

(RANDOM) TREES OF INTERMEDIATE VOLUME GROWTH

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ABSTRACT. For every sufficiently well-behaved function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that grows at least linearly and at most exponentially we construct a tree T of uniform volume growth g , that is,

$$C_1 \cdot g(r/4) \leq |B_G(v, r)| \leq C_2 \cdot g(4r), \quad \text{for all } r \geq 0 \text{ and } v \in V(T),$$

where $B_G(v, r)$ denotes the ball of radius r centered at a vertex v . In particular, this yields examples of trees of uniform intermediate (*i.e.*, super-polynomial and sub-exponential) volume growth.

We use this construction to provide first examples of unimodular random rooted trees of uniform intermediate growth, answering a question by Itai Benjamini. We find a peculiar change in structural properties for these trees at growth $r^{\log \log r}$.

1. INTRODUCTION

Given a simple graph G , a vertex $v \in V(G)$ and $r \geq 0$, the set

$$B_G(v, r) := \{w \in V(G) \mid d_G(v, w) \leq r\}$$

is called the *ball* of radius r around v . The growth of the cardinality of these balls as r increases is known as the *growth behavior* or *volume growth* of G at the vertex v . The two extreme cases of such growth are exhibited by two instructive examples, the regular trees (of exponential growth) and the lattice graphs (of polynomial growth). It is an ongoing endeavor to map the possible growth behaviors in various graph classes, the most famous example potentially being Cayley graphs of finitely generated groups, the central object of study in geometric group theory (see *e.g.* [10]). Examples of major results in this regard are the existence of Cayley graphs of *intermediate* growth (that is, super-polynomial but sub-exponential) [9], and the proof that vertex-transitive graphs can have polynomial growth only for integer exponents [13, Theorem 2].

Cayley graphs (and more generally vertex-transitive graphs) automatically have the same growth at every vertex. In other graph classes this must be imposed manually: we say that a graph G is of *uniform* growth if its growth does not vary too much between vertices. Following [5], the precise formulation is as follows: there is a function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and constants $c_1, C_1, c_2, C_2 \in \mathbb{R}_{> 0}$ so that

$$(1.1) \quad C_1 \cdot g(c_1 r) \leq |B_G(v, r)| \leq C_2 \cdot g(c_2 r) \quad \text{for all } r \geq 0 \text{ and } v \in V(G).$$

The graph G is then said to be of *uniform growth* g .

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In this article we construct infinite tree graphs T of uniform volume growth for a wide variety of growth behaviors, including intermediate growth and polynomial growth for non-integer exponents.

We subsequently demonstrate how our construction gives rise to *unimodular random rooted trees* for the same wide range of growth behaviour, answering a question by Itai Benjamini (private communication). We probe the structure of these trees and find a threshold phenomenon happening roughly at the growth rate $r^{\log \log r}$. We identify unimodular trees of intermediate growth where only the root is random, *i.e.*, they are almost surely (a.s.) isomorphic to a fixed deterministic tree.

Our work follows a history of studies on the growth rate of graphs, and particularly of trees, on which we focus here. The first (unimodular) trees of uniform polynomial growth (with the goal of turning them into planar triangulations of arbitrary polynomial growth) were constructed by Benjamini and Schramm [7] (also for non-integer exponents). Particular attention to exponential growth for trees was given by Timár [12], where the focus was on the existence of a well-defined basis for the exponential rate (which they call the *exponential growth rate* of a the graph, which is distinct from our use of this term). Quite recent advancement in this regard was made by Abert, Fraczyk and Hayes [1], further clarifying when this rate is well-defined. Intermediate but not necessarily uniform growth in trees has been studied by Amir and Yang [3] as well as the references given therein.

1.1. Main results. We establish the existence of deterministic and unimodular random rooted trees with volume growth g for a wide variety of functions $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ between (and including) polynomial and exponential growth. The precise statements are as follows:

Theorem 1. *If $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a tree T of uniform growth g .*

Theorem 2. *If $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a unimodular random rooted tree (\mathcal{T}, ν) of uniform growth g .*

Super-additivity and log-concavity can be understood as formalizing the intuitive constraints on prescribed growth to be “at least linear” and “at most exponential”, while at the same time preventing certain pathologies such as too strong oscillations in the growth behavior.

We further prove a structure theorem (**Theorem 3**) which describes the structure of the unimodular trees we construct and how it depends on the prescribed growth rate. Its precise formulation requires some terminology, but the core message is as follows:

- (i) for uniform growth *above* $r^{\alpha \log \log r}$ with $\alpha > 1$ the constructed unimodular tree is reminiscent of the classic canopy tree. In particular, it is 1-ended;
- (ii) for uniform growth *below* $r^{\log \log r}$ the constructed unimodular tree has probability zero to coincide with any particular deterministic tree, *i.e.*, every countable set of rooted trees is attained with probability zero. Also, it is a.s. 1-ended or 2-ended.

1.2. General notes on notation. All graphs in this article are simple and potentially infinite. For a graph G we write $V(G)$ for its vertex set and $E(G)$ for its edge set. For $v, w \in V(G)$ we write $vw \in E(G)$ for a connecting edge and $d_G(v, w)$ for their graph-theoretic distance in G .

For a graph of uniform growth as in (1.1) one generally distinguishes

- *uniform polynomial growth* if $g(r) = \exp(\mathcal{O}(\log r))$,
- *uniform exponential growth* if $g(r) = \exp(\Omega(r))$,
- *uniform intermediate growth* if $g(r) = \exp(o(r))$ and $g(r) = \exp(\omega(\log r))$,

where we used the Landau symbols $\mathcal{O}, \Omega, o, \omega$ as usual.

1.3. Overview. Section 2 provides the main construction: a recursively defined sequence T_n of finite trees as well as its limit tree T , which we later show to be of uniform growth. The growth of T can be finely controlled using a sequence of parameters $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}_{\geq 1}$. We provide intuition for the connection between this sequence and the growth of T , supplemented with several examples.

In Section 3 we initially detail the choice of δ_n to aim for a particular growth prescribed by some super-additive function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. We then show that, subject to some technical conditions, T is indeed of uniform growth g (Theorem 3.4):

$$C_1 \cdot g(r/4) \leq |B_G(v, r)| \leq C_2 \cdot g(4r).$$

We conclude this section proving that the technical conditions are always satisfied if g is (eventually) log-concave (Theorem 3.6). This proves Theorem 1.

In Section 4 we recall the necessary terminology on unimodular random rooted graphs and Benjamini-Schramm limits. We investigate convergence (in the Benjamini-Schramm sense) of the sequence T_n and find that its limit is indeed a unimodular tree of uniform growth, proving Theorem 2. We probe the structure of these limits in Theorem 3. We lastly construct a unimodular tree of uniform intermediate growth that is a.s. a unique deterministic tree (with randomly chosen root).

2. THE CONSTRUCTION

For each integer sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ with $\delta_n \geq 1$ we construct a tree $T = T(\delta_1, \delta_2, \dots)$. The choice of sequence will determine the growth rate of T . The tree T is constructed as a limit object of the following sequence of trees T_n where $n \geq 0$:

Construction 2.1. The trees T_n are defined recursively. In each tree we distinguish two special types of vertices: a *center*, and a set of so-called *apocentric vertices* (or outermost or peripheral vertices), both will be defined alongside the trees:

- (i) T_0 is the tree consisting of a single vertex. This vertex is both the center of T_0 as well as its only apocentric vertex.
- (ii) The tree T_n is built from $\delta_n + 1$ disjoint copies $\tau_0, \tau_1, \dots, \tau_{\delta_n}$ of T_{n-1} that we join into a single tree by adding the following edges: for each $i \in \{1, \dots, \delta_n\}$ add an edge between the center of τ_i and some apocentric vertex of τ_0 . There is a choice in selecting these apocentric vertices of τ_0 (and note that we can choose the same apocentric vertex more than once), but we shall require that these adjacencies are distributed in a maximally uniform way among the apocentric vertices of τ_0 (we postpone a rigorous definition of this until we introduced suitable notation; see Remark 2.5).

It remains to define the distinguished vertices of T_n : the center of T_n is the center of τ_0 ; the apocentric vertices of T_n are the apocentric vertices of $\tau_1, \dots, \tau_{\delta_n}$.

See Figure 1 for an illustration of this recursive definition.

The following three properties follow immediately from the recursive definition:

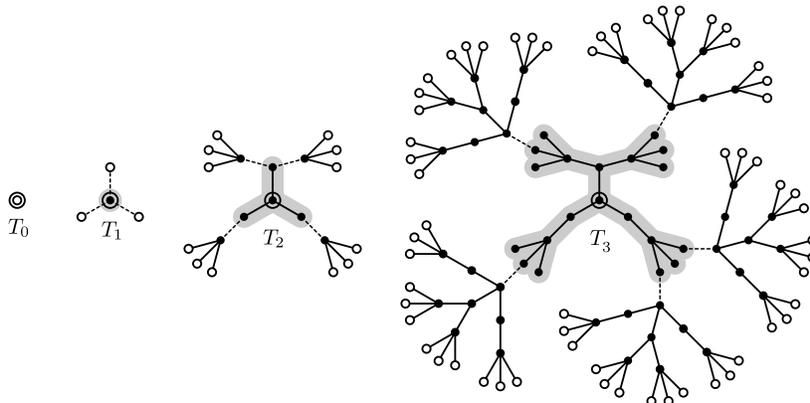


Figure 1. Illustration of the first four trees T_0, \dots, T_3 defined using the sequence $\delta_n := n+2$. The ringed vertex is the center, and the white vertices are the apocentric vertices in the respective tree. The highlighted subgraph is the central copy τ_0 in T_n . The dashed lines are the new edges added to connect the copies to form a single tree.

Observation 2.2.

