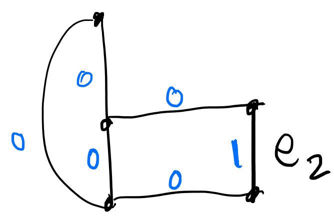


$C_1 - e_1$

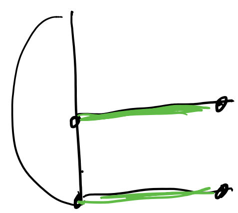


$\cup$

C_2

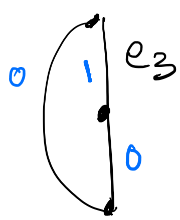


$C_2 - e_2$

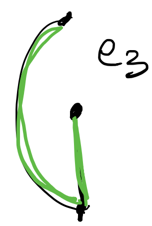


$\cup$

C_3



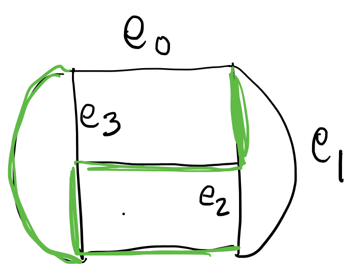
$C_3 - e_3$



In original graph

Green edges

= maximal tree



So a vertex $= (g, G, T, e_0, \dots, e_{n-1})$

T a max. tree in G

$e_0, \dots, e_{n-1} =$ ordering of edges in $G-T$

Face relations identify

$(g, G, T, e_0, \dots, e_{n-1})$ with $(c_T g, G/T, e_0 \cdots e_{n-1})$

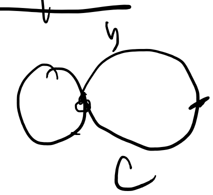
$G/T = R$ is a rose, so all vertices v are of the form $(r, R, e_0, \dots, e_{n-1})$, ie lie in rose faces of $\Sigma(g, G)$.

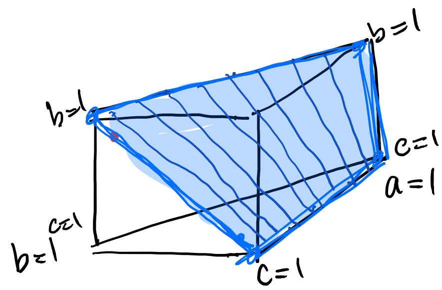
Different maximal trees T' give different rose faces of $\Sigma(g,G)$, each with $n!$ vertices $(g, G, T', e'_0, \dots, e'_{n-1})$, so there are

$$n! \cdot (\#\text{ max trees in } G)$$

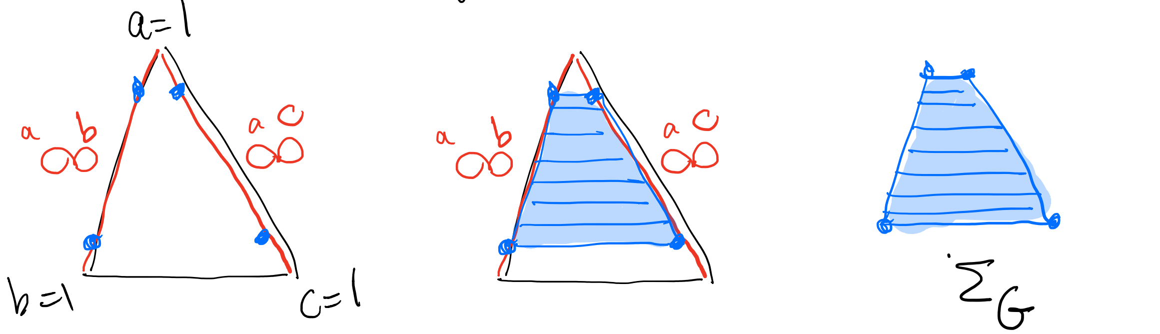
vertices in $\Sigma(g,G)$ (also in $S(g,G)$)

Examples

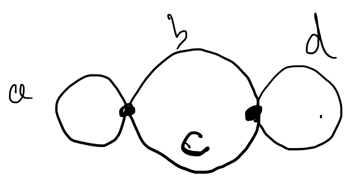
(1) 



