L^1-smoothing for the Ornstein-Uhlenbeck semigroup

K. Ball, F. Barthe, W. Bednorz, K. Oleszkiewicz and P. Wolff *

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Abstract

Given a probability density, we estimate the rate of decay of the measure of the level sets of its evolutes by the Ornstein-Uhlenbeck semigroup. It is faster than what follows from the preservation of mass and Markov's inequality.

1 Introduction

Let $N \geq 1$. For $t \geq 0$, consider the probability measure $\mu_t = \frac{1-e^{-t}}{2}\delta_{-1} + \frac{1+e^{-t}}{2}\delta_1$. We simply write μ for $\mu_\infty = \frac{1}{2}(\delta_{-1} + \delta_1)$. On the (multiplicative) group $\{-1,1\}^N$, we consider the semigroup of operators $(T_t)_{t\geq 0}$ defined for functions $f : \{-1,1\}^N \to \mathbb{R}$ by

$$T_t f = f * \mu_t^N.$$

In other words,

$$T_t f(x) = \int f(x \cdot y) K_t(y)\, d\mu^N(y),$$

where $K_t(y) = \prod_{i=1}^N (1 + e^{-t} y_i)$. For $A \subset \{1, \ldots, N\}$, we define $w_A : \{-1,1\}^N \to \mathbb{R}$ by $w_A(y) = \prod_{i=1}^N y_i$ with the convention $w_\emptyset = 1$. This family, known as the Walsh system, forms an orthonormal basis of $L^2(\{-1;1\}^N, \mu^N)$. Expanding the product in the definition of the kernel K_t one readily checks that $T_t w_A = e^{-t\,\mathrm{card(A)}} w_A$.

The above formulations show that $T_s \circ T_t = T_{s+t}$, that T_t is self-adjoint in L^2 and preserves positivity and integrals (with respect to μ^N). As a consequence T_t is a contraction from $L^p = L^p(\{-1;1\}^N, \mu^N)$ into itself: $\|T_t f\|_p \leq \|f\|_p$ for $p \geq 1$. Actually, the hypercontractive estimate of Bonami [2] and Beckner [1] tells more: if $1 < p < q < +\infty$ and $e^{2t} \geq \frac{q-1}{p-1}$, then

$$\|T_t f\|_q \leq \|f\|_p.$$

Hence the semigroup improves the integrability of functions in L^p provided $p > 1$. A challenging problem is to understand the improving effects of T_t on functions $f \in L^1$. In the paper [5], Talagrand asks the following question: for $t > 0$, is there a function $\psi_t : [1, +\infty) \to (0, +\infty)$ with $\lim_{u\to+\infty} \psi_t(u) = +\infty$, such that for every $N \geq 1$ and every function f on $\{-1,1\}^N$ with $\|f\|_1 \leq 1$, and all $u > 1$,

$$\mu^N\left(\{x,\ |T_t f(x)| > u\}\right) \leq \frac{1}{u\,\psi_t(u)}? \tag{1}$$

This would be a strong improvement on the following simple consequence of Markov's inequality and the contractivity property:

$$\mu^N\left(\{x,\ |T_t f(x)| > u\}\right) \leq \frac{\|T_t f\|_1}{u} \leq \frac{\|f\|_1}{u}.$$

Talagrand actually asks a more specific question with $\psi_t(u) = c(t)\sqrt{\log(u)}$ and he observes that one cannot expect a faster rate in u. Question (1) is still open; only in some special cases an affirmative answer is known (see the last chapter). Its difficulty is essentially due to the lack of convexity of the tail

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condition. Nevertheless, the paper [5] contains a similar result for the averaged operator $M := \int_0^1 T_t\, dt$: there exists K such that for all N and $u > 1$,

$$\mu^N\big(\{x;\ |Mf(x)| \geq u\|f\|_1\}\big) \leq K\frac{\log\log u}{\log u}.$$

The goal of this note is to study the analogue of Question (1) in Gauss space.