- (i) T_n has exactly $(\delta_1 + 1) \cdots (\delta_n + 1)$ vertices.
- (ii) T_n has exactly $\delta_1 \cdots \delta_n$ apocentric vertices, all of which are leaves of the tree (but not all leaves are necessarily apocentric).
- (iii) the distance from the center of T_n to any of its apocentric vertices is $2^n - 1$.

One way to construct a “limit” of the T_n is the following:

Construction 2.3. For each $n \geq 1$ identify T_n with one of its copies $\tau_0, \tau_1, \dots, \tau_{\delta_{n+1}}$ in T_{n+1} . In this way we obtain an inclusion chain $T_0 \subset T_1 \subset T_2 \subset \dots$ and the union $T = T(\delta_1, \delta_2, \dots) := \bigcup_{n \geq 0} T_n$ is an infinite tree.

For later use we distinguish three natural types of limits:

- the *centric limit* always identifies T_n with the “central copy” τ_0 in T_{n+1} . This limit comes with a designated vertex $x^* \in V(T_0) \subset V(T)$, the *global center*.
- *apocentric limits* always identify T_n with an “apocentric copy” τ_i in T_{n+1} .
- *balanced limits* make both central and apocentric identifications infinitely often.

In [Section 4](#) we discuss a different and arguably more canonical way to take a limit (the Benjamini-Schramm limit) that avoids arbitrary identifications.

Our core claim is now that for “most” sequences $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ and independent of the type of the limit, the tree T has a uniform volume growth of some sort, and that with a careful choice of the sequence we can model a wide range of growth behaviors, including polynomial, intermediate and exponential.

The following example computation gives us a first idea of the connection between the sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ and the growth rate of T . For this, let T be the centric limit with global center $x^* \in V(T)$. By [Observation 2.2 \(iii\)](#) the ball of radius $r = 2^n - 1$ in T , centered at x^* , is exactly $T_n \subset T$. By [Observation 2.2 \(i\)](#) it follows

$$(2.1) \quad |B_T(x^*, r)| = |T_n| = (\delta_1 + 1) \cdots (\delta_n + 1).$$

Thus, if we aim for, say, $B_T(x^*, r) \approx g(r)$ with a given growth function $g: \mathbb{N}_0 \rightarrow \mathbb{N}_0$, then (2.1) suggests to use a sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ for which $(\delta_1 + 1) \cdots (\delta_n + 1)$ approximates $g(2^n - 1)$. In practice it turns out more convenient to approximate $g(2^n)$ (the computations are nicer and we still can prove uniform growth), and so we are lead to

$$(2.2) \quad \delta_n + 1 \approx \frac{g(2^n)}{g(2^{n-1})},$$

where we necessarily introduce an error when rounding the right side to an integer. To establish uniform growth with a prescribed growth rate g it remains to prove

- the error introduced by rounding the right side of (2.2) is manageable.
- an estimation close to (2.1) holds for radii r that are not of the form $2^n - 1$.
- an estimation close to (2.1) holds for general limit trees and around vertices other than a designated “global center”.

These points are addressed in the next section.

The remainder of this section is used to introduce helpful notation, clarify the phrase “maximally uniform distribution of adjacencies” used in Construction 2.1 and provide examples.

Notation 2.4. By Construction 2.1 (ii) for every $n \in \mathbb{N}$, T_n (and each tree isomorphic to T_n) comes with a canonical decomposition into copies of T_{n-1} . Recursively we obtain a canonical decomposition of T_n into copies of T_m for each $m \leq n$ (see Figure 2). We shall use the notation $\tau \prec_m T_n$ to indicate that τ is such a canonical copy of T_m , or $\tau \prec T_n$ if m is not relevant.

A copy $\tau \prec T_n$ is called *central* if it contains the center of T_n ; it is called *apocentric* if it shares apocentric vertices with T_n (and one can easily show that then all apocentric vertices of τ are apocentric in T_n).

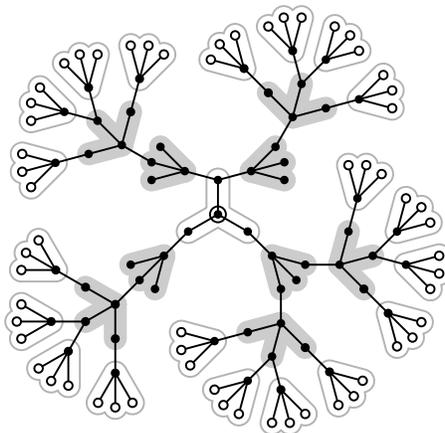


Figure 2. All canonical copies $\tau \prec_1 T_3$ are highlighted. The central and apocentric copies are highlighted in white, the others in gray.

With this notation in place we can clarify our use of “maximal uniform distribution of adjacencies” in Construction 2.1 (ii).

Remark 2.5. Let $\tau_0 \prec_{n-1} T_n$ be the central copy. Then there are exactly δ_n “outwards edges” connecting τ_0 to the apocentric copies $\tau_1, \dots, \tau_{\delta_n} \prec_{n-1} T_n$. “Maximal uniform distribution” means that every apocentric copy $\tau \prec_m \tau_0$ (where $m \leq n-1$) intersects the expected number of these edges (up to rounding). More precisely, if E_τ is the number of the “outwards edges” that have an end in τ , then

$$(2.3) \quad \left\lfloor \frac{\delta_1 \cdots \delta_m}{\delta_1 \cdots \delta_{n-1}} \cdot \delta_n \right\rfloor \leq E_\tau \leq \left\lceil \frac{\delta_1 \cdots \delta_m}{\delta_1 \cdots \delta_{n-1}} \cdot \delta_n \right\rceil,$$

where $(\delta_1 \cdots \delta_m)/(\delta_1 \cdots \delta_{n-1})$ is exactly the fraction of apocentric vertices of τ_0 that are also in τ (cf. [Observation 2.2 \(ii\)](#)).

It is not hard to see that in each step of [Construction 2.1](#) this distribution can be achieved by adding the “outwards edges” one by one. The reader can verify that the steps shown in [Figure 1](#) are in accordance with a maximally uniform distribution.

We close with three examples demonstrating the versatility of [Construction 2.3](#).

Example 2.6 (Polynomial growth). If we aim for polynomial growth $g(r) = r^\alpha$, $\alpha \in \mathbb{N}$ then the heuristics [\(2.2\)](#) suggests to use a constant sequence $\delta_n := 2^\alpha - 1$.

In fact, the corresponding trees T_n embed nicely into the α -th power¹ of the α -dimensional lattice graph (shown in [Figure 3](#) for $\alpha = 2$).

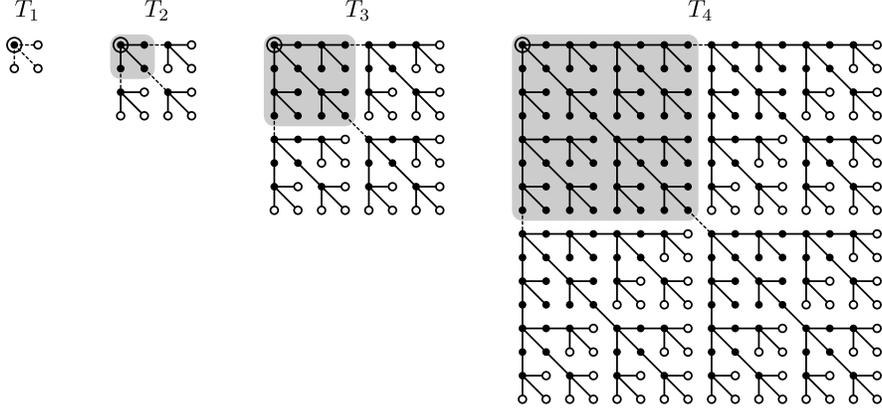


Figure 3. Embedding of the tree $T(3, 3, \dots)$ into the square of the 2-dimensional lattice graph.

More generally, for any constant sequence $\delta_n := c$ we expect to find polynomial volume growth, potentially with a non-integer exponent $\log(c+1)$.

Example 2.7 (Exponential growth). For the sequence $\delta_n := d^{2^n - 1}$, $d \in \mathbb{N}$ the centric limit T is the d -ary tree, in particular, of exponential volume growth (see [Figure 4](#) for the case $d = 2$, *i.e.*, the binary tree). In fact, using [\(2.1\)](#) for $r = 2^n$ (where $x^* \in V(T)$ is the global center) we find

$$|B_T(x^*, r-1)| = (\delta_1 + 1) \cdots (\delta_n + 1) = \prod_{k=1}^n (d^{2^{k-1}} + 1) = \sum_{i=0}^{2^n - 1} d^i = \frac{d^{2^n} - 1}{d - 1} = \frac{d^r - 1}{d - 1}.$$

¹Recall, the α -th power of G is a graph G^α with vertex set $V(G)$ and an edge between any two vertices whose distance in G is at most α .

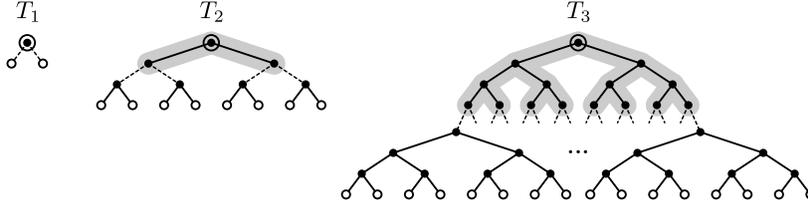


Figure 4. The binary tree as constructed from [Construction 2.1](#) using the doubly exponential sequence $\delta_n = 2^{2^{n-1}}$.

Extrapolating from [Example 2.6](#) and [Example 2.7](#), it seems reasonable that unbounded sequences $\delta_1, \delta_2, \delta_3, \dots$ with a growth sufficiently below doubly exponential result in intermediate volume growth.