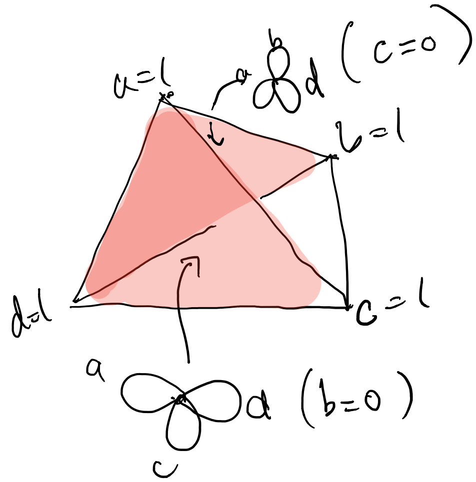
Embed a permutahedron in each rose face of $\sigma(g, G)$; then Σ_G is homeomorphic to the convex hull of all the vertices you see:

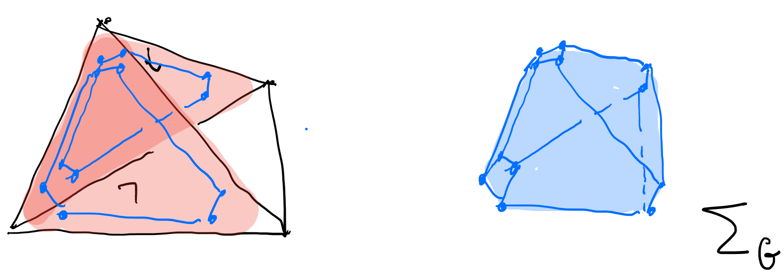


(2)



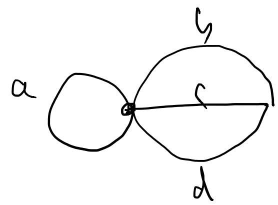
T = b or c

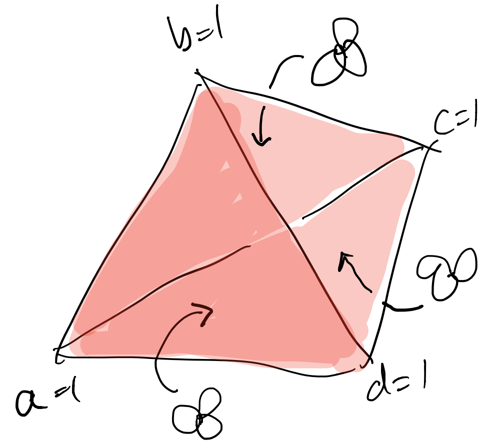




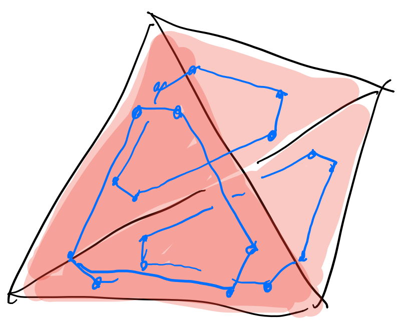
$2 \cdot 6 = 12$ vertices

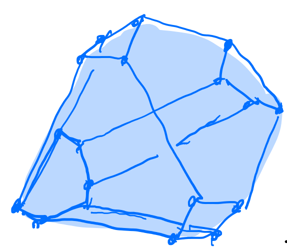
3





$T = b, c$ and d

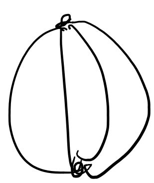




Σ_G

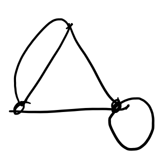
$3 \cdot 6 = 18$ vertices

4



4 trees $\Rightarrow$ $4 \cdot 6 = 24$ vertices!

