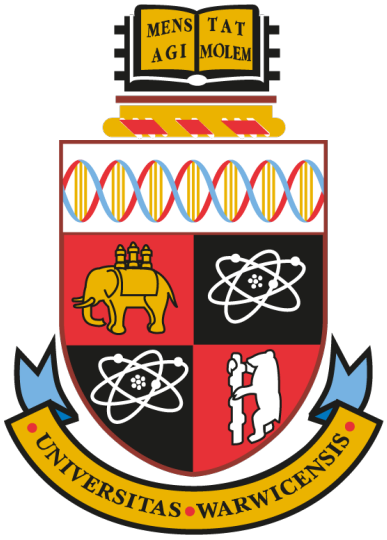


Infinite sumsets via ergodic theory

by

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Mathematics. It has been composed by myself and has not been submitted in any previous application for any degree. The work presented was carried out by myself under the supervision of Joel Moreira. The work presented in Chapter 3 is work that has been done in collaboration with Tristán Radić, the work presented in Chapter 4 has been done in collaboration with Dimitrios Charamaras, Andreas Mountakis and Tristán Radić and the work presented in Chapter 7 has been done in collaboration with Felipe Hernández and Tristán Radić.

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- **Revisiting sums and products in countable and finite fields** (2025). *Ergodic Theory Dynam. Systems* Published online 2025:1-32.
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Abstract

Until recently, virtually no results were known concerning the existence of infinite arithmetic configurations in sets of integers with positive density. The first breakthroughs in this thematic area were achieved in [49] by Moreira, Richter and Robertson and the same authors with Kra in [41, 43, 45]. By introducing new ergodic-theoretic methods, they verified conjectures of Erdős about infinite sumsets, and established far-reaching density analogues of Hindman's finite sums theorem.

Building upon this foundation, we resolve various conjectures and several open problems in this exciting area of research. One of our results extends the main theorem of [41], confirming [44, Conjecture 3.10]. Namely, we show that for any set $A \subset \mathbb{N}$ with positive upper Banach density and any $\ell, m \in \mathbb{N}$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ so that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 < b_2\} + t \subset A$.

Under the same assumptions, we show the existence of an infinite set B and a sequence $(t_k)_{k\in\mathbb{N}}$ of naturals such that $\left\{\sum_{n\in F} n : F \subset B, |F| = k\right\} \subset A - t_k$, for every $k \in \mathbb{N}$. This strengthens the density finite sums theorem of [45] and, in the context of density analogues to Hindman's theorem, is the first result to guarantee an unbounded number of summands generated by a fixed infinite set. As a corollary of this, we obtain a sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers such that $B_1 + \cdots + B_k \subset A$, for every $k \in \mathbb{N}$, thus extending the main result of [43] and verifying [44, Conjecture 2.10].

We also study unrestricted sumsets, which are not generally contained in every set of positive density. We fully characterise, in terms of natural asymptotic densities, the existence of an infinite $B \subset \mathbb{N}$ and $t \in \{0, 1\}$ with $B + B + t \subset A$: such B and t exist whenever $\underline{\mathrm{d}}(A) > 1/2$ or $\overline{\mathrm{d}}(A) > 3/4$. Moreover, we show that for any $\ell, m \in \mathbb{N}$, any set $A \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) > 1/2$ contains a configuration $\{mb_1 + \ell b_2 : b_1, b_2 \in B,\ b_1 \leq b_2\} + t$ for some infinite B and $t \in \mathbb{N}$.

We furthermore expand this framework to countable abelian groups. Under natural, and necessary, assumptions on the doubling map $D : g \mapsto g + g$ of such a group $(G, +)$, we show that for any Følner sequence $\Phi = (\Phi_N)_N$ in G, if $A \subset G$ is such that $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$, then there is an infinite set $B \subset G$ and some $t \in G$ for which $B + B + t \subset A$, where $\Phi/2 = (D^{-1}(\Phi_N))_N$.

For the aforementioned results, the density thresholds are sharp – restricting to the cases where this statement is a non-trivial one. We show this by constructing sets that attain the lower bounds but do not contain the related sumsets.