Example 2.8 (Intermediate growth). For $\delta_n := (n+3)^\alpha - 1$, $\alpha \in \mathbb{N}$ we can compute this explicitly (see [Figure 1](#) for the case $\alpha = 1$). If T is the centric limit with global center $x^* \in V(T)$ and $r = 2^n$, then:

$$\begin{aligned} |B_T(x^*, r-1)| &= (\delta_1 + 1) \cdots (\delta_n + 1) = \left(\frac{1}{6}(n+3)!\right)^\alpha \sim (n!n^3)^\alpha \sim (n^n e^{-n} n^{7/2})^\alpha \\ &= r^{\alpha \log \log r} r^{-\alpha/\ln 2} (\log r)^{7\alpha/2} \end{aligned}$$

We therefore expect this choice of sequence to lead to a tree of uniform intermediate volume growth. Trees constructed from $\delta_n \sim n^\alpha$ present an interesting boundary case in [Section 4](#) when we discuss unimodular random trees (see also [Theorem 3](#)).

3. UNIFORM VOLUME GROWTH

We fix an increasing function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that shall serve as our target growth rate. The goal of this section is to construct an appropriate sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ for which we can show that $T := T(\delta_1, \delta_2, \dots)$ is of uniform volume growth g . We shall show in particular that this is feasible if g is super-additive and log-concave, thus proving [Theorem 1](#).

Recall that the uniform growth rate of a graph must be at least linear (like in a path graph) and at most exponential (like in a regular tree). It would therefore be reasonable to impose these constraints on the target growth rate g right away. However, we shall not do this: these constraints will also emerge naturally during our journey towards the main theorem of this section. In fact, we shall uncover even more precise constraints expressed in terms of the sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ (see [Remarks 3.2](#) and [3.3](#)) and we discuss how they relate to the natural constraints of g growing at least linear and at most exponential.

Following our motivation from the last section we firstly try to approximate $g(2^n)$ by $(\delta_1 + 1) \cdots (\delta_n + 1)$.

Lemma 3.1. *There exists a sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ so that*

$$(3.1) \quad (1-c) \cdot g(2^n) < (\delta_1 + 1) \cdots (\delta_n + 1) < (1+c) \cdot g(2^n),$$

for all $n \geq 0$, where $c := \sup_n g(2^n)/g(2^{n+1}) \leq 1$.

Proof. For this proof we abbreviate $\bar{\delta}_n := \delta_n + 1$. We set $\bar{\delta}_1 := g(2)$ and recursively

$$\bar{\delta}_{n+1} := \begin{cases} \lceil g(2^{n+1})/g(2^n) \rceil & \text{if } \bar{\delta}_1 \cdots \bar{\delta}_n \leq g(2^n) \\ \lfloor g(2^{n+1})/g(2^n) \rfloor & \text{if } \bar{\delta}_1 \cdots \bar{\delta}_n > g(2^n) \end{cases}.$$

This sequence satisfies (3.1) as we show by induction on n : the induction base is clear from the definition of $\bar{\delta}_1$. Next, if (3.1) holds for n , and $\bar{\delta}_1 \cdots \bar{\delta}_n \leq g(2^n)$, then

$$\begin{aligned} \bar{\delta}_1 \cdots \bar{\delta}_n \cdot \bar{\delta}_{n+1} &\leq g(2^n) \cdot \left\lceil \frac{g(2^{n+1})}{g(2^n)} \right\rceil < g(2^n) \cdot \left(\frac{g(2^{n+1})}{g(2^n)} + 1 \right) \\ &= \left(1 + \frac{g(2^n)}{g(2^{n+1})} \right) \cdot g(2^{n+1}) \leq (1+c) \cdot g(2^{n+1}), \\ \bar{\delta}_1 \cdots \bar{\delta}_n \cdot \bar{\delta}_{n+1} &> (1-c) \cdot g(2^n) \cdot \left\lceil \frac{g(2^{n+1})}{g(2^n)} \right\rceil \geq (1-c) \cdot g(2^n) \cdot \frac{g(2^{n+1})}{g(2^n)} \\ &= (1-c) \cdot g(2^{n+1}). \end{aligned}$$

An analogous argument applies in the case $\bar{\delta}_1 \cdots \bar{\delta}_n > g(2^n)$. \square

Remark 3.2. Recall that **Construction 2.1** requires $\delta_n \geq 1$ for all $n \geq 1$ in order for $T(\delta_1, \delta_2, \dots)$ to be well-defined. The sequence provided by the proof of **Lemma 3.1** does not generally have this property unless further requirements on g are met. One possible condition is *super-additivity*, which turns out to also formalize the idea of “ g growing at least linearly”.

Super-additivity means $g(r_1 + r_2) \geq g(r_1) + g(r_2)$ and indeed implies $g(2^{n+1}) = g(2^n + 2^n) \geq 2g(2^n)$, and hence

$$\delta_{n+1} + 1 \geq \left\lceil \frac{g(2^{n+1})}{g(2^n)} \right\rceil \geq 2.$$

This moreover implies $c := \sup_n g(2^n)/g(2^{n+1}) \leq 1/2 < 1$ and guarantees that the lower bound in **Lemma 3.1** is not vacuous.

Remark 3.3. Recall that graphs of uniform growth are necessarily of bounded degree. But again, the sequence constructed in the proof of **Lemma 3.1** does not necessarily produce such a tree unless further requirements on g are met. We elaborate how these requirements can be interpreted as “ g growing at most exponentially”.

The tree T_{n-1} has exactly $\delta_1 \cdots \delta_{n-1}$ apocentric vertices. That is, each apocentric vertex of the central copy $\tau_0 \prec_{n-1} T_n$ is adjacent, on average, to

$$(3.2) \quad \Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}$$

apocentric copies $\prec_{n-1} T_n$. By the maximal uniform distribution of adjacencies (*cf.* **Remark 2.5**) the maximum degree among the apocentric vertices of τ_0 is exactly $\lceil \Delta(n) \rceil + 1$. Thus, since T contains T_n for each $n \geq 0$, T is of bounded degree only if $\bar{\Delta} := \sup_n \lceil \Delta(n) \rceil + 1 < \infty$. We can use this to put an upper bound on the growth of the sequence $\delta_1, \delta_2, \delta_3, \dots$ for $n \geq 2$ and using (3.2) it holds

$$\frac{\delta_n}{\delta_1 \cdots \delta_{n-1}} \leq \bar{\Delta} - 1 \implies \delta_n \leq ((\bar{\Delta} - 1)\delta_1)^{2^{n-2}}.$$

Comparing with **Example 2.7** suggests that g grows at most exponentially.

It remains to prove the main result of this section: establishing the exact growth rate of $T = T(\delta_1, \delta_2, \dots)$ in terms of g . As we shall see, bounded degree (as discussed in **Remark 3.3**) is not sufficient to prove uniform volume growth. Instead we require an additional criterion that can also be concisely expressed in terms of the function

Δ . In terms of g this can be interpreted as preventing too strong oscillations in the growth of g (cf. [Theorem 3.6](#)).

In order to state the main result we recollect that $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a strictly increasing super-additive function. The sequence $\delta_1, \delta_2, \delta_3, \dots \in \mathbb{N}$ is chosen to satisfy

$$(1 - c) \cdot g(2^n) < (\delta_1 + 1) \cdots (\delta_n + 1) < (1 + c) \cdot g(2^n),$$

for some $c \leq 1/2$ (according to [Lemma 3.1](#)) and satisfies $\delta_n \geq 1$ by [Remark 3.2](#). In particular, $T = T(\delta_1, \delta_2, \dots)$ exists. Finally, $\Delta(n)$ is defined as in [\(3.2\)](#). The main result then reads as follows:

Theorem 3.4. *Fix $v \in V(T)$ and $r \geq 0$. Then holds*

- (i) $|B_T(v, r)| \geq (1 - c) \cdot g(r/4) \geq \frac{1}{2} g(r/4)$.
- (ii) If $\bar{\Delta} := \sup_n [\Delta(n)] + 1 < \infty$, then T is of maximum degree $\bar{\Delta}$ and

$$|B_T(v, r)| \leq 2\bar{\Delta} \cdot (1 + c)^2 \cdot g(2r)^2 \leq \frac{9}{2} \bar{\Delta} \cdot g(2r)^2.$$
- (iii) If $\Gamma := \sup_{m \geq n} [\Delta(m)/\Delta(n)] < \infty$, then $\bar{\Delta} < \infty$ and

$$\begin{aligned} |B_T(v, r)| &\leq 2(1 + c) \cdot (\Gamma \cdot g(4r) - (\Gamma - 1) \cdot g(2r)) \\ &\leq 3(\Gamma \cdot g(4r) - (\Gamma - 1) \cdot g(2r)) \\ &\leq 3\Gamma \cdot g(4r). \end{aligned}$$

In particular, if $\Gamma < \infty$ then T is of uniform volume growth g .

Note that the lower bound holds unconditionally. The conditions $\bar{\Delta} < \infty$ and $\Gamma < \infty$ for the upper bounds are technical, but there are natural criteria in terms of g that imply both, such as being *log-concave* (see [Theorem 3.6](#) below).

Proof of Theorem 3.4. For a vertex $w \in V(T)$ we use the notation $\tau_n(w)$ to denote the unique copy $\prec_n T$ that contains w .

To prove (i) choose $n \in \mathbb{N}_0$ with $2^{n+1} \leq r \leq 2^{n+2}$. Each vertex of $\tau_n(v)$ can be reached from v in at most $2(2^n - 1)$ steps: at most $2^n - 1$ steps from v to the center of $\tau_n(v)$, and at most $2^n - 1$ steps from the center to any other vertex of $\tau_n(v)$. This yields

$$\begin{aligned} |B_T(v, r)| &\geq |B_T(v, 2^{n+1})| > |B_T(v, 2(2^n - 1))| \geq |T_n| \geq (1 - c) \cdot g(2^n) \\ &\geq (1 - c) \cdot g(r/4). \end{aligned}$$

Here and in the following the final form of the bound is obtained by using $c \leq 1/2$.