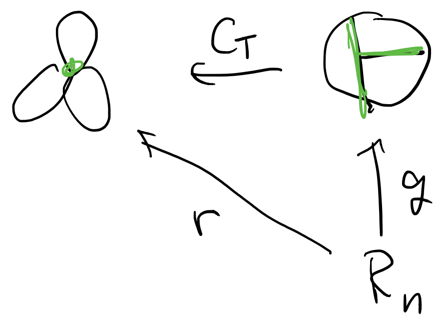
5



5 trees $\Rightarrow$ $5 \cdot 6 = 30$ vertices ...

$v = (r, R, e_0, \ldots, e_{n-1})$

If $\exists\, T \subset G$ st $(r, R) = (c_T \circ g, G/T)$:



say (g, G) is a blowup of (r, R)

If (g, G) is a blowup of (r, R)
$\Sigma(r, R)$ is a face of $\Sigma(g, G)$, so
<u>all</u> the vertices of $\Sigma(r, R)$ are in $\Sigma(g, G)$

so all the vertices of $\Delta(g, G)$ are adjacent to v in S_n.

ie $\quad lk_{S_n}(v) = lk_{S_n}(r, R, e_0, \ldots, e_{n-1})$

$= \{(g, G, T', e'_0, \ldots, e'_{n-1}) \mid (g, G) \text{ is a blowup of } (r, R)\}$

We want to order the vertices of $\mathcal{S}_n$, $(v_i)_{i\in\mathbb{N}}$, in such a way that

$$lk_+ v \simeq \bigvee S^{2n-4} \text{ or contractible}$$

for all vertices v.

Given any set α of cyclic words in F_n and any point $x = (g, G = C_0 \supset C_1 \supset \cdots \supset C_k)$ in $\overline{O}_n$ we defined

$$l_i(\alpha) = l_i^\alpha(x) = \text{length of } g(\alpha) \text{ in metric on } C_i$$

$$l^\alpha(x) = (l_0^\alpha(x), \ldots, l_k^\alpha(x), 0, \ldots, 0) \in \mathbb{R}_+^n$$

and $\lambda^\alpha(x) = \sum_{i=0}^{n-1} l_i^\alpha(x)$

in particular, for any vertex $v = (g, R, e_0 \ldots e_{n-1})$

$$l_i^\alpha(v) = \# \text{ of times } g(\alpha) \text{ crosses } e_i$$

$$\lambda^\alpha(v) = \text{(cyclic) word length of } g(\alpha)$$

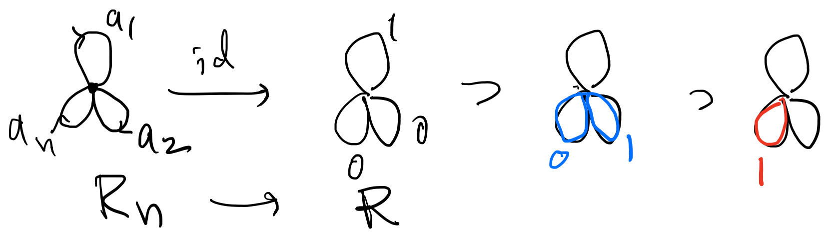
Given any sequence $\mathcal{A} = (w_0, w_1, w_2, \dots)$

Define $L_i^{\mathcal{A}}(x) = (l_i^{w_0}(x), l_i^{w_1}(x), l_i^{w_2}(x), \dots)$

and $L^{\mathcal{A}}(x) = (L_0^{\mathcal{A}}(x), \dots L_k^{\mathcal{A}}(x), 0, \dots, 0)$

$$\in (\mathbb{R}_+^{\mathcal{A}})^n$$

All of these are ways of measuring the "complexity" of the map g, which we use as a measure of distance to the vertex $(id, R_n, e_0, \dots, e_{n-1})$ below:



Any homeomorphism $h: R \to R$ permutes and inverts the edges of R

$R_n \xrightarrow{h \circ id} R$ is some vertex of $\Sigma(g, R)$

α_0 = primitive unoriented cyclic words of length 1 or 2
$= \{ a_1, a_2 \ldots, a_n, \ a_1 a_2, \ a_1 a_2^{-1}, \ldots, a_{n-1} a_n^{-1} \}$

Prop (Culler-V 86) There are only finitely many vertices v with $\lambda^{\alpha_0}(v) < N$, for any $N \in \mathbb{N}$. The minimum occurs at vertices of $\Sigma(id, R)$.