Chapter 1

Introduction

In this thesis, we investigate various problems of additive-combinatorial nature, by mainly using tools and methods from ergodic theory. The problems we study are chiefly concerned with the existence of infinite configurations, and in particular, sumsets, inside sets of integers, and other abelian groups, with positive density.

Our principal contributions can be summarised as follows. First, we establish weighted infinite sumset results in sets of positive density. Second, we develop density analogues of Hindman's finite sums theorem. Third, we determine sharp density conditions for the existence of unrestricted sumsets in a general abelian group setting. The underlying common approach is to translate these problems into statements about recurrence and convergence of orbits in dynamical systems.

The contents of this work thus fall primarily under the scope of ergodic Ramsey theory, a relatively young branch of mathematics – originating in Furstenberg's foundational work [18] – in which ergodic-theoretic techniques are used to prove Ramsey-theoretic and, generally, arithmetic-combinatorial statements. The developments presented herein add yet another witness, to a plethora of pre-existing ones, illustrating the strength and utility of this perspective.

1.1 Background and motivation

1.1.1 Arithmetic Ramsey theory

Ramsey theory is a branch of combinatorics studying the broad phenomenon of order appearance in substructures of a given structure, or structure preservation under finite partitions of a structured object. In the narrower intersection of the field with arithmetic or additive combinatorics, the key objects of enquiry are *partition regular configurations*, that is, configurations that must exist in one cell of any finite

partition of, say, the natural numbers.

A prototypical result in arithmetic Ramsey theory is a well-known theorem of Schur [56] – via which Schur proved that Fermat's last theorem fails in modular arithmetic (see the discussion in Bergelson's survey [6, Section 1]) – showing that for any finite *colouring* (i.e. partition) $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists a colour class C_i, and distinct $x, y \in \mathbb{N}$ such that $\{x, y, x+y\} \subset C_i$, or $\{x, y, x+y\}$ is *monochromatic*. A more concise way to phrase this is that the configuration $\{x, y, x+y\}$ is *partition regular*.

Perhaps the most classical result, and the starting point for a huge amount of research, in the field is van der Waerden's theorem on arithmetic progressions [61], which states that arithmetic progressions of arbitrary length, i.e. sets of the form $\{x, x+y, x+2y, \ldots, x+ky\}$, are partition regular in $\mathbb{N}$.

Finally, we also recall the result to which Ramsey theory owes its name, namely, Ramsey's theorem [54], as it also happens to be pertinent to our work. A strong version of it, which is of relevance for us, affirms that for any finite colouring of $\binom{\mathbb{N}}{2}$, there is an infinite set $S \subset \mathbb{N}$ such that $\binom{S}{2}$, i.e. the set of cardinality 2 subsets of S, is monochromatic. A direct corollary of this is the fact that for any finite colouring $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists an infinite set $B \subset \mathbb{N}$ for which

$$B \oplus B := \{b_1 + b_2 : b_1, b_2 \in B,\ b_1 \neq b_2\} = \{b_1 + b_2 : b_1, b_2 \in B,\ b_1 < b_2\}$$

is monochromatic. It was observed by Hindman [27] that the restriction $b_1 \neq b_2$ is necessary for this result to hold.

1.1.2 Hindman's theorem and Erdős's insight

A particular, and already very interesting, class of partition regular configurations is those that arise as solutions of systems of homogeneous equations. For clarity, a system of homogeneous equations with integer coefficients is partition regular if the system has a monochromatic solution for any finite colouring of the natural numbers. Notice that both Schur's theorem and van der Waerden's theorem can be formulated in this terminology, as the configurations under consideration in each case are solutions of systems of linear equations. Both results are special instances of Rado's celebrated theorem [53] characterizing all (finite) partition regular systems of homogeneous linear equations.

These being examples of finite partition regular configurations, a natural question – even more so, in view of Ramsey's theorem – is what non-trivial infinite configurations are partition regular. In his landmark work [26], Hindman provided

the first example of an infinite partition regular system of equations.

