A very short proof of the functional equation for ζ

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This note contains a short proof of Riemann's [R] functional equation for the zeta function. The argument is computationally similar to the Hankel contour argument that was one of Riemann's own, and that is called the "second method" in Titchmarsh [T]. But it is made simpler to follow by the symmetries of sine: the fact that sine is a periodic and an odd function. In this form the argument seems to be new or at least not widely known. We begin with a simple lemma which uses integration by parts to introduce the sine (or hyperbolic sine).

Lemma 1. *For $s > 1$*

$$\Gamma(s)\zeta(s) = \frac{1}{4s}\int_0^\infty \frac{t^s}{\sinh^2(t/2)}\,dt.$$

Proof

$$\begin{aligned}\Gamma(s)\zeta(s) &= \sum_1^\infty \frac{1}{n^s}\int_0^\infty e^{-t}t^{s-1}\,dt = \sum_1^\infty \int_0^\infty e^{-nt}t^{s-1}\,dt \\ &= \int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt = \frac{1}{s}\int_0^\infty \frac{e^t\,t^s}{(e^t-1)^2}\,dt \\ &= \frac{1}{4s}\int_0^\infty \frac{t^s}{\sinh^2(t/2)}\,dt.\end{aligned}$$

□

We now move on to the theorem itself.

Theorem 2. *The function $s \mapsto (s-1)\zeta(s)$ defined on $(1,\infty)$ has an analytic continuation to the entire complex plane and for all s we have*

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$$

with the usual convention regarding the removable singularities.

Proof Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{\pi^2 z^{1-s}}{\sin^2 \pi z}\,dz$$

where the contour C is a vertical line between 0 and 1, say ($\Re z = 1/2$), traversed upwards. In this and the rest of the argument we take powers z^{-s} only for complex numbers satisfying $\Re z \geq 0$ so there is no ambiguity. Because of the rapid growth of $\sin \pi z$ as z moves away from the real axis, this function is easily seen to be an entire function of s.

If $s > 1$ then the factor $(x+iy)^{1-s} \to 0$ uniformly in y as $x \to \infty$ while the factor $\sin^2 \pi(x+iy)$ is periodic in x, so we can compute the integral as the negative of the sum of the residues of

$$\frac{\pi^2 z^{1-s}}{\sin^2 \pi z}$$

at the positive integers. The residue at n is $(1-s)/n^s$ so we find that the integral is $(s-1)\zeta(s)$.

On the other hand, if $s < 0$ the integral makes sense at $z = 0$ so we may shift the contour onto the imaginary axis. By splitting into the upper and lower halves of the axis we get that the integral is

$$\begin{aligned}
\frac{(e^{i\pi/2})^{2-s} - (e^{-i\pi/2})^{2-s}}{2\pi i}\int_0^\infty \frac{\pi^2 y^{1-s}}{\sin^2 \pi i y}\,dy &= -\pi\sin(\pi s/2)\int_0^\infty \frac{y^{1-s}}{\sinh^2 \pi y}\,dy \\
&= \frac{-\pi\sin(\pi s/2)}{(2\pi)^{2-s}}\int_0^\infty \frac{t^{1-s}}{\sinh^2(t/2)}\,dt \\
&= \frac{-\pi\sin(\pi s/2)}{(2\pi)^{2-s}} 4(1-s)\Gamma(1-s)\zeta(1-s) \\
&= 2(2\pi)^{s-1}\sin(\pi s/2)(s-1)\Gamma(1-s)\zeta(1-s)
\end{aligned}$$

where the last but one identity uses the lemma. □

References

[R] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsb. der Berliner Akad., (1858).

[T] E. C. Titchmarsh, revised by D. R. Heath-Brown, The Riemann Zeta-Function, Oxford science publications, OUP (1986).

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