To prove (ii) choose $n \in \mathbb{N}$ with $2^{n-1} \leq r \leq 2^n$. Let x be the center of $\tau_n(v)$, and, if it exists, let w be the unique neighbor of x outside of $\tau_n(v)$. Which other copies $\prec_n T$ are reachable from v within 2^n steps? The following list is exhaustive:

- $\tau_n(v)$ and $\tau_n(w)$,
- a copy $\tau \prec_n T$ adjacent to an apocentric vertex of $\tau_n(v)$ resp. $\tau_n(w)$.

Both $\tau_n(v)$ and $\tau_n(w)$ have not more than $|T_n|$ apocentric vertices, each of which is adjacent to at most $\bar{\Delta} - 1$ copies $\tau \prec_n T$. In conclusion, at most $2\bar{\Delta} \cdot |T_n|$ copies of T_n are reachable from v in 2^n steps, containing a total of at most $2\bar{\Delta} \cdot |T_n|^2$ vertices. We therefore find

$$\begin{aligned} |B_T(v, r)| &\leq |B_T(v, 2^n)| \leq 2\bar{\Delta} \cdot |T_n|^2 \leq 2\bar{\Delta} \cdot (1 + c)^2 \cdot g(2^n)^2 \\ &\leq 2\bar{\Delta} \cdot (1 + c)^2 \cdot g(2r)^2. \end{aligned}$$

One might suggest a much better estimation for the number of copies $\tau \prec_n T$ reachable through apocentric vertices of $\tau_n(v)$ resp. $\tau_n(w)$: namely δ_{n+1} . However, this is only true if $\tau_n(v)$ is a central copy in $\tau_{n+1}(v)$. In general, $\tau_n(v)$ can as well be apocentric in $\tau_{n+1}(v)$, or can even be apocentric in $\tau_m(v)$ for some very large $m > n$. If so, then the number of copies $\prec_n T$ reachable through apocentric vertices of $\tau_n(v)$ is more plausibly related to δ_{m+1} than δ_{n+1} , which is the reason for the very crude estimation above. With a more careful analysis we can prove the second upper bound.

To show (iii) let $m \geq n$ be maximal so that $\tau_n(v)$ is apocentric in $\tau_m(v)$. Such an m might not exist (e.g. when T is an apocentric limit), in which case the apocentric vertices of $\tau_n(v)$ are leaves in T and no other copies $\prec_n T$ are reachable through them. If m exists however, then $\tau_m(v)$ is necessarily central in $\tau_{m+1}(v)$. In particular, the $\delta_1 \cdots \delta_m$ apocentric vertices of $\tau_m(v)$ are adjacent to δ_{m+1} apocentric copies $\prec_m \tau_{m+1}(v)$. By the maximally uniform distribution of adjacencies we can estimate how many of these copies are reachable through the $\delta_1 \cdots \delta_n$ apocentric vertices of $\tau_n(v)$. Following [Remark 2.5](#) this number is at most

$$\begin{aligned} \left\lceil \frac{\delta_1 \cdots \delta_n}{\delta_1 \cdots \delta_m} \cdot \delta_{m+1} \right\rceil &= \left\lceil \frac{\delta_{m+1}}{\delta_1 \cdots \delta_m} \cdot \left(\frac{\delta_{n+1}}{\delta_1 \cdots \delta_n} \right)^{-1} \cdot \delta_{n+1} \right\rceil \\ &= \left\lceil \frac{\Delta(m+1)}{\Delta(n+1)} \cdot \delta_{n+1} \right\rceil \leq \Gamma \cdot \delta_{n+1}. \end{aligned}$$

This yields an improved upper bound on the number of copies $\tau \prec_n T$ adjacent to apocentric vertices of $\tau_n(v)$. The same argument applies to $\tau_n(w)$. Including $\tau_n(v)$ and $\tau_n(w)$ there are then at most $2\Gamma \cdot \delta_{n+1} + 2$ copies $\prec_n T$ reachable from v in 2^n steps, and thus:

$$\begin{aligned} |B_T(v, r)| &\leq |B_T(v, 2^n)| \leq (2\Gamma \cdot \delta_{n+1} + 2)|T_n| \\ &\leq 2(\Gamma \cdot (\delta_{n+1} + 1)|T_n| - (\Gamma - 1) \cdot |T_n|) \\ &\leq 2(\Gamma \cdot |T_{n+1}| - (\Gamma - 1) \cdot |T_n|) \\ &\leq 2(c+1) \cdot (\Gamma \cdot g(2^{n+1}) - (\Gamma - 1) \cdot g(2^n)). \\ &\leq 2(c+1) \cdot (\Gamma \cdot g(4r) - (\Gamma - 1) \cdot g(2r)). \end{aligned}$$

□

It appears non-trivial to actually prescribe a natural growth rate g for which Γ diverges, and so we feel confident that [Theorem 3.4](#) applies quite generally. Yet, for the remainder of this section we discuss criteria that are sufficient to imply $\Gamma < \infty$.

Proposition 3.5. $\Gamma < \infty$ holds in any of the following cases:

- (i) $\delta_{n+1} \leq \delta_n^2$ eventually.
- (ii) the sequence of δ_n is bounded.

Proof. Note that $\Delta(n+1) = \delta_{n+1}/\delta_n^2 \cdot \Delta(n)$. Assuming (i), $\Delta(n)$ will eventually be non-increasing and so either $\Gamma \leq 1$ or the value Γ is attained only on a finite initial segment, thus, is finite.

For part (ii) observe $\Delta(m)/\Delta(n) = \delta_m/(\delta_n^2 \cdot \delta_{n+1} \cdots \delta_{m-1}) \leq \max_i \delta_i$. □

Condition (ii) of [Proposition 3.5](#) holds, for example, when g is a polynomial (see [Example 2.6](#)).

It remains to provide a criterion for $\Gamma < \infty$ in terms of g . Below we show that g being *log-concave* is sufficient, where log-concave means

$$\alpha \log g(x) + \beta \log g(y) \leq \log g(\alpha x + \beta y)$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Intuitively, being log-concave is sufficient because it prevents both super-exponential growth and strong oscillations in the growth rate.

Theorem 3.6. *If g is eventually log-concave, then $\Gamma < \infty$.*

Proof. We start from a well-chosen instance of log-concavity: set

$$\alpha = 1/3, \quad \beta = 2/3, \quad x = 2^{n+1}, \quad y = 2^{n-1}.$$

and verify $\alpha x + \beta y = 2^n$. Since g is eventually log-concave, for sufficiently large n holds

$$\frac{1}{3} \log g(2^{n+1}) + \frac{2}{3} \log g(2^{n-1}) \leq \log g(2^n).$$

This further rearranges to

$$\log g(2^{n+1}) + 2 \log g(2^{n-1}) \leq 3 \log g(2^n) \implies \frac{g(2^{n+1})}{g(2^n)} \leq \left(\frac{g(2^n)}{g(2^{n-1})} \right)^2.$$

Recall that roughly $\delta_n \approx g(2^n)/g(2^{n-1})$, and so the right-most inequality resembles $\delta_{n+1} \leq \delta_n^2$. If this were exact then $\Gamma < \infty$ would already follow from [Proposition 3.5 \(i\)](#). However, the actual definition of δ_n from the proof of [Lemma 3.1](#) only yields

$$(3.3) \quad \delta_{n+1} \leq \left\lfloor \frac{g(2^{n+1})}{g(2^n)} \right\rfloor \leq \frac{g(2^{n+1})}{g(2^n)} \leq \left(\frac{g(2^n)}{g(2^{n-1})} \right)^2 \leq \left\lceil \frac{g(2^n)}{g(2^{n-1})} \right\rceil^2 \leq (\delta_n + 2)^2.$$

This turns out to be sufficient for proving $\Gamma < \infty$, though the argument becomes more technical. Define $\gamma: \mathbb{N} \rightarrow \mathbb{R}$ with

$$\gamma(1) := 25, \quad \gamma(\delta) := \prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta^{2^k}} \right)^2, \quad \text{whenever } \delta \geq 2.$$

To keep this proof focused, most technicalities surrounding this function have been moved to [Appendix A](#). This includes a proof of convergence of the infinite product ([Lemma A.1](#)) as well as a proof of the following crucial property (*) of γ : for any two integers $\delta_1, \delta_2 \geq 1$ with $\delta_1 \leq (\delta_2 + 2)^2$ holds $\delta_1 \gamma(\delta_1) \leq \delta_2^2 \gamma(\delta_2)$ ([Lemma A.5](#)).

Lastly, we define $\text{pot}(n) := \gamma(\delta_n) \Delta(n)$, intended to represent the *potential* for Δ to increase, as we shall see that it is an upper bound on $\Delta(m)$ for *all* $m \geq n$. This follows from two observations. Firstly, $\Delta(m) < \text{pot}(m)$ since $\gamma(\delta) > 1$. Secondly, $\text{pot}(n)$ is decreasing in n : since $\delta_{n+1} \leq (\delta_n + 2)^2$ by (3.3) (if n is sufficiently large), we can apply property (*) to find

$$\text{pot}(n+1) = \gamma(\delta_{n+1}) \Delta(n+1) = \gamma(\delta_{n+1}) \frac{\delta_{n+1}}{\delta_n^2} \Delta(n) \stackrel{(*)}{\leq} \gamma(\delta_n) \Delta(n) = \text{pot}(n).$$

To summarize, for all sufficiently large $m \geq n$ holds

$$\Delta(m) < \text{pot}(m) \leq \text{pot}(n) = \Delta(n) \gamma(\delta_n) \implies \frac{\Delta(m)}{\Delta(n)} \leq \gamma(\delta_n) \leq 25,$$

where we used that $\gamma(\delta)$ is decreasing in δ ([Corollary A.3](#)) and so attains its maximum at $\gamma(1) = 25$. This proves that Γ is finite. \square

As a corollary of [Theorem 3.4](#) and [Theorem 3.6](#) we have proven

Theorem 1. *If $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a tree T of uniform growth g .*

4. UNIMODULAR RANDOM TREES

In the remainder of the article we apply the theory of *Benjamini-Schramm limits* to the sequence T_n (**Construction 2.1**); we obtain *unimodular random rooted trees of uniform intermediate growth*, answering a question by Itai Benjamini. We investigate the structure of the limit graphs and provide an instance that is supported on a single deterministic tree.

