Little Surprises for my Combinatorics Undergrads

December 10, 2020

A collection of problems aimed at students who are learning how to count.

1 The Pigeonhole Principle

VE

1.We paint each point on the plane either red or green. Given a particular toothpick, is it always possible to place its ends on points of the same colour?

2.Prove that every graph has two vertices of the same degree.

3.Prove that, in every set of $n+1$ integers, there are two whose difference is divisible by n.

E

4.Given the numbers $\{1, ..., 2n\}$, can we choose $n+1$ among them that are pairwise not relatively prime?

5.Given a square and nine lines such that each line cuts the square in two pieces with ratio of areas r (same for all lines), show that three of the lines meet at a single point.

M

6.Does every convex polyhedron have two faces with the same number of edges?

7.Prove that, if we draw the diagonals of a 21-gon, an angle of measure $\leq 1^o$ is formed.

H

8.Prove that, for every set X of n real numbers, there is non-empty $S \subseteq X$ and $m \in Z$ such that $|m + \Sigma_{s\in S} s| \leq 1/(n+1)$.

9.Show that there is a Fibonacci number divisible by 1000.

10.Fifty lines go through the common centre of two concentric circles, dividing each to a hundred distinct arcs. We colour half of the arcs of the inner circle green and the rest red. Then, we colour the arcs of the outer circle red and green (not necessarily half and half). Prove that we can rotate the outer circle in such a way that at least half of its arcs lie exactly above arcs of the inner circle that have the same colour.

VH

11.Let $S = \{x_1, ..., x_n\}$ be a set of real numbers. For each $I \subseteq S$, define $s(I) = \Sigma_{x\in I} x$. Suppose that the function s takes at least 1.8^n values. Prove that the number of sets I for which $s(I) = 2019$ does not exceed 1.7^n.

2 Generating Functions

VE

1.Calculate the generating function of $\binom{n+k}{k}$ for fixed k.
2.Calculate the generating function of $\binom{n}{k}$ for fixed n.
3.A gumball dispenser gives a red gumball for 2 dimes and a blue gumball for 1 dime. If the dispenser holds infinite red gumballs but only 7 blue gumballs, what is the generating function for the number of possible ways to spend n dimes at it?

E

4.Prove that there are $\Sigma_{i=0}^{[n/2]} \binom{n-i}{i}$ ways to pave a 2xn chessboard with dominoes.
5.Let F_n be the Fibonacci numbers and $k \in N$. What is the generating function of the sequence A_n, where $A_n = F_n$ if $k|n$ and 0 otherwise?

M

6.How many ways are there to colour the elements of $[n]$ with blue, green and red so that no two consecutive elements are both green or both blue?
7.How many spanning trees does the 2xn Manhattan graph have?
8.Prove that $\Sigma_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

H

9.A k-fountain of coins is a placement of rows of coins such that i) each coin, except for those of the first row, touch exactly two coins of the previous row, ii) every row is connected and iii) the first row has k coins. How many k-fountains are there?
10.Prove that $\Sigma_{k\geq 0}(-1)^{n-k} \binom{2n}{k}^2 = \binom{2n}{n}$.
11.Find the sequence a_r with recursive formula $\frac{(r-1)(ra_{r-1}-a_r)a_{r-2}}{a_r a_{r-1}} = 1$, where $a_0 = a_1 = 1$.

3 Graphs

In the following, $\delta(G)$ will denote the minimum degree of a graph, $\Delta(G)$ will be the maximum degree and $\epsilon(G)$ will denote the density $\frac{|E(G)|}{|V(G)|}$ of a graph.

VE

1.Prove that the sum of the degrees of the vertices of a graph is an even number (hand-shaking lemma).
2.Show that there are at most $deg(v)(\Delta(G) - 1)^{l-1}$ paths of length l starting from a vertex v in a graph.