Next
List all conjugacy classes in F_n in order of increasing size: (any order within that..)
$\mathcal{C} = (a_1, a_2, \ldots, a_1 a_2, \ldots, a_1 a_2 a_3, \ldots)$
$= (w_1, w_2, w_3, \ldots \ldots)$

Thm (Chiswell, Culler-Morgan, Alperin-Bass)
$$L^{\mathcal{C}}(v) = L^{\mathcal{C}}(w) \Rightarrow v = w.$$

(actually, can get away with only considering primitive conjugacy classes — ie classes of words $\varphi(a_1)$ for φ an automorphism.)

Now define our Morse function:

$$\mu(v) = (\lambda^{\alpha_0}(v), L^{\mathcal{B}}(v)) \in \mathbb{N} \times (\mathbb{N}^{\mathcal{B}})^n$$

$$= (\lambda^{\alpha_0}(v), l_0^{w_1}(v), l_0^{w_2}(v), \ldots, l_1^{w_1}(v), l_1^{w_2}(v), \ldots, l_k^{w_1}(v), l_k^{w_2}(v), \ldots, l_{n-1}^{w_1}(v), l_{n-1}^{w_2}(v), \ldots)$$

$\mathbb{N} \times (\mathbb{N}^{\mathcal{B}})^n$ is ordered lexicographically:
$\mu(v) < \mu(v')$ if the coord of $\mu(v)$ $<$ the coord of $\mu(v')$ at the first place they differ.

By the quoted theorems, we can use μ to totally order the vertices of $\overline{O}_n = V(\mathcal{S}_n)$:
There are only finitely many v with

$\chi^{\alpha_0}(v)$ minimal, and $L^{\mathcal{C}}$ totally orders them.
Only fin. many v with $\chi^{\alpha_0}(v) < k$ for each k
so can order the v:

$v_1, v_2, v_3, \ldots$

and write S_n as an increasing union
of $K_i = \text{span}\langle v_1, \ldots, v_i \rangle$ compact, etc.

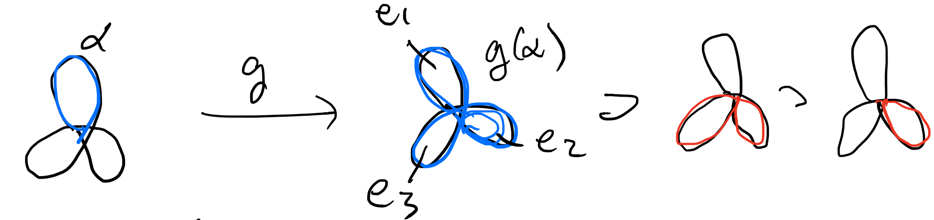
It remains to show $lk_+(v)$ is contractible
or $\simeq \bigvee S^{2n-4}$.

Easy case: Suppose v is not maximal
in $\Sigma(r, R)$, ie reordering the e_i makes
μ larger, say $\mu(v') > \mu(v)$.

Since every vertex of $\Sigma(r, R)$ is in every
$\Sigma(g, G)$ adjacent to v, v' is adjacent
to every vertex of $lk_+(v)$ in S_n, ie
$lk_+(v)$ is a cone on v', so is contractible.

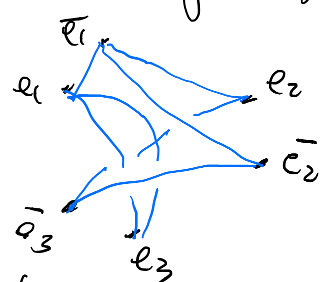
If v is maximal in its rose face,
we actually have to work.