A *rooted graph* is a pair of the form (G, o) , where G is a graph and $o \in V(G)$. For a definition of *random rooted graphs* and their basic properties we follow [7]: firstly, there is a natural topology on the set of rooted graphs – the *local topology* – induced by the metric

$$\begin{aligned} \text{dist}((G, o), (G', o')) &:= 2^{-R} \quad \text{if } B_G(o, r) \cong B_{G'}(o', r) \text{ for all } 0 \leq r \leq R \\ &\quad \text{and } B_G(o, R+1) \not\cong B_{G'}(o', R+1), \end{aligned}$$

where it is understood that $B_G(o, r)$ is rooted at o and that isomorphisms between rooted graphs preserve roots.

A random rooted graph (G, o) is a Borel probability measure (for the local topology) on the set of locally finite, connected rooted graphs. We call (G, o) *finite* if the set of infinite rooted graphs has (G, o) -measure zero. If in addition the conditional distribution of the root in (G, o) over each finite graph is uniform, then (G, o) is called *unbiased*.

Given a sequence (G_n, o_n) of unbiased random rooted graphs, a random rooted graph (G, o) is said to be the *Benjamini-Schramm limit* of (G_n, o_n) if for every rooted graph (H, ω) and natural number $r \geq 0$ we have

$$\lim_{n \rightarrow \infty} P(B_{G_n}(o_n, r) \cong (H, \omega)) = P(B_G(o, r) \cong (H, \omega)).$$

Note that, if it exists, (G, o) is the unique limit. If a random rooted graph is the Benjamini-Schramm limit of some sequence, we say that it is *sofic*. One can show that a set of graphs of uniformly bounded degree is compact in the local topology, and thus, a sequence (G_n, o_n) of uniformly bounded degree always has a convergent subsequence.

We say that a random rooted graph (G, o) is of uniform growth $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, if there are constants $c_1, C_1, c_2, C_2 \in \mathbb{R}_{>0}$ so that a.s.

$$C_1 \cdot g(c_1 r) \leq |B_G(o, r)| \leq C_2 \cdot g(c_2 r), \quad \text{for all } r \geq 0.$$

Note that this is stronger than being a.s. of uniform growth g : we require the constants to be not only independent of the root, but independent of the underlying tree as well.

The original formulation of Benjamini's question asks for *unimodular* trees of intermediate growth. Unimodularity generalizes the concept of a uniformly chosen root to graphs that are not necessarily finite: a random rooted graph (G, o) is unimodular if it obeys the *mass transport principle*, i.e.,

$$\mathbb{E} \left[\sum_{x \in V(G)} f(G, o, x) \right] = \mathbb{E} \left[\sum_{x \in V(G)} f(G, x, o) \right]$$

for every *transport function* f , which, for our purpose, are sufficiently defined as Borel functions over doubly-pointed graphs that output non-negative real numbers (for a precise definition we direct the reader to [11]). The function f simulates mass transport between vertices, and the mass transport principle states, roughly, that

the root o sends, on average, as much mass to other vertices as it receives from them. Unimodular graphs are significant in the theory of random graphs and encompass some important classes, most notably, all sofic graphs. Conversely, the famous *Aldous-Lyons conjecture* asks whether all unimodular random graphs are sofic [11].

In [Section 3](#), given a suitable function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, we constructed a rooted tree (T, x^*) of uniform growth g (x^* being the global center, *i.e.*, the unique vertex of $T_0 \subset T$). Can we turn it into a unimodular tree of uniform growth g ? Naturally, we can interpret (T, x^*) as the random rooted graph a.s. being this particular rooted tree. It is known that for unimodular graphs the conditional distribution of the root over each underlying graph has positive probability on each vertex orbit [8, Proposition 12]. Thus, if T is not vertex-transitive, (T, x^*) cannot be unimodular.

That being said, unimodular trees with uniform growth g can be obtained from the defining sequence $T_n, n \geq 0$ ([Construction 2.1](#)). Let (T_n, o_n) denote the unbiased random rooted graph that is a.s. isomorphic to T_n .

Proposition 4.1. *Let $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be super-additive and log-concave. Then the sequence (T_n, o_n) has a subsequence that converges in the Benjamini-Schramm sense to a unimodular random rooted tree of uniform growth g .*

Proof. By [Theorem 3.6](#) the sequence (T_n, o_n) has uniformly bounded degree, and, since the set of graphs with uniformly bounded degree is compact in the local topology, has a subsequence (T_{n_k}, o_{n_k}) Benjamini-Schramm convergent to a random rooted tree (\mathcal{T}, ι) . By the definition of Benjamini-Schramm limit, every ball $B_{\mathcal{T}}(\iota, r)$ is a.s. isomorphic to a ball in T_{n_k} for k large enough. However, every ball of radius $r \leq 2^{n_k} - 1$ in T_{n_k} is isomorphic to a ball in some limit tree T (*cf.* [Construction 2.3](#)) and by [Theorem 3.4](#) therefore admits the bounds

$$C_1 \cdot g(c_1 r) \leq |B_{\mathcal{T}}(\iota, r)| \leq C_2 \cdot g(c_2 r),$$

with constants that only depend on the sequence of T_n .

In conclusion, (\mathcal{T}, ι) exhibits uniform growth of order g . Since (\mathcal{T}, ι) is sofic (*i.e.*, it is the Benjamini-Schramm limit of unbiased graphs), it is also unimodular. \square

At this point we have proven

Theorem 2. *If $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a unimodular random rooted tree (\mathcal{T}, ι) of uniform growth g .*

We can be more precise about the convergence of the sequence T_n as it is actually not necessary to restrict to a subsequence:

Proposition 4.2. *The sequence $T_n, n \geq 0$ is Benjamini-Schramm convergent.*

Proof. Fix $r \geq 1$ and a graph H . Call a vertex $v \in V(T_n)$ *good* if $B_{T_n}(v, r) \cong H$. Let P_n be the probability that a uniformly chosen vertex of T_n is good. We show that P_n is a Cauchy sequence, hence convergent.

Let s_n be the number of good vertices in T_n . Since T_n consists of $\delta_n + 1$ copies of T_{n-1} , one might expect $s_n \approx (\delta_n + 1)s_{n-1}$. However, besides the copies, T_n also consists of δ_n new edges, and so for $\tau \prec_{n-1} T_n$ and $v \in V(\tau)$, the r -ball around v in $\tau \cong T_{n-1}$ might look different from the r -ball around v in T_n . Still, this can only happen if v is r -close to one of the $2\delta_n$ ends of the new edges. Let C_r be an upper bound on the size of an r -ball in T (which exists since T is of bounded degree, *cf.* [Remark 3.3](#)). We can then conclude

$$|s_n - (\delta_n + 1)s_{n-1}| \leq 2C_r \delta_n.$$

Since $P_n = s_n/|T_n|$, division by $|T_n| = (\delta_n + 1)|T_{n-1}|$ yields

$$|P_n - P_{n-1}| \leq \frac{\delta_n}{\delta_n + 1} \frac{2C_r}{|T_{n-1}|} \leq \frac{2C_r}{|T_{n-1}|}.$$

It follows for all $k \geq 1$

$$\begin{aligned} |P_{n+k} - P_n| &\leq |P_{n+k} - P_{n+k-1}| + \cdots + |P_{n+1} - P_n| \\ &\leq 2C_r \left(\frac{1}{|T_{n+k-1}|} + \cdots + \frac{1}{|T_n|} \right) \leq 2C_r \sum_{i=n}^{\infty} \frac{1}{|T_i|}. \end{aligned}$$

Since $\delta_n \geq 1$ and $|T_n| = (\delta_1 + 1) \cdots (\delta_n + 1)$ we have $|T_n| \geq 2^n$. It therefore follows

$$|P_{n+k} - P_n| \leq \frac{4C_r}{2^n},$$

and P_n is indeed a Cauchy sequence. \square

The constructions [Proposition 4.1](#) resp. [Proposition 4.2](#) are not particularly enlightening with regards to the structure of the obtained limit tree $(T_n, o_n) \rightarrow (\mathcal{T}, \iota)$. We shall now take some time to explore its structure.

The following definitions will turn out useful: let $p_n := \delta_n/(\delta_n + 1)$, and note that p_n is precisely the probability that for a uniformly chosen vertex $v \in V(T_k)$, $k \geq n$ we have $v \in \tau \prec_{n-1} \tau' \prec_n T_k$ where τ is apocentric in τ' . We say that the sequence (T_n, o_n) is *apocentric* if

$$\prod_{n=1}^{\infty} p_n > 0.$$

Otherwise, we say that it is *balanced*. Note that a sequence is apocentric if and only if the same is true for each of its subsequences.

A *generalized canopy tree*² is the Benjamini-Schramm limit of an apocentric sequence. Analogously, a *balanced tree* is the Benjamini-Schramm limit of a balanced sequence. We shall see shortly that these notions capture two classes of growth with qualitatively different Benjamini-Schramm limits, and that the terminology is chosen suggestively and with foresight.