2 Gaussian setting

Let $n \geq 1$. We work on $\mathbb{R}^n$ with its canonical Euclidean structure $(\langle\cdot,\cdot\rangle, |\cdot|)$. Denote by γ_n the standard Gaussian probability measure on $\mathbb{R}^n$:

$$\gamma_n(dx) = e^{-|x|^2/2}\frac{dx}{(2\pi)^{n/2}}.$$

Let G be a standard Gaussian random vector, with distribution γ_n. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then the Ornstein-Uhlenbeck semigroup $(U_t)_{t\geq 0}$ is defined by

$$\begin{aligned} U_tf(x) &= Ef\big(e^{-t}x + \sqrt{1-e^{-2t}}G\big) \\ &= \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}y\big)e^{-y^2/2}\frac{dy}{(2\pi)^{n/2}} \\ &= (1-e^{-2t})^{-n/2}\int_{\mathbb{R}^n} f(z)e^{-\frac{(z-e^{-t}x)^2}{2(1-e^{-2t})}}\frac{dz}{(2\pi)^{n/2}} \\ &= (1-e^{-2t})^{-n/2}e^{x^2/2}\int_{\mathbb{R}^n} f(z)e^{-\frac{e^{-2t}}{2(1-e^{-2t})}(z-e^tx)^2}d\gamma_n(z), \end{aligned}$$

when f is nonnegative or belongs to $L^1(\gamma_n)$. The operators U_t preserve positivity and mean. They are self-adjoint in $L^2(\gamma_n)$. By Nelson's hypercontractivity theorem [3], U_t is a contraction from $L^p(\gamma_n)$ to $L^q(\gamma_n)$ provided $1 < p \leq q$ and $(p-1)e^{2t} \geq q-1$. It is natural to ask the analogue of Question (1) for U_t: does there exist a function ψ_t with $\lim_{u\to+\infty}\psi_t(u) = +\infty$ such that for all n and all nonnegative or γ_n-integrable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\gamma_n\left(\left\{x,\ |U_tf(x)| > u\|f\|_{L^1(\gamma_n)}\right\}\right) \leq \frac{1}{u\,\psi_t(u)}? \tag{2}$$

This inequality would actually follow from Talagrand's conjecture on the discrete cube. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded, consider the function $g : \{-1,1\}^{nk} \to \mathbb{R}$ defined by

$$g\big((x_{i,j})_{i\leq n, j\leq k}\big) = f\left(\frac{x_{1,1}+\cdots+x_{1,k}}{\sqrt{k}}, \ldots, \frac{x_{n,1}+\cdots+x_{n,k}}{\sqrt{k}}\right).$$

By the Central Limit Theorem, when k goes to infinity, the distribution of g under μ^{nk} tends to the one of f under γ_n, while the distribution of T_tg under μ^{nk} tends to the one of U_tf under γ_n (see e.g. [1]). This allows to pass from (1) for g to (2) for f. The above argument uses boundedness and continuity. These assumptions can be removed by a classical truncation argument, and using the semigroup property: $U_tf = U_{t/2}U_{t/2}f$ where $U_{t/2}f$ is automatically continuous. We omit the details.

To conclude this introduction, let us provide evidence that the functions $\psi_t(u)$ in (2) cannot grow faster than $\sqrt{\log u}$. We will do this for $n = 1$, which implies the general case (by choosing functions depending on only one variable). We have showed that

$$U_tf(x) = \int_{\mathbb{R}} Q_t(x,z)f(z)\,d\gamma_1(z), \tag{3}$$

where

$$Q_t(x,z) = (1-e^{-2t})^{-\frac{1}{2}}\exp\left(\frac{1}{2}\Big(x^2 - \frac{(z-e^tx)^2}{e^{2t}-1}\Big)\right).$$

We are going to choose specific functions $f \geq 0$ with $\int f\, d\gamma_1 = 1$ for which $U_t f$ can be explicitly computed. Note that the previous formula allows to extend the definition of U_t to (nonnegative) measures ν with $\int \varphi\, d\nu < +\infty$ where $\varphi(t) = e^{-t^2/2}/\sqrt{2\pi}$ is the Gaussian density. The simplest choice is then to take normalized Dirac masses $\widetilde{\delta}_y := \varphi(y)^{-1}\delta_y$ as test measures. Obviously $\int \varphi\, d\widetilde{\delta}_y = 1$ and $U_t\widetilde{\delta}_y = Q_t(\cdot, y)$. Actually, by the semigroup property, $Q_t(\cdot, y) = U_{t/2}U_{t/2}\widetilde{\delta}_y = U_{t/2}Q_{t/2}(\cdot, y)$, where $x \mapsto Q_{t/2}(x, y)$ is a nonnegative function with unit Gaussian integral. Hence,

$$\Big\{Q_t(\cdot, y);\ y \in \mathbb{R}\Big\} \subset \Big\{U_{t/2}f;\ f \geq 0 \text{ and } \int f\, d\gamma_1 = 1\Big\}.$$