How to think about $l_i^\alpha(v)$?

v:


Hard to see $l_i^\alpha(v)$ in this picture

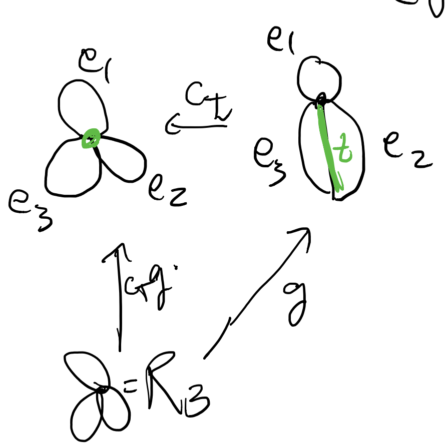
Instead cut graph at midpts of edges, let blue edges loose

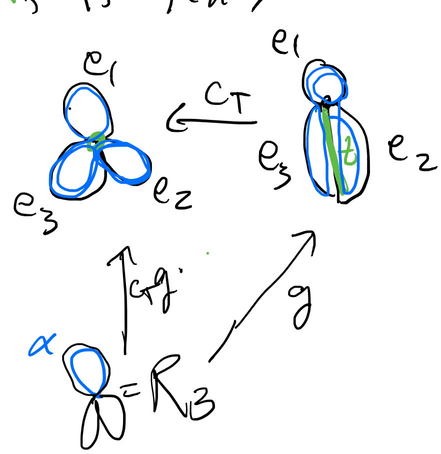


= **star graph** of $g(\alpha)$

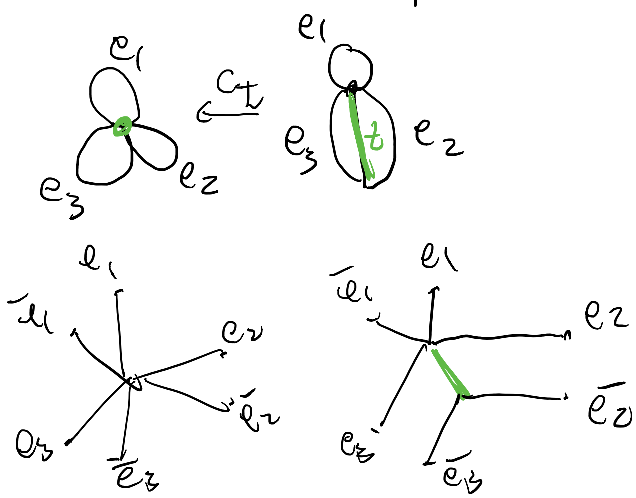
$l_i^\alpha(v)$ = valence of e_i (or $\bar{e}_i$) in the star graph:

If v is described as a vertex of $\Sigma(g, G)$, ie $v = (g, G, T, e_1, \dots, e_n)$:

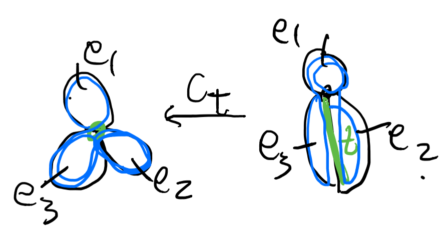


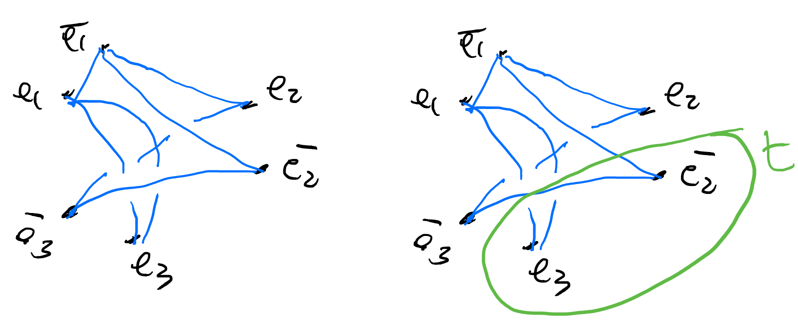


Cut edges e_i as before:



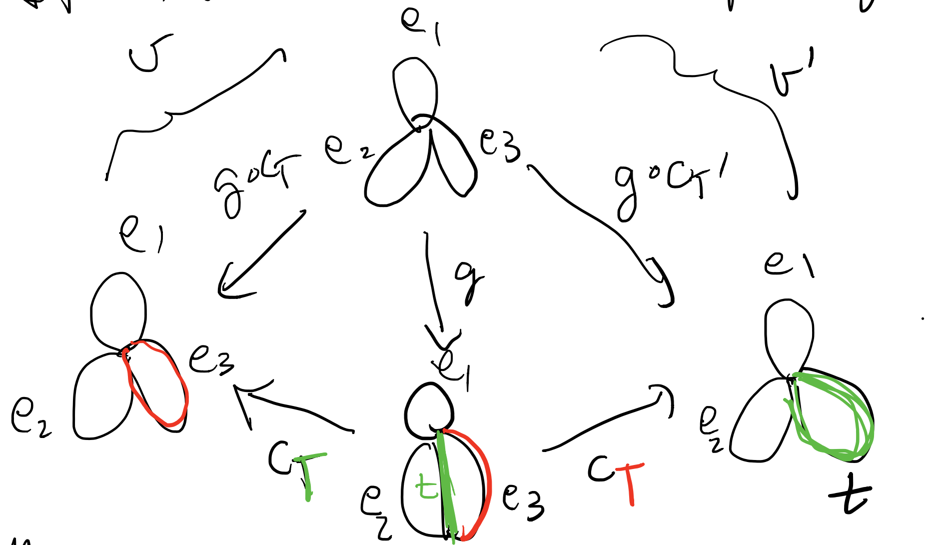
With $g(\alpha)$ in the picture:





# of times $g(\alpha)$ crosses t = # of intersections of star graph with partition circle.

If v, v' are both vertices of $\Sigma(g, G)$, eg



then

$$L^{\mathcal{G}}(v) = \left(L^{\mathcal{G}}_{e_1}(v),\ L^{\mathcal{G}}_{e_2}(v),\ L^{\mathcal{G}}_{e_3}(v) \right)$$

$$< L^{\mathcal{G}}(v') = \left(L^{\mathcal{G}}_{e_1}(v),\ L^{\mathcal{G}}_{e_2}(v),\ L^{\mathcal{G}}_{t}(v) \right)$$

means $L^{\mathcal{G}}_{e_3}(v) < L^{\mathcal{G}}_{t}(v)$

Write $|e_3| < |t|$

here t <u>separates</u> e_3 from $\bar{e}_3$, so e_3 is a maximal tree in G' and we can collapse it.

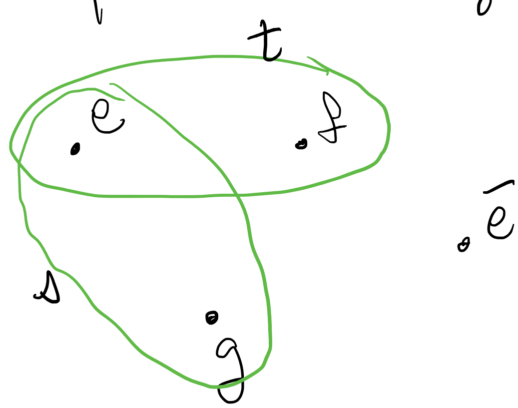
Q: Can you always find a vertex v' adjacent to v which has larger $\mu(v')$?

translates to:

Can you always find a partition circle that intersects <u>more</u> than the valence of some vertex it separates? (more measured in $\mathbb{R}_+^6$!)

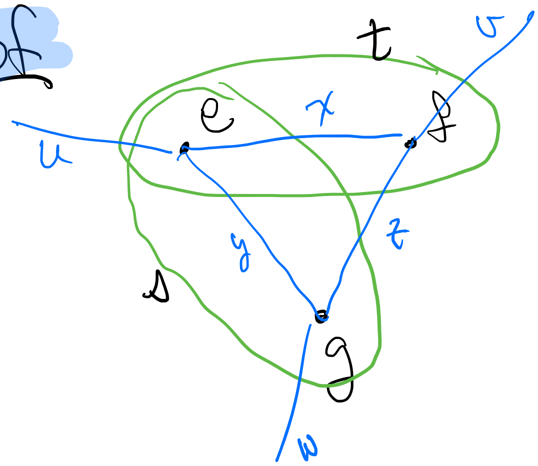
<u>A: yes</u>:

Look at any triple e, f, g
It separates something from its twin, say e:



<u>claim</u>: one of $|s| > |g|$ or $|t| > |f|$
(possibly both)

pf



$|s| = u + \cancel{x} + w + \cancel{z}$ $\quad$ $|f| = \cancel{x} + \cancel{z} + v$

$|t| = u + \cancel{y} + v + \cancel{z}$ $\quad$ $|g| = \cancel{y} + \cancel{z} + w$

$|s| < |f| \Rightarrow u + w < v \Rightarrow u < v - w$

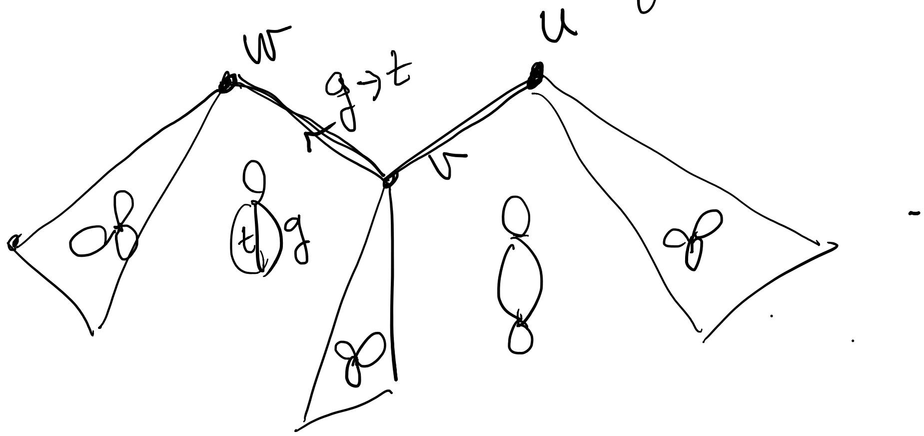
$|t| < |g| \Rightarrow u + v < w \Rightarrow u < w - v$

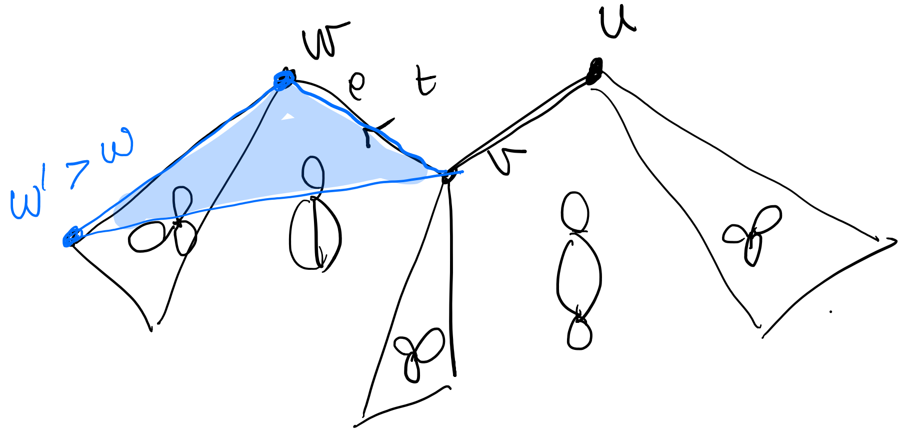
$\Rightarrow u = 0$ ↯

(<u>some</u> word uses eh, for $h \neq \bar{f}, \bar{g}$)

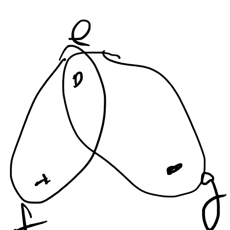
So if we "trade" f for s or g for t, we increase μ.