Proposition 4.3. (\mathcal{T}, ι) is a generalized canopy tree if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\delta_n} < \infty.$$

In particular, if $\delta \in \Omega(n^\alpha)$ for some $\alpha > 1$ then (\mathcal{T}, ι) is a generalized canopy tree, and if $\delta_n \in \mathcal{O}(n)$ then (\mathcal{T}, ι) is balanced.

Proof. The product $\prod_n p_n$ is positive if and only if $(\prod_n p_n)^{-1} = \prod_n (1 + 1/\delta_n)$ converges. As is well-known, the latter converges if and only if $\sum_n 1/\delta_n$ converges. \square

In terms of g this threshold can be located roughly at $g(r) = r^{\alpha \log \log r}$ between $\alpha = 1$ and $\alpha > 1$. In fact, note that

$$g(2^n) = (2^n)^{\alpha \log \log 2^n} = n^{\alpha n}$$

²Recall that the *canopy tree* is the Benjamini-Schramm limit of a sequence of full binary trees of increasing depths. The canopy tree is the single most famous example of a generalized canopy tree.

and according to [Lemma 3.1](#)

$$\delta_n \sim \frac{g(2^{n+1})}{g(2^n)} = (n+1)^\alpha \cdot \left(1 + \frac{1}{n}\right)^{\alpha n} \sim (n+1)^\alpha e^\alpha \sim n^\alpha.$$

It remains to establish the qualitative differences between generalized canopy trees and balanced trees.

Theorem 3 (Structure Theorem).

- (i) *If (\mathcal{T}, \imath) is a generalized canopy tree, then it is a.s. an apocentric limit. In particular, it is 1-ended.*
- (ii) *If (\mathcal{T}, \imath) is a balanced tree, then it is a.s. a balanced limit and it is a.s. 1-ended or 2-ended. In particular, if $\delta_n \neq 1$ eventually, then (\mathcal{T}, \imath) is a.s. 1-ended and the probability for being isomorphic to any particular tree is 0.*

Proof. We start with the proof of (i). For an apocentric sequence, any tail product $p^{(k)} := \prod_{n=k+1}^{\infty} p_n$ is positive. Additionally, $p^{(k)}$ is the limit of the probability that o_n lies in an apocentric $\tau \prec_k T_n$ as $n \rightarrow \infty$. So, consider the event $A_{k,r}$ that the ball $B_{\mathcal{T}}(\imath, r)$ is isomorphic to a ball centered at a vertex of an apocentric $\tau \prec_k T_n$ (for n large enough that the center of T_n is not contained in the ball of radius r centered at the center of τ) and note that, for every $r \geq 0$, $P(A_{k,r}) \geq p^{(k)}$. Let $A_k := \bigcap_{r \geq 0} A_{k,r}$. That is, A_k is the event that for every $r \geq 0$ the ball $B_{\mathcal{T}}(\imath, r)$ is isomorphic to a ball centered at an apocentric $\tau \prec_k T_n$ for n large enough. Since $A_{k,r}$ is decreasing in r , we have

$$P(A_k) = \lim_{r \rightarrow \infty} P(A_{k,r}) \geq p^{(k)}.$$

Since A_k is increasing in k ,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{k \rightarrow \infty} P(A_k) \geq \lim_{k \rightarrow \infty} p^{(k)} = 1.$$

Therefore, there exists a.s. $k \in \mathbb{N}_0$ such that the event A_k occurs, implying that (\mathcal{T}, \imath) is an apocentric limit.

Let $v \in V(\mathcal{T})$ and let $m \geq k$ be such that a copy $\tau \prec_m \mathcal{T}$ contains both \imath and v . Take r large enough so that $N(\tau) \subset B_{\mathcal{T}}(\imath, r)$ and recall that (since A_k is increasing and $m \geq k$) $B_{\mathcal{T}}(\imath, r)$ is isomorphic to a ball of a tree T_n centered at a vertex of an apocentric $\tau' \prec_m T_n$. Since the apocentric vertices of τ' are leaves in T_n , the apocentric vertices of τ are leaves in \mathcal{T} , so the only vertex of τ that has a neighbor outside of τ is its center. Only one of the connected components of $\mathcal{T} \setminus v$ contains the center of τ , so the rest are subtrees of τ , therefore finite. Hence, \mathcal{T} is 1-ended.

Subsequently, we proceed with the proof of (ii). For a balanced sequence, let $p_{(k)} := \prod_{n=1}^k p_n$ and note that $p_{(k)}$ is the probability that a uniformly chosen vertex of a tree T_n , $n \geq k$ is apocentric in a copy $\tau \prec_k T_n$. Such a vertex will be called a *k-boundary vertex*, and any ball centered at it will be called a *k-boundary ball*. Let $B_{k,r}$ be the event that the ball $B_{\mathcal{T}}(\imath, r)$ is isomorphic to a *k-boundary ball* of T_n for n large enough. Note that, for $r \geq 2^{k+1} - 1$, the only vertices that are the centers of balls isomorphic to *k-boundary balls* are indeed *k-boundary vertices*, hence $P(B_{k,r}) \leq p_{(k)}$. Also, observe that $B_{k,r}$ is a decreasing sequence in r . Let $B_k := \bigcap_{r \geq 0} B_{k,r}$. That is, B_k is the event that for every $r \geq 0$ the ball $B_{\mathcal{T}}(\imath, r)$ is isomorphic to a ball centered at an apocentric vertex of a copy $\tau \prec_k T_n$ that is a

leaf of T_n . We deduce that

$$P(B_k) = \lim_{r \rightarrow \infty} P(B_{k,r}) \leq \lim_{r \rightarrow \infty} p^{(k)} = p^{(k)}.$$

Since B_k is also a decreasing sequence in k ,

$$P(\liminf_{k \rightarrow \infty} B_k) = P\left(\bigcap_{k \geq 0} B_k\right) = \lim_{k \rightarrow \infty} P(B_k) \leq \lim_{k \rightarrow \infty} p^{(k)} = 0.$$

Therefore \imath lies a.s. in a sequence of copies $\tau \prec_{k_n} \tau' \prec_{k_{n+1}} \mathcal{T}$ with τ central in τ' . The proof that \imath lies a.s. in a sequence of copies $\tau \prec_{l_n} \tau' \prec_{l_{n+1}} \mathcal{T}$ with τ apocentric in τ' is completely analogous. We have shown that (\mathcal{T}, \imath) is a.s. a balanced limit.

We now proceed to deduce the number of ends of a balanced random rooted tree. It is well-known result that every unimodular random rooted tree with average root degree 2 has a.s. either one or two ends. However, this set happens to include all the Benjamini-Schramm limits of sequences of finite trees, and in particular \mathcal{T} . Indeed, the average degree of a tree on n vertices is $2 - \frac{1}{n}$, and, since $\lim_{n \rightarrow \infty} P(B_{T_n}(o_n, 1) \cong S) = P(B_{\mathcal{T}}(\imath, 1) \cong S)$ for every star S , we obtain

$$E(\deg(\imath)) = \lim_{n \rightarrow \infty} \sum_{m=1}^{\bar{\Delta}} m P(B_{T_n}(o_n, 1) \cong S_m) = \lim_{n \rightarrow \infty} 2 - \frac{1}{n} = 2,$$

where we recall that $\bar{\Delta}$ is the uniform upper bound for the vertex degrees of the sequence T_n .

We have shown that balanced trees have a.s. one or two ends. If (\mathcal{T}, \imath) has two ends, then there exists a vertex in \mathcal{T} that belongs to a double ray, so by [8, Proposition 11] \imath belongs to a double ray with positive probability. One ray R_1 is obtained by starting at \imath and moving to the center of τ' for each instance of $\imath \in \tau \prec_n \tau' \prec_{n+1} \mathcal{T}$ where τ is apocentric in τ' . Let k_n be the sequence for the terms of which we have $\imath \in \tau \prec_{k_n} \tau' \prec_{k_{n+1}} \mathcal{T}$ and τ is central in τ' . Say that $\tau \prec_{k_n} \mathcal{T}$ is *reachable* if either $n = 0$ and $\tau = \imath$ or the center of τ is adjacent to an apocentric vertex of a reachable $\tau' \prec_{k_{n-1}} \mathcal{T}$. Then \imath emits another ray R_2 disjoint from R_1 if and only if for each k_n there exists a reachable $\tau \prec_{k_n} \mathcal{T}$. By calculating the average number of reachable copies $\tau \prec_{k_n} \mathcal{T}$ and taking the limit as $n \rightarrow \infty$ we find that the average number of rays disjoint from R_1 that start from \imath is

$$\mathcal{R} := \prod_{n=1}^{\infty} \frac{\delta_{k_n}}{\prod_{m=k_{n-1}}^{k_n-1} \delta_m}.$$

Suppose that $\delta_n > 1$ for every $n > c$. Using [Proposition 4.3](#), we have $\delta_n = o(n^2)$ and we observe that $\frac{k_n^2}{2^{k_n - n - c}}$ is eventually an upper bound for the partial products of \mathcal{R} . For $\imath \in \tau \prec_n \tau' \prec_{n+1} \mathcal{T}$, let X_n be the random variable that equals 1 if τ is central in τ' and 0 otherwise. Define $X(k) = \sum_{n=1}^k X_n$ and note that the X_i are independent, and that $X(k_n) = n$. By Hoeffding's inequality,

$$P(n - (1 - p_{k_n})k_n > p_{k_n}k_n - 3 \log k_n - c) \leq e^{-\frac{2(p_{k_n}k_n - 3 \log k_n - c)^2}{k_n}} \leq e^{-\frac{k_n}{8}}.$$

Since $\sum_{n=1}^{\infty} e^{-\frac{k_n}{8}} < \infty$, by the Borel-Cantelli Lemma it a.s. eventually holds that $n \leq k_n - 3 \log k_n - c$ for every n , implying that $\mathcal{R} \leq \lim_{n \rightarrow \infty} \frac{1}{k_n} = 0$. Therefore, a.s. no such rays exist, and we have that \mathcal{T} is 1-ended.