Fix $t > 0$ and let $u > (1 - e^{-2t})^{-1/2}$. Then using $Q_t(x, y) = Q_t(y, x)$ and setting $v = u\sqrt{1 - e^{-2t}}$ one readily gets that

$$\begin{aligned}\{x;\ Q_t(x, y) > u\} &= \left\{x,\ \exp\left(\frac{1}{2}\Big(y^2 - \frac{(x - e^t y)^2}{e^{2t} - 1}\Big)\right) > v\right\} \\ &= \left(e^t y - \sqrt{(e^{2t} - 1)(y^2 - 2\log v)_+};\ e^t y + \sqrt{(e^{2t} - 1)(y^2 - 2\log v)_+}\right).\end{aligned}$$

For the particular choice $y = y_0 := e^t\sqrt{2\log v}$, one gets

$$\{x;\ Q_t(x, y_0) > u\} = \left(\sqrt{2\log v};\ (2e^{2t} - 1)\sqrt{2\log v}\right).$$

Since for $0 < a < b$, $\gamma_1((a, b)) \geq \int_a^b \frac{s}{b} e^{-s^2/2} ds/\sqrt{2\pi} = \frac{e^{-a^2/2} - e^{-b^2/2}}{b\sqrt{2\pi}}$, we can deduce that

$$\gamma_1\big(\{x;\ Q_t(x, y_0) > u\}\big) \geq \frac{1}{2\sqrt{2\pi}(2e^{2t} - 1)\sqrt{\log v}}\left(\frac{1}{v} - \frac{1}{v^{(2e^{2t}-1)^2}}\right).$$

Combining the above observations yields

$$\liminf_{u \to +\infty} u\sqrt{\log u}\ \sup\Big\{\gamma_1\big(\{x;\ U_{t/2}f(x) > u\}\big);\ f \geq 0 \text{ and } \int f\, d\gamma_1 = 1\Big\} > 0.$$

Hence $\psi_t(u)$ in (2) cannot grow faster than $\sqrt{\log u}$.

Using the same one-dimensional test functions and similar calculations, one can check that for $t > 0$, the image by U_t of the unit ball $B_1 = \{f \in L^1(\gamma_n);\ \|f\|_1 \leq 1\}$ is not uniformly integrable, that is:

$$\liminf_{c \to +\infty}\ \sup_{f \in B_1} \int |U_t f|\, \mathbf{1}_{|U_t f| > c}\, d\gamma_n > 0.$$

Consequently $U_t : L^1(\gamma_n) \to L^\phi(\gamma_n)$ is not continuous when ϕ is a Young function with $\lim_{t \to +\infty} \phi(t)/t = +\infty$. Next, we turn to positive results.

3 Main results

In the rest of this section $B(a, r)$ denotes the closed ball of center a and radius r, while $C(a, r_1, r_2) = \{x \in \mathbb{R}^n;\ r_1 \leq |x - a| \leq r_2\}$. We start with an easy inclusion of the upper level-sets of $U_t f$.

Lemma 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be such that $\int f\, d\gamma_n = 1$. Then for all $t, u > 0$,*

$$\Big\{x \in \mathbb{R}^n;\ U_t f(x) > u\Big\} \subset B\Big(0, \sqrt{\big(2\log u + n\log(1 - e^{-2t})\big)_+}\Big)^c.$$

Proof. As already explained

$$U_t f(x) = (1 - e^{-2t})^{-n/2} e^{x^2/2} \int_{\mathbb{R}^n} f(z) e^{-\frac{e^{-2t}}{2(1 - e^{-2t})}(z - e^t x)^2} d\gamma_n(z).$$

Consequently $U_t f(x) \leq (1 - e^{-2t})^{-n/2} e^{x^2/2} \int f\, d\gamma_n$. Our normalization hypothesis then implies that $\big\{x;\ U_t f(x) > u\big\} \subset \big\{x;\ |x|^2 > 2\log u + n\log(1 - e^{-2t})\big\}$. ☐

The probability measure of complements of balls appearing in the above lemma can be estimated thanks to the following classical fact.