Theorem A (Hindman's finite sums theorem)**.** Let $\mathbb{N} = C_1 \cup \cdots \cup C_r$ be a finite colouring. Then, there exists a colour class C_i and an infinite set $B \subset \mathbb{N}$ such that

$$\left\{ \sum_{n \in F} n : \emptyset \neq F \subset B,\ |F| < \infty \right\} \subset C_i. \tag{1.1.1}$$

As we hinted at in the beginning of the introduction, a principle of Ramsey theory is that order emerges so long as one considers large enough subsets of a structured object – the notion of largeness depending on the order under consideration. Specifically for arithmetic configurations in $\mathbb{N}$, an important notion of largeness is positive upper Banach density; that density being defined for a set $A \subset \mathbb{N}$ as

$$\mathrm{d}^*(A) := \limsup_{N-M \to \infty} \frac{|A \cap \{M, M+1, \ldots, N-1\}|}{N-M}.$$

A representative example of this principle is Szemerédi's famous theorem [58], according to which any set with positive upper Banach density contains arbitrarily long arithmetic progressions. This is a highly non-trivial strengthening of van der Waerden's theorem, since in a finite colouring of $\mathbb{N}$ at least one of the colours has positive upper Banach density, and was conjectured by Erdős and Turán [17], almost forty years before its proof.

Another influential insight of Erdős was that Hindman's finite sums theorem should enjoy a density counterpart. Since the set of odd numbers $A = 2\mathbb{N} - 1$ has density $1/2$ and does not contain configurations as in the LHS of (1.1.1), a direct density analogue – one paralleling the relation between the theorems of Szemerédi and van der Waerden – is not possible. Moreover, an example of Straus (published not by him but by others; see, e.g., [5, Theorem 2.20]) placed even more limitations on what could potentially be a density version of Hindman's theorem. In particular, Straus showed that there are sets $A \subset \mathbb{N}$ of density arbitrarily close to 1, no shift of which, i.e. $A - t$, for $t \in \mathbb{N}$, contain infinite sumsets as in (1.1.1). This led Erdős to conjecture, on different occasions [14, 15, 16], that for any set $A \subset \mathbb{N}$ with positive density,

(i) there are infinite sets $B, C \subset \mathbb{N}$ such that $B + C \subset A$ and

(ii) there is an infinite set $B \subset \mathbb{N}$ and a shift $t \in \mathbb{N}$, such that $B \oplus B + t \subset A$.

Apart from a weaker result of Nathanson [50], both conjectures saw no progress for many years. The first partial result in this direction was established in [12].

In this work, Di Nasso, Goldbring, Jin, Leth, Lupini and Mahlburg employed nonstandard analysis and ideas from ergodic theory in order to verify (i) for sets with $\mathrm{d}^*(A) > 1/2$.

1.1.3 Ergodic Ramsey theory and recent breakthroughs

The beginning of ergodic Ramsey theory is traced back to Furstenberg's highly influential ergodic proof of Szemerédi's theorem [18]. The first step in Furstenberg's argument was to translate the statement of Szemerédi's theorem into a statement about multiple recurrence in measure-preserving systems, which he went on to prove via a stronger multiple ergodic theorem. This translation step, although not very technical, was arguably a very insightful part of the proof and has since been termed *Furstenberg's correspondence principle*. To phrase this principle rather vaguely: the quadruple $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathrm{d}^*, T)$ behaves like a measure-preserving system. Here $\mathcal{P}(\mathbb{N})$ denotes the power-set of the integers, and T is a shift map on $\mathbb{N}$, specifically, translation by 1. More precisely, Furstenberg proved the following result.

Lemma B (Furstenberg's correspondence principle)**.** Given $A \subset \mathbb{N}$ there exists a measure-preserving system $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a Borel probability space, and a set $E \in \mathcal{B}$ with $\mu(E) = \mathrm{d}^*(A)$, such that for every $m \in \mathbb{N}$ and any integers $n_1, \ldots, n_m \in \mathbb{N}$, we have that

$$\mu(E \cap T^{-n_1}E \cap \cdots \cap T^{-n_m}E) \leq \mathrm{d}^*(A \cap (A - n_1) \cap \cdots \cap (A - n_m)) \tag{1.1.2}$$

A few years after [12] was published, in a major breakthrough, Moreira, Richter and Robertson [49] confirmed the full $B + C$ sumset conjecture of Erdős, using a mixture of combinatorial and nonstandard analysis, and an ergodic structural result. Host [30], subsequently gave a shorter and more classical ergodic proof of the same conjecture.