Finally, note that, if $\delta_n > 1$ eventually, then each identification chain yields a different sequence of balls $|B_{\mathcal{T}}(\imath, r)|$, i.e. a different rooted limit tree. Also, for a

balanced sequence in particular, $\prod_{n=1}^{\infty} \mathcal{P}_n = 0$, where each \mathcal{P}_n can be equal to either p_n or $1 - p_n$. That is to say, each particular identification chain, and therefore from the previous observation each rooted limit tree of [Construction 2.3](#), is sampled from our balanced tree (\mathcal{T}, ι) with probability 0. Since each such tree has only countable vertices, any particular tree has probability 0 to be the underlying tree of (\mathcal{T}, ι) . \square

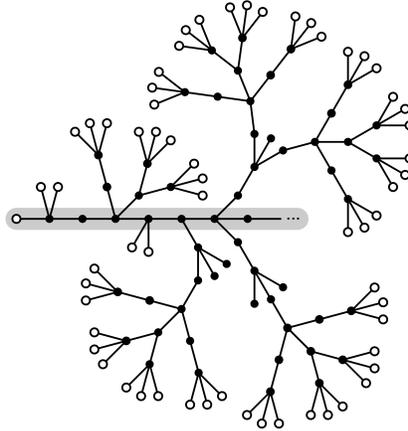


Figure 5. The balanced tree corresponding to the sequence $\delta_n = n + 2$. It is locally indistinguishable from a generalised canopy tree.

The significance of the distinction worked out in [Theorem 3](#) becomes more apparent with an example: generalized canopy trees (even of intermediate growth) can be a.s. isomorphic to a single deterministic tree. In essence, such limits can be seen as deterministic trees with a randomly chosen root. This is not possible for balanced trees (except for trivial cases).

Example 4.4. Define recursively $\delta_1 := 1, \delta_2 := 2$ and $\delta_{n+1} := \delta_n \delta_{n-1}$. In explicit form this reads

$$\delta_n = 2^{F_n}, \quad \text{for all } n \geq 0,$$

where F_n denotes the n -th Fibonacci number starting with $F_1 = 0, F_2 = 1$.

Let $T_n, n \geq 0$ be the sequence of trees according to [Construction 2.1](#). By [Proposition 4.2](#) (T_n, o_n) converges in the Benjamini-Schramm sense to a random rooted tree (\mathcal{T}, ι) of uniform volume growth. From $|T_n| = (\delta_1 + 1) \cdots (\delta_n + 1)$ one can estimate

$$\frac{1}{2} D^{r^\alpha} \leq |T_n| \leq \frac{1}{2} r \cdot D^{r^\alpha}, \quad \text{with } D := 2^{\varphi^2/\sqrt{5}} \approx 2.251 \text{ and } \alpha := \log \varphi \approx 0.6942.$$

Here $\varphi \approx 1.618$ denotes the golden ratio. The growth is therefore *intermediate*.

We claim that (\mathcal{T}, ι) is a.s. isomorphic to a particular deterministic tree: from [Proposition 4.3](#) we can see that (\mathcal{T}, ι) is a generalized canopy tree, and by [Theorem 3](#) it is a.s. an apocentric limit of the T_n . But as we show below ([Lemma 4.5](#)), the T_n are highly symmetric in that any two apocentric copies $\tau, \tau' \prec_{n-1} T_n$ are in fact indistinguishable by symmetry. In consequence, there exists (up to symmetry) only one possible inclusion chain leading to an apocentric limit, and \mathcal{T} is the unique tree obtained in this way.

In the following we establish the symmetry of T_n in an even stronger form:

Lemma 4.5. *With the sequence of [Example 4.4](#) the trees T_n are transitive on apocentric copies $\tau \prec_m T_n$ for each $m \leq n$, i.e., for any two apocentric copies $\tau_1, \tau_2 \prec_m T_n$ there is a symmetry $\phi \in \text{Aut}(T_n)$ sending $\phi(\tau_1) = \tau_2$.*

Proof. The proof is by induction on n . The cases $n \in \{0, 1\}$ are trivially verified and form the induction basis. Suppose that the statement is proven up to some $n \geq 1$; we prove the statement for T_{n+1} .

Let $\tau_0 \prec_n T_{n+1}$ denote the central copy, and let $\tau_1, \dots, \tau_{\delta_{n+1}} \prec_n T_{n+1}$ denote the apocentric copies. From our specific choice of sequence follows that the apocentric copies $\prec_n T_{n+1}$ are in one-to-one relation with the apocentric copies $\prec_{n-2} \tau_0$: by the “maximally uniform distribution of adjacencies” (cf. [Remark 2.5](#), especially equation [\(2.3\)](#)) the expected number of “outwards adjacencies” of an apocentric copy $\hat{\tau} \prec_{n-2} \tau_0$ is exactly

$$\frac{\delta_1 \cdots \delta_{n-2}}{\delta_1 \cdots \delta_n} \cdot \delta_{n+1} = \frac{\delta_1 \cdots \delta_{n-2}}{\delta_1 \cdots \delta_n} \cdot \delta_{n-1} \delta_n = 1.$$

In other words, there are exactly δ_{n+1} apocentric copies $\hat{\tau}_1, \dots, \hat{\tau}_{\delta_{n+1}} \prec_{n-2} \tau_0$ and they are in one-to-one relation with the τ_i . We can assume an enumeration so that $\hat{\tau}_i$ and τ_i are connected by an edge, its end in $\hat{\tau}_i$ we call x_i .

Fix two apocentric copies $\rho_1, \rho_2 \prec_m T_{n+1}$ for some $m \leq n+1$. The goal is to construct a symmetry of T_{n+1} that sends ρ_1 onto ρ_2 .

Let $i_k \in \mathbb{N}$ be indices so that $\rho_k \prec_m \tau_{i_k}$. By induction hypothesis there is a symmetry $\phi_0 \in \text{Aut}(\tau_0)$ that maps $\hat{\tau}_{i_1}$ onto $\hat{\tau}_{i_2}$. More generally, ϕ_0 permutes the trees $\hat{\tau}_1, \dots, \hat{\tau}_n$, that is, ϕ_0 sends $\hat{\tau}_i$ onto $\hat{\tau}_{\sigma(i)}$ for some permutation $\sigma \in \text{Sym}(n)$. It holds $\sigma(i_1) = i_2$. Note however that ϕ_0 does not necessarily map x_i onto $x_{\sigma(i)}$. We can fix this: invoking the induction hypothesis again, for each $i \in \{1, \dots, n\}$ exists a symmetry $\phi_1^i \in \text{Aut}(\hat{\tau}_i)$ that sends $\phi_0(x_i)$ onto $x_{\sigma(i)}$. The map

$$\phi_2(x) := \begin{cases} (\phi_1^i \circ \phi_0)(x) & \text{if } x \in \hat{\tau}_i \\ \phi_0(x) & \text{otherwise} \end{cases}.$$

is then a symmetry of τ_0 that sends the pair $(\hat{\tau}_i, x_i)$ onto $(\hat{\tau}_{\sigma(i)}, x_{\sigma(i)})$. Having this, ϕ_2 extends to a symmetry $\phi_3 \in \text{Aut}(T_{n+1})$ that necessarily sends τ_i onto $\tau_{\sigma(i)}$. In particular, it sends τ_{i_1} (which contains ρ_1) onto τ_{i_2} (which contains ρ_2). As before, ϕ_3 does not necessarily send ρ_1 onto ρ_2 right away; but also as before, this can be fixed: by induction hypothesis there is a symmetry $\phi_4 \in \text{Aut}(\tau_{i_2})$ that sends $\phi_3(\rho_1)$ onto ρ_2 . The map

$$\phi_5(x) := \begin{cases} (\phi_4 \circ \phi_3)(x) & \text{if } x \in \tau_{i_1} \\ \phi_3(x) & \text{otherwise} \end{cases}.$$

is a symmetry of T_{n+1} that sends ρ_1 onto ρ_2 . □

Unimodular random rooted trees that are a.s. isomorphic to a unique tree of even smaller uniform growth can be constructed by setting $\delta_{n+1} := \delta_n \delta_{n-1}$ for only some n , and $\delta_{n+1} := \delta_n$ otherwise. However, since Benjamini-Schramm limits of balanced sequences have measure zero on every countable set of trees, this approach cannot yield examples with uniform growth below $r^{\log \log r}$. One might wonder whether this is an artifact of our construction or a general phenomenon (see also [Question 5.8](#)).

5. CONCLUDING REMARKS AND OPEN QUESTIONS

Given a suitable function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, we constructed deterministic trees as well as unimodular random rooted trees of uniform volume growth g . We conclude this article with some related remarks and open question.

5.1. Planar triangulations. Historically, the interest in such trees has one of its roots in the, back then, curious observation, initially from physics, that planar triangulations can have non-quadratic uniform growth [2, 4]. In their landmark paper [7] Benjamini and Schramm also demonstrated how any tree of a particular growth can be turned into such a triangulation with a similar growth (it is worth mentioning that planar triangulation of sub-quadratic uniform growth are now known to be always “tree-like” [6]). We recall this construction briefly:

Construction 5.1. Suppose T is a tree of maximum degree $\bar{\Delta}$. Fix a triangulated sphere with at least $\bar{\Delta}$ pairwise disjoint triangles. Take copies of this sphere, one for each vertex of T , and identify two spheres along a triangle when the associated vertices are adjacent in T . This yields a planar triangulation. If T is of uniform growth g , so is this triangulation.

