102.18 The *problème des ménages* revisited

The *problème des ménages* (the couples problem) asks to count the number of ways that n male-female couples can sit around a circular table so that no one sits next to her partner or someone of the same gender.

The problem was first stated in an equivalent but different form by Tait [1, p. 159] in the framework of knot theory:

> "How many arrangements are there of n letters, when A cannot be in the first or second place, B not in the second or third, &c."

It is easy to see that the *ménage* problem is equivalent to Tait's arrangement problem: first sit around the table, in $2n!$ ways, the men or alternatively the women, leaving an empty space between any two, then count the number of ways to sit the members of the other gender.

Recurrence relations that compute the answer to Tait's question were soon given by Cayley and Muir [2, 3, 4, 5]. Almost fifteen years later, Lucas [6], evidently unaware of the work of Tait, Cayley and Muir, posed the problem in the formulation of husbands and wives, named it *problème des ménages* and supplied a recurrence relation already described by Cayley and Muir. But it was not until forty-three years later that an explicit formula for Tait's problem was given by Touchard [7], alas without a proof. (For historic accounts, see Kaplansky and Riordan [8] and Dutka [9].)

The first proof for Touchard's explicit formula was given nine years later (sixty-five years later than Tait's question) by Kaplansky [10]. Specifically he showed that the number of permutations of $\{1, \ldots, n\}$ that differ in all places from both the identity permutation and the cyclic permutation $(1 \to 2 \to \ldots \to n \to 1)$ is

$$\sum_{r=0}^{n} (-1)^r \frac{2n}{2n-r} \binom{2n-r}{r} (n-r)! \tag{1}$$

For the proof, he used the principle of inclusion and exclusion in a way that in [11] is characterised as 'simple but not straightforward'. As for the solution for the formulation in terms of husbands and wives, Kaplansky and Riordan [8] in a later exposition wrote:

> "We begin by fixing the positions of husbands and wives, say wives, for courtesy's sake."

Forty-three years later (more than a century later than Tait), Bogart and Doyle [11] gave the first proof of the explicit formula for the *ménage* problem, *not* starting from a reduction to Tait's problem. Interestingly, they characterised their proof as 'non-sexist'. (See also a related article in *The New York Times* [12].)

The method Bogart and Doyle used is indeed clever and simple. They started by counting the number of ways to place n non-overlapping dominos on a cycle with $2n$ positions (see Figure 1). This can be done in d_r ways, where

$$d_r = \frac{2n}{2n-r} \binom{2n-r}{r}. \tag{2}$$

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This calculation, as Bogart and Doyle write, is a routine combinatorial problem, which they leave as an exercise and also give a reference. (We give below an easy proof for completeness.) Then they used the principle of inclusion and exclusion by first counting in how many ways the members of the couples can be seated, so that no two members of the same gender are adjacent, and at least r of the couples each occupy a single domino. However, this counting takes some effort. Here is how to avoid even this, perhaps at the expense of political correctness!

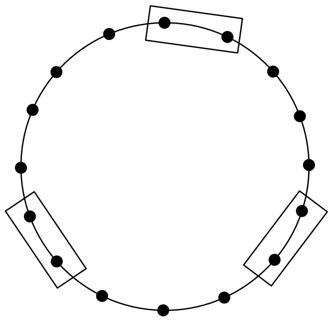

FIGURE 1: A cycle with 16 places and 3 non-overlapping dominos

Alternative proof using both dominos and the reduction to Tait's problem

As in the solution of Kaplansky and Riordan [8], we start by first seating the n women in a circle of $2n$ positions, leaving an empty position between any two of them. There are $2n!$ ways to do this (the factor of 2 being due to the 2 possibilities of the starting point). Then, aiming at using the principle of inclusion and exclusion again, we place, in d_r ways, r *transparent* non-overlapping dominos on the cycle. Now comes what apparently escaped Bogart and Doyle: there are $(n - r)!$ ways to allocate the n men so that at least r of them are seated next to their spouses; indeed r *uniquely determined* men will take the free spaces in respective r dominos (the dominos being transparent, so we know what lies beneath them); the rest will be arbitrarily assigned to the remaining empty spaces of the cycle. So all in all, by the principle of inclusion and exclusion, the answer to the *problème des ménages* is

$$2n! \sum_{r=0}^{n} (-1)^r \frac{2n}{2n - r} \binom{2n - r}{r} (n - r)! \, .$$

Proof of (2): Pick a starting place on the cycle: $2n$ ways; place the identical dominos with an undetermined number of empty spaces in each of the r arcs between any two: these arcs should be filled with $2n - 2r$ empty spaces; determine the number of empty spaces in each arc between two consecutive dominos by throwing $2n - 2r$ identical balls into r distinguished bins: $\binom{2n - r - 1}{r - 1}$ ways; divide by r since in the counting so far, each of the circularly placed r identical dominos counted separately as the first one, to get finally

$$d_r = \frac{2n}{r}\binom{2n-r-1}{r-1} = \frac{2n}{2n-r}\binom{2n-r}{r}.$$

Conclusion

It is quite unfortunate that putting some non-mathematical remarks of a 1943 proof for a 19th century problem under a contemporary social lens hindered the very clever idea of dominos to show its full simplifying power; all that was necessary was to look behind the r dominos to see the women behind them!

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