Lemma 2. *For all $n \in \mathbb{N}^*$ there exists a constant c_n such that for all $u \geq \sqrt{2n}$ it holds*

$$\gamma_n\big(B(0,u)^c\big) \leq c_n u^{n-2} e^{-u^2/2}.$$

Actually, when $n \leq 2$ this is valid for all $u > 0$. Also one may take $c_1 = \sqrt{2/\pi}$.

Proof. Polar integration gives that

$$\gamma_n\big(B(0,u)^c\big) = (2\pi)^{-n/2} \cdot n\,\mathrm{vol}_n(B(0,1)) \int_u^{+\infty} r^{n-1} e^{-r^2/2}\,dr.$$

For $u^2 \geq 2n-4$ the map $r \mapsto r^{n-2}e^{-r^2/4}$ is non-decreasing on (u,∞). Thus we may bound the last integral:

$$\int_u^\infty r^{n-2}e^{-r^2/4} \cdot re^{-r^2/4}\,dr \leq \int_u^\infty u^{n-2}e^{-u^2/4} \cdot re^{-r^2/4}\,dr = 2u^{n-2}e^{-u^2/2}.$$

□

Combining the previous statements gives a satisfactory estimate in dimension 1, which improves on the Markov estimate $\gamma_n(U_t f \geq u) \leq \min(1, 1/u)$ if f is non-negative with integral 1.

Proposition 3. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be integrable. Then for all $t > 0$ and $v > 1$,*

$$\gamma_1\Big(\Big\{x; U_t f(x) > v\frac{\int f\,d\gamma_1}{\sqrt{1-e^{-2t}}}\Big\}\Big) \leq \frac{1}{v\sqrt{\pi \log v}}.$$

In higher dimension, the above reasoning gives a weaker estimate than Markov's inequality. However a more precise approach allows to get a slightly weaker decay for the level sets of $U_t f$. Our main result is stated next. It contains a dimensional dependence that we were not able to remove.

Theorem 4. *Let $n \geq 2$ and $t > 0$. Then there exists a constant $K(n,t)$ such that for all non-negative functions f defined on $\mathbb{R}^n$ with $\int f\,d\gamma_n = 1$, for all $u > 10$,*

$$\gamma_n\left(\Big\{x \in \mathbb{R}^n;\ U_t f(x) > u\Big\}\right) \leq K(n,t)\frac{\log\log u}{u\sqrt{\log u}}.$$

Proof. Note that it is enough to show the inequality for u larger than some number $u_0(n,t) > 10$ depending only of n and t. We will just write that we choose u large enough, but an explicit value of $u_0(n,t)$ can be obtained from our argument. Let us define

$$\begin{aligned} R_1 &= R_1(u,n,t) := \big(2\log u + n\log(1-e^{-2t})\big)_+^{\frac{1}{2}}, \\ R_2 &= R_2(u,n) := \big(2\log u + (n-1)\log\log u\big)^{\frac{1}{2}}. \end{aligned}$$

It is clear that for u large enough $R_2 > \sqrt{2n}$ and also $R_2 > R_1 > 0$. So by Lemma 2,

$$\begin{aligned} \gamma_n(B(0,R_2)^c) &\leq c_n e^{-R_2^2/2} R_2^{n-2} = \frac{c_n}{u} \frac{\big(2\log u + (n-1)\log\log u\big)^{\frac{n-2}{2}}}{\big(\log u\big)^{\frac{n-1}{2}}} \\ &\leq \frac{c_n}{u} \frac{\big((n+1)\log u\big)^{\frac{n-2}{2}}}{\big(\log u\big)^{\frac{n-1}{2}}} = \frac{c_n'}{u\sqrt{\log u}}. \end{aligned}$$