Having established the existence of trees for a variety of growth rates, we can conclude the existence of planar triangulations with the same wide range of growth behaviors. In fact, we can say slightly more: previously known triangulations of polynomial growth are planar, but not necessarily triangulations of the plane, *i.e.*, they are not necessarily homeomorphic to \mathbb{R}^2 . For this to be true, the tree T needs to be *one-ended*, which is the case *e.g.* for [Construction 2.3](#) with apocentric limits.

5.2. Subgraphs of uniform growth. Our initial approach for constructing trees of uniform intermediate growth was to start from just any graph of intermediate growth (such as a Cayley graph of the Grigorchuk group [9]), and extract a spanning tree that inherits this growth in some way. Ironically, working out the details of this extraction led to an understanding of the desired trees that allowed us constructing them without a need for the ambient graph. Still, the question remained:

Question 5.2. Given a graph G of uniform growth g , is there a spanning tree (or just any embedded tree) of the same uniform growth?

This question can be modified, weakened, and generalized in various forms:

Question 5.3. Given a graph G of uniform growth g , does it contain a planar subgraph of the same uniform growth? What about other types of subgraphs defined by excluded minors?

Question 5.4. Fix $k \in \mathbb{N}$. Given a graph G of uniform growth g , does G contain a subgraph H that is *not* k -connected and has the same uniform growth g ?

Question 5.5. If G is of uniform growth g , and $h \in o(g)$ (perhaps assuming some niceness conditions such as super-additivity and log-concavity), is there a subgraph $H \subset G$ of uniform growth h ?

We emphasize that an embedded tree $T \subset G$ of the same uniform growth (as we ask for in [Question 5.2](#)) is not necessarily quasi-isometric to G .

Example 5.6. Consider a family of cycles chained together to form the graph G as depicted in [Figure 6](#). G is the homeomorphic image of the infinite path with fibers of size at most two, hence of linear growth. Removing one edge from each cycle yields a spanning tree of linear growth. However, G has infinite perimeter, and hence no quasi-isometric spanning tree.

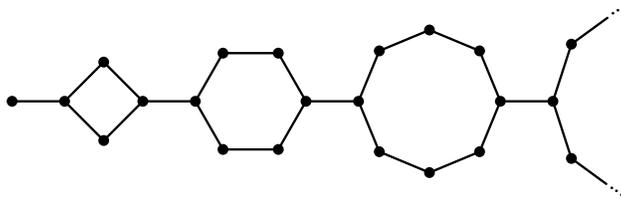


Figure 6. A graph with a spanning tree of the same uniform growth but with no quasi-isometric spanning tree.

5.3. Beyond the construction. The unimodular random rooted graphs of uniform volume growth constructed in [Section 4](#) were obtained as Benjamini-Schramm limits of the particular sequence T_n . We found a curious threshold roughly at growth $r^{\log \log r}$ and it remains open whether this is an artifact of our construction or in how far this points to a more fundamental phase change phenomenon in unimodular trees of uniform growth.

Question 5.7. To what extent are unimodular trees with growths on either side of the threshold $r^{\log \log r}$ structurally different?

In light of [Example 4.4](#) and the discussion below it we also wonder the following:

Question 5.8. Is there a random rooted tree (\mathcal{T}, ι) (other than the doubly-infinite path) that satisfies simultaneously all of the following three properties?:

- (i) (\mathcal{T}, ι) is unimodular.
- (ii) (\mathcal{T}, ι) is a.s. isomorphic to a particular deterministic tree.
- (iii) (\mathcal{T}, ι) is of uniform volume growth $g \in O(r^{\log \log r})$.

Questions about threshold phenomena and essentially deterministic unimodular graphs are of course equally meaningful for graph classes other than trees.

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APPENDIX A. EXISTENCE AND PROPERTIES OF γ

Recall the function $\gamma: \mathbb{N} \rightarrow \mathbb{R}$ introduced in the proof of [Theorem 3.6](#), defined by $\gamma(1) := 25$ and

$$(A.1) \quad \gamma(\delta) := \prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta^{2^k}}\right)^2, \quad \text{whenever } \delta \geq 2.$$

It is evident from the definition that $\gamma(\delta) > 1$. We show that it is well-defined:

Lemma A.1. $\gamma(\delta)$ is well-defined, i.e., the infinite product (A.1) converges.

Proof. $\gamma(\delta)$ is the square of

$$\prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta^{2^k}}\right) = \sum_{i=0}^{\infty} \frac{2^{b(i)}}{\delta^i},$$

where $b(i)$ denotes the number of 1's in the binary representation of i . Since $b(i) \leq \lfloor \log_2(i) \rfloor + 1 \leq \frac{i}{2}$ for $i \geq 10$, a geometric series estimation yields

$$(A.2) \quad \prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta^{2^k}}\right) \leq \sum_{i=0}^9 \frac{2^{b(i)}}{\delta^i} + \sum_{i=10}^{\infty} \frac{\sqrt{2}^i}{\delta^i} \leq \sum_{i=0}^9 \frac{2^{b(i)}}{\delta^i} + \frac{2 + \sqrt{2}}{32},$$

which is finite. □

Corollary A.2. The following bounds apply:

$$\begin{aligned} 9 &< \gamma(2) < 25/2 \\ 25/9 &< \gamma(3) < 5. \end{aligned}$$

Proof. The upper bounds can be verified from the proof of (A.2). The lower bounds follow via

$$\begin{aligned} \gamma(2) &> \left(1 + \frac{2}{2}\right)^2 \left(1 + \frac{2}{4}\right)^2 = 9, \\ \gamma(3) &> \left(1 + \frac{2}{3}\right)^2 = \frac{25}{9}. \end{aligned}$$

□

Corollary A.3. $\gamma(\delta)$ is strictly decreasing in δ .

Proof. For $2 \leq \delta_1 < \delta_2$ follows $\gamma(\delta_1) < \gamma(\delta_2)$ straight from the definition. For $\delta_1 = 1$ we use the bounds in [Corollary A.2](#) and find $\gamma(1) = 25 > 25/2 > \gamma(2)$. □

Proposition A.4. $\delta^2 \gamma(\delta)$ is strictly increasing for $\delta \geq 3$.

Proof. The function in question is the square of

$$f(\delta) := \delta \prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta^{2^k}}\right).$$

If we consider δ as a continuous parameter, we can compute the derivative of $\log f(\delta)$ w.r.t. δ , which yields

$$(\log f(\delta))' = \frac{1}{\delta} - \frac{2}{\delta} \sum_{k=0}^{\infty} \frac{2^k}{\delta^{2^k} + 2}.$$

For $\delta^2\gamma(\delta)$ to be increasing, we require this expression to be positive, which, after some rearranging, is equivalent to

$$\sum_{k=0}^{\infty} \frac{2^k}{\delta^{2^k} + 2} < \frac{1}{2}.$$

The sum is clearly decreasing in δ and so it suffices to verify the case $\delta = 3$:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{2^k}{3^{2^k} + 2} &= \frac{1}{5} + \frac{2}{9} + \sum_{k=2}^{\infty} \frac{2^k}{3^{2^k} + 2} \\ &< \frac{21}{55} + \sum_{k=2}^{\infty} \frac{2^k}{3^{2^k}} < \frac{21}{55} + \sum_{\ell=4}^{\infty} \frac{\ell}{3^{\ell}} = \frac{21}{55} + \frac{1}{12} < \frac{1}{2}. \end{aligned}$$

□

The following property of γ is used in the proof of [Theorem 3.6](#).

Lemma A.5. *If $\delta_1 \leq (\delta_2 + 2)^2$ then $\delta_1\gamma(\delta_1) \leq \delta_2^2\gamma(\delta_2)$.*

Proof. We proceed by case analysis. A use of [Corollary A.3](#) (i.e., γ is decreasing) is indicated by (*), a use of [Proposition A.4](#) (i.e., $\delta^2\gamma(\delta)$ is increasing) by (**). We also use the bounds proven in [Corollary A.2](#).

(1) If $\delta_1 \geq \delta_2^2$ then

$$\begin{aligned} \delta_1\gamma(\delta_1) &\stackrel{(*)}{\leq} (\delta_2 + 2)^2\gamma(\delta_2^2) = \delta_2^2 \left(1 + \frac{2}{\delta_2}\right)^2 \prod_{k=0}^{\infty} \left(1 + \frac{2}{(\delta_2^2)^{2^k}}\right)^2 \\ &= \delta_2^2 \prod_{k=0}^{\infty} \left(1 + \frac{2}{\delta_2^{2^k}}\right)^2 = \delta_2^2\gamma(\delta_2). \end{aligned}$$

(2) If $\delta_2 \leq \delta_1 < \delta_2^2$ then $\delta_1\gamma(\delta_1) \stackrel{(*)}{<} \delta_2^2\gamma(\delta_2)$ follows by factor-wise comparison

(3) If $3 \leq \delta_1 \leq \delta_2$ then

$$\delta_1\gamma(\delta_1) < \delta_1^2\gamma(\delta_1) \stackrel{(**)}{\leq} \delta_2^2\gamma(\delta_2).$$

(4) If $\delta_1 = 2$ and $\delta_2 \geq 3$ then

$$\delta_1\gamma(\delta_1) = 2\gamma(2) < 25 < 3^2\gamma(3) \stackrel{(**)}{\leq} \delta_2^2\gamma(\delta_2).$$

(5) If $\delta_1 = 1$ and $\delta_2 \geq 3$ then

$$\delta_1\gamma(\delta_1) = 1\gamma(1) = 25 < 3^2\gamma(3) \stackrel{(**)}{\leq} \delta_2^2\gamma(\delta_2).$$

(6) if $\delta_1 = 1$ and $\delta_2 = 2$ then

$$\delta_1\gamma(\delta_1) = 1\gamma(1) = 25 < 36 < 2^2\gamma(2).$$

□

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