The next advancement was established in [43] by Kra, Moreira, Richter and Robertson, who gave a new proof of Erdős's conjecture (i), and moreover proved that for any $k \in \mathbb{N}$, any set with positive upper Banach density contains infinite sumsets of the form $B_1 + \cdots + B_k$.

The resolution of the second aforementioned conjecture of Erdős, came from Kra, Moreira, Richter and Robertson [41] using once again methods from ergodic theory and building on the techniques developed earlier in [43] by the same authors.

Theorem C ([41, Theorem 1.2])**.** For any $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ there exists some

infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$ such that

$$\{b_1 + b_2 \colon b_1, b_2 \in B,\ b_1 \neq b_2\} + t \subset A.$$

Finally, in compelling work [45], using structure theory of multiple ergodic averages, Kra, Moreira, Richter and Robertson proved the following truncated density version of Hindman's theorem.

Theorem D ([45, Theorem 1.1])**.** Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, for every $k \in \mathbb{N}$, there exists an infinite set $B_k \subset \mathbb{N}$ and an integer $t_k \geq 0$ such that

$$\left\{ \sum_{n \in F} n \colon\ F \subset B_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A.$$

In each of these results, a version of the correspondence principle was implemented in order to translate the combinatorial statements into dynamical ones, specifically about the convergence of special orbits in ergodic systems; an example of this translation is outlined in the following section.

1.2 Infinite sumsets in sets of positive density

1.2.1 Asymmetric infinite sumsets

With our first main result, published in solo-authored work [38], we extend the methods developed in [41] in order to strengthen the central result therein, that is, Theorem C, and verify [44, Conjecture 3.10].

Theorem 1.2.1. *For any $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ and $\ell, m \in \mathbb{N}$, there exists an infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} + t \subset A.$$

Observe that Theorem C is a special case of Theorem 1.2.1 for the case $\ell = m = 1$. Our proof of Theorem 1.2.1, presented in Chapter 6, is ergodic-theoretic in nature and it relies on extensions of the beautiful argument crafted in [41] for the proof of Theorem C. The main setup for it is laid out in Section 6.4 and it is completed in Section 6.5, along with a slightly stronger result (see Remark 6.5.3).

To give a preliminary sense of how dynamics enter the picture, given a set $A \subset \mathbb{N}$, we first relate the inclusion

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 < b_2\} + t \subset A$$

to the existence of a specific topological dynamical system (X, T), and a triple $(a, x_1, x_2) \in X^3$, with $a \in X$ being a predetermined point, satisfying the convergence

$$\lim_{i\to\infty} \left(T^\ell \times T^m\right)^{n_i} (a, x_1) = (x_1, x_2), \tag{1.2.1}$$

along a sequence $(n_i)_{i\in\mathbb{N}}$. We call triples of points satisfying (1.2.1), $(\ell, m)-$Erdős progressions – which may justify the unorthodox order in the notation of Theorem 1.2.1. In Section 6.3.1 we explain how a classical version of Furstenberg's correspondence principle, which is similar to Lemma B, allows us to translate Theorem 1.2.1 to a dynamical statement of the above form, namely, Theorem 6.3.2. The proof of the latter is carried out by constructing a special measure on X^3 supported on triples of points satisfying (1.2.1) and the bulk of the argument is to verify some properties of this measure that allow this to happen.

1.2.2 Density analogues of Hindman's finite sums theorem

The most technical results in this thesis are presented in Chapter 7 and relate to extensions of Theorem D, towards more density versions of Hindman's finite sums theorem.

To bridge the gap between Theorem D and the counterexample of Straus we mentioned in Section 1.1.2, Kra, Moreira, Richter and Robertson asked whether a sequence of shifts – instead of a single shift – would allow for an unbounded number of summands generated by one infinite set.