By Lemma 1, $\{x;\, U_t f(x) > u\}$ is a subset of $B(0,R_1)^c$. Hence we may write, using Markov's inequality on the annulus $C(0,R_1,R_2)$ and the self-adjointness of U_t:

$$\begin{aligned}
\gamma_n\big(\{x;\ U_t f(x) > u\}\big) &\leq \gamma_n\big(\{x;\ U_t f(x) > u\} \cap B(0,R_2)\big) + \gamma_n\big(B(0,R_2)^c\big) \\
&= \gamma_n\Big(\{x;\ U_t f(x) > u\} \cap B(0,R_1)^c \cap B(0,R_2)\Big) + \gamma_n\big(B(0,R_2)^c\big) \\
&\leq \int \frac{U_t f}{u} \mathbf{1}_{C(0,R_1,R_2)} d\gamma_n + \gamma_n\big(B(0,R_2)^c\big) \\
&= \frac{1}{u} \int \big(U_t \mathbf{1}_{C(0,R_1,R_2)}\big)\, f\, d\gamma_n + \gamma_n\big(B(0,R_2)^c\big) \\
&\leq \frac{1}{u} \|U_t \mathbf{1}_{C(0,R_1,R_2)}\|_\infty + \frac{c_n'}{u\sqrt{\log u}}.
\end{aligned}$$

To prove the theorem, it remains to show that $\|U_t \mathbf{1}_{C(0,R_1,R_2)}\|_\infty = O\Big(\frac{\log\log u}{\sqrt{\log u}}\Big)$. First note that for any set $A \subset \mathbb{R}^n$ and all $x \in \mathbb{R}^n$,

$$U_t \mathbf{1}_A(x) = E\mathbf{1}_A\big(e^{-t}x + \sqrt{1-e^{-2t}}G\big) = P\Big(G \in \frac{A - e^{-t}x}{\sqrt{1-e^{-2t}}}\Big) = \gamma_n\Big(\frac{A - e^{-t}x}{\sqrt{1-e^{-2t}}}\Big).$$

Therefore

$$\|U_t \mathbf{1}_{C(0,R_1,R_2)}\|_\infty = \sup_{a \in \mathbb{R}^n} \gamma_n\big(C(a, \widetilde{R}_1, \widetilde{R}_2)\big),$$

where $\widetilde{R}_i := R_i/\sqrt{1-e^{-2t}}$. The main idea is the above shells can be covered by a thin slab and the complement of a large ball. Set

$$r = r(u) := 2(\log\log u)^{\frac{1}{2}},$$

then for u large enough, Lemma 2 yields

$$\gamma_n\big(B(0,r)^c\big) \leq c_n e^{-r^2/2} r^{n-2} = c_n 2^{n-2} \frac{(\log\log u)^{\frac{n-2}{2}}}{(\log u)^2} \leq c_n'' \frac{\log\log u}{\sqrt{\log u}}.$$

For an arbitrary point $a \in \mathbb{R}^n$,

$$\begin{aligned}
\gamma_n\big(C(a,\widetilde{R}_1,\widetilde{R}_2)\big) &\leq \gamma_n\big(C(a,\widetilde{R}_1,\widetilde{R}_2) \cap B(0,r)\big) + \gamma_n\big(B(0,r)^c\big) \\
&\leq \gamma_n\big(C(a,\widetilde{R}_1,\widetilde{R}_2) \cap B(0,r)\big) + c_n'' \frac{\log\log u}{\sqrt{\log u}}.
\end{aligned}$$

For u large enough, $r < \widetilde{R}_1$, and the forthcoming Lemma 5 ensures that $C(a,\widetilde{R}_1,\widetilde{R}_2) \cap B(0,r)$ is contained in a strip S of width

$$w := \widetilde{R}_2 - \sqrt{\widetilde{R}_1^2 - r^2}.$$

By the product properties of the Gaussian measure, $\gamma_n(S)$ coincides with the one-dimensional Gaussian measure of an interval of length w. Therefore it is not bigger than $w/\sqrt{2\pi} \leq w$. Hence

$$\begin{aligned}
&\gamma_n\big(C(a,\widetilde{R}_1,\widetilde{R}_2) \cap B(0,r)\big) \leq \widetilde{R}_2 - \sqrt{\widetilde{R}_1^2 - r^2} \\
= &\sqrt{\frac{2\log u + (n-1)\log\log u}{1-e^{-2t}}} - \sqrt{\frac{2\log u + n\log(1-e^{-2t})}{1-e^{-2t}} - 4\log\log u} \\
\leq &\frac{(n-1+4(1-e^{-2t}))\log\log u - n\log(1-e^{-2t})}{\sqrt{1-e^{-2t}}\sqrt{2\log u + (n-1)\log\log u}} \leq \kappa(n,t) \frac{\log\log u}{\sqrt{\log u}},
\end{aligned}$$

where the last inequality is valid for u large enough. The proof of the theorem is therefore complete. □

Lemma 5. *Let $0 < r < \rho_1 < \rho_2$ and $a, b \in \mathbb{R}^n$, then the set*

$$C(a,\rho_1,\rho_2) \cap B(b,r)$$

is contained in a strip of width at most $\rho_2 - \sqrt{\rho_1^2 - r^2}$.