So $\mathcal{U}_{e+v}$ is always non-$\emptyset$



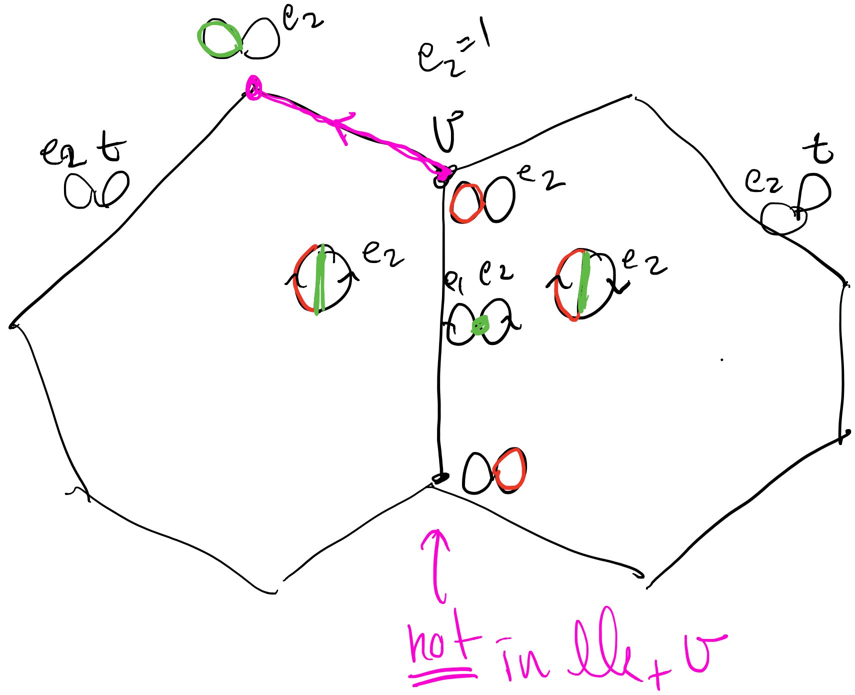


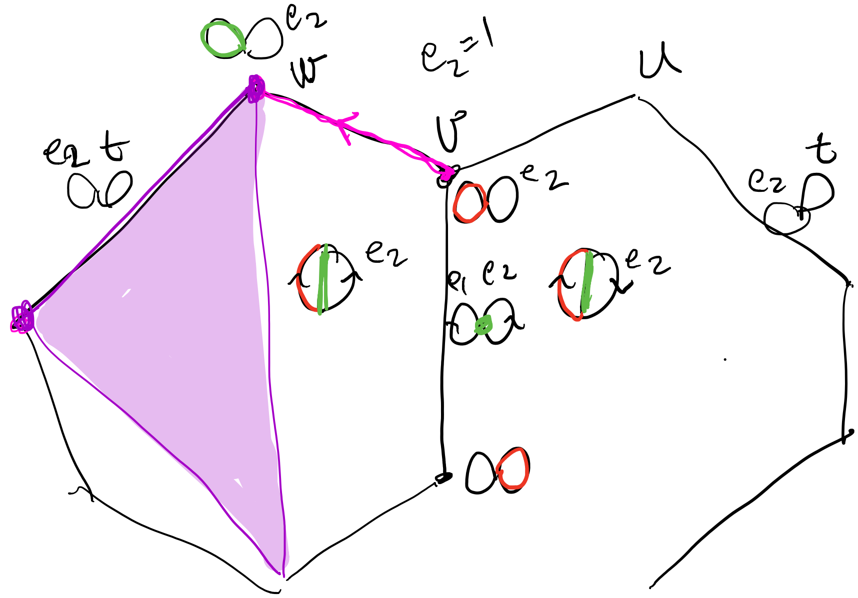
If $e = e_0$ in



Then since e_i w/ $i \geq 1$ got replaced —
we are still at a vertex w/ $e_0 = 1$

$\underline{n=2}$: we wanted to show $lk_+ v \simeq \vee S^0$ or contractible. All we needed to do was show it is non-$\emptyset$:





so $lk_+ v \simeq 1$ (= cone on w)

or $\simeq S^0$ (= $c(w) \sqcup c(u)$)

n=3: We need to show $lk_+ v \simeq \bigvee S^2$,

Idea retract $lk^{sn}_+ v$ onto $lk^{e_0=1}_+ (v)$ then use an inductive argument

The induction used by Bestvina and Feighn is intricate! I think there is a better way but haven't managed to get the details right yet.