E

3.For $\epsilon, \delta > 0$, show that there are infinite graphs with $\delta(G) \leq \delta$ and $\epsilon(G) \geq \epsilon$.
4.Prove that the perimeter (length of longest cycle) of a graph is at least $\delta(G)+1$.
5.A graph with no cycles (tree) has at least two vertices of degree 1 (leaves).
6.How many edges are there in a tree on n vertices? Show that, if $\epsilon(G) \geq 1$, then G has a cycle.

7.What is the longest cycle in an n-dimensional cube? Construct a longest cycle of the 5-dimensional cube.
8.Which trees are isomorphic to their own complements (self-complementary)?
9.Show that, for each graph, either itself or its complement has diameter at most 3.
<u>M</u>
10.Show that every graph has a bipartite subgraph with at least $|E(G)|/2$ edges.
11.Prove that the dual of a planar Eulerian graph is bipartite.
12.Prove that in every connected graph there is a closed walk traversing each edge exactly twice.
<u>H</u>
13.If a graph with only odd-degree vertices has a Hamiltonian cycle, then prove that it has another.

4 Graphs - Connectivity

<u>VE</u>
1. Given a 2-connected graph G and vertices x, y, z, show that there is a path from x to y passing from z.
2. Prove that, for any two 2-connected graphs G, H, if $V(G) \cap V(H) \geq 2$, then $G \cup H$ is 2-connected.
<u>E</u>
3.Show that every graph has at least 2 non-cut vertices.
<u>M</u>
4.Show that in every k-connected graph there is a cycle of length $\geq 2k$.
5.Prove that, if a graph is k-connected, then it is also k-edge-connected. Does the inverse hold?
<u>H</u>
6.Prove that, for every k vertices in a k-connected graph, there is a cycle going through all of them.

5 Ramsey Theory

<u>VE</u>
1.Let S be an infinite set of points on the plane. Prove that there is an infinite set $A \subseteq S$ which is either contained in a line, or does not contain any three collinear points.
2.Prove constructively that $R(s) \geq (s-1)^2$.
3.Show that $R(4,3) \leq 10$.
<u>E</u>
4.Prove that there exists some number $N(k)$ such that every sequence of $N(k)$

numbers contains either a non-increasing or a non-decreasing subsequence of length k. Show that $N(k) > (k-1)^2$ by counterexample.
5.Show that $R_k(3) \leq k(R_{k-1}(3)-1)+2$.
6.Show that $R(4,3) \leq 9$.
<u>M</u>
7.Show that $R_k(3) \leq [ek!]+1$.
<u>H</u>
8.We colour the natural numbers with k colours. Prove that there is an initial subset $[n]$ of the naturals which includes k numbers and their sum, all coloured the same.

6 Time Complexity

<u>VE</u>
1.Show that the classes P and NP are closed under finite union, finite intersection, and finite concatenation.
2.Show that PATH= $\{< G,t,s > |$there is a path from s to t in the graph $G\} \in P$.
<u>E</u>
3.Show that RELATIVELY PRIME= $\{< x,y > |$x,y are relatively prime$\} \in P$.
4.Show that, if $P = NP$, then all non-trivial languages of P are NP-complete.
<u>M</u>
5.Show that PRIME= $\{p|p$ is prime$\} \in NP$.
6.Show that HAMILTONIAN PATH= $\{< G,t,s > |$there is a hamiltonian path from s to t in the graph $G\} \in NP$.
7.Show that CLIQUE= $\{< G,k > |G$ has a clique of size $k\}$ is NP-complete.
8.Show that VERTEX COVER= $\{< G,k > |G$ has a vertex cover with k vertices $\}$ is NP-complete.
<u>H</u>
9.Show that HAMILTONIAN PATH is NP-complete.
10.Prove the implication $P = NP \Rightarrow EXP = NEXP$.
<u>VH</u>
11.Read the 2002 paper "PRIME is in P" (presupposes some knowledge of algebra).
12.Prove that GENERALISED POKEMON is NP-complete (presupposes some knowledge of Pokemon).