Proof. Assume that the intersection is not empty. Then without loss of generality, $a = 0$ and $b = te_1$ with $t > 0$. Let z be an arbitrary point in the intersection. Obviously $z_1 \leq |z| \leq \rho_2$. Next, since $z \in B(b, r)$ and $|z| \geq \rho_1$, one gets

$$r^2 \geq |z - te_1|^2 = |z|^2 - 2tz_1 + t^2 \geq \rho_1^2 - 2tz_1 + t^2.$$

Hence by the arithmetic mean-geometric mean inequality

$$z_1 \geq \frac{1}{2}\left(\frac{\rho_1^2 - r^2}{t} + t\right) \geq \sqrt{\rho_1^2 - r^2}.$$

Summarizing, $z \in \left[\sqrt{\rho_1^2 - r^2}, \rho_2\right] \times \mathbb{R}^{n-1}$. □

4 Product functions on the discrete cube

Finally, we provide an affirmative answer to the Question (1) in the case of functions with product structure.

Proposition 6. *Assume that functions $f_1, f_2, \dots, f_N : \{-1, 1\} \to [0, \infty)$ satisfy $\int f_i \, d\mu = 1$ for $i = 1, 2, \dots, N$. Let $f = f_1 \otimes f_2 \otimes \dots \otimes f_N$, i.e. $f(x) = \prod_{i=1}^N f_i(x_i)$. Then for every $t > 0$ there exists a positive constant c_t such that for all $u > 1$ there is*

$$\mu^N\left(\{x,\ |T_t f(x)| > u\}\right) \leq \frac{c_t}{u\sqrt{\log u}}.$$

Proof. The above result is immediately implied by the following inequality.

Proposition 7. *([4]) Let $b > a > 0$. Let $X_1, X_2, \dots, X_N$ be independent non-negative random variables such that $EX_i = 1$ and $a \leq X_i \leq b$ a.s. for $i = 1, 2, \dots, N$. Then for every $u > 1$ we have*

$$P(\prod_{i=1}^N X_i > u) \leq Cu^{-1}(1 + \log u)^{-1/2},$$

where C is a positive constant which depends only on a and b.

Indeed, $T_t f = T_t f_1 \otimes T_t f_2 \otimes \dots \otimes T_t f_N$, where $T_t f_i : \{-1, 1\} \to [1 - e^{-t}, 1 + e^{-t}]$ satisfy $\int T_t f_i \, d\mu = \int f_i \, d\mu = 1$ for $i = 1, 2, \dots, N$. Thus random variables $X_1, X_2, \dots, X_N$ defined on the probability space $(\{-1, 1\}^N, \mu^N)$ by $X_i(x) = T_t f_i(x_i)$ satisfy assumptions of Proposition 7 with $a = 1 - e^{-t}$ and $b = 1 + e^{-t}$ while $f = \prod_{i=1}^N X_i$.

□

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K. Ball: Department of Mathematics University College London. Gower Street. London WC1E 6BT. UNITED KINGDOM. Email: kmb@math.ucl.ac.uk

F. Barthe: Institut de Mathématiques de Toulouse (CNRS UMR 5219). Université Paul Sabatier. 31062 Toulouse cedex 09. FRANCE. Email: barthe@math.univ-toulouse.fr

W. Bednorz: Institute of Mathematics. University of Warsaw. Banacha 2. 02-097 Warszawa. POLAND. Email: W.Bednorz@mimuw.edu.pl

K. Oleszkiewicz: Institute of Mathematics. University of Warsaw. Banacha 2. 02-097 Warszawa. POLAND *and* Institute of Mathematics. Polish Academy of Sciences. Śniadeckich 8. 00-956 Warszawa. POLAND. Email: K.Oleszkiewicz@mimuw.edu.pl

P. Wolff: Department of Mathematics. Case Western Reserve University. 10900 Euclid Avenue Cleveland, OH 44106-7058. USA. E-mail: pawel.wolff@case.edu