Question 1.2.2 ([44, Question 2.11])**.** Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Does there exist an infinite set $B \subset \mathbb{N}$ and a sequence of shifts $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\left\{ \sum_{n\in F} n \colon \ F \subset B \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N} \ ?$$

In joint work with Hernández and Radić [24], we answer Question 1.2.2 in the negative (see Proposition 7.3.2) by constructing a set of density arbitrarily close to 1 for which even a weaker version of the conclusion fails. However, our main results, Theorems 1.2.3 and 1.2.4, extend Theorem D in two different directions, achieving in each case a certain degree of independence between the infinite set B and the number of summands $k \in \mathbb{N}$.

Theorem 1.2.3. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$ and $\ell \geq 0$ an integer. Then,*

there exist an infinite set $B_\ell \subset \mathbb{N}$ *and a sequence* $(t_{\ell,k})_{k\in\mathbb{N}} \subset \mathbb{N}$ *such that*

$$\left\{ \sum_{n\in F} n\colon \ F \subset B_\ell \text{ with } k \leq |F| \leq k+\ell \right\} + t_{\ell,k} \subset A, \text{ for every } k \in \mathbb{N}.$$

In Section 7.3, we show that Theorem 1.2.3 is optimal in the sense that one cannot choose the set B_ℓ independently of ℓ. We also show that the sequence of shifts $(t_{\ell,k})_{k\in\mathbb{N}}$ cannot, in general, be bounded.

Theorem 1.2.4. *Let* $A \subset \mathbb{N}$ *be a set with* $\mathrm{d}^*(A) > 0$. *Then, there exist an infinite set* B *and sequences* $(s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *such that*

$$\left\{ \sum_{n\in F} n\colon \ F \subset B + s_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

Letting $B_k = B + s_k$, for every $k \in \mathbb{N}$, we see that the strength of this result over Theorem D is that the sets $(B_k)_{k\in\mathbb{N}}$ are shifts of a single infinite set $B \subset \mathbb{N}$. Our construction in Section 7.3 shows that one cannot in general remove the shift sequences $(t_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$, as well as the necessity of them being unbounded. The second auxiliary sequence in Theorem 1.2.4, as opposed to merely shifting the set A, is naturally introduced to overcome the further local obstructions that arise in our answer to Question 1.2.2.

Recall that, the main result of [43] shows that for any $k \in \mathbb{N}$, any set of positive density contains a sumset of the form $B_1 + \cdots + B_k$, where $B_1, \ldots, B_k$ are infinite. As a corollary of our main theorems, we strengthen this by removing the dependence of the infinite sets on k. This positively answers [43, Question 8.1] and verifies [44, Conjecture 2.10].

Corollary 1.2.5. *Let* $A \subset \mathbb{N}$ *be a set with* $\mathrm{d}^*(A) > 0$. *Then, there exists an infinite sequence* $(B_n)_{n\in\mathbb{N}}$ *of infinite sets of natural numbers such that*

$$B_1 + B_2 + \cdots + B_k \subset A, \text{ for every } k \in \mathbb{N}.$$

We remark that Theorem 1.2.3 for $\ell = 0$ already implies Corollary 1.2.5 and the details can be found in Section 7.4.1. Similarly, Theorem 1.2.4 also implies Corollary 1.2.5.

A high-level outline of the ergodic machinery built to prove Theorems 1.2.3 and 1.2.4 is presented in Section 7.2.2.

1.3 Unrestricted sumsets in sets of large density

For the purposes of contextualizing our next main results, we redirect our attention to Theorem C. In particular, we point out that the restriction $b_1 \neq b_2$ is necessary and the nature of the problem changes substantially when it is removed. Indeed, it is not difficult to exhibit a set of full upper Banach density and positive lower density that does not contain infinite sumsets of the form $B+B+t := \{b_1+b_2 : b_1, b_2 \in B\}+t$ (see, e.g., [44, Example 2.3]).

A natural question then is whether such unrestricted sumsets are guaranteed in sets that satisfy stronger density assumptions. We refer to *the (unrestricted) $B+B$ problem* as the general problem of finding a density scheme – initially in the integer setting, and later on in countable abelian groups – for which any set with large enough density contains a sumset of the form $B+B+t$.

1.3.1 Sumsets of the form $B+B$ in the integers

The focus of Chapter 3, which is based on the joint work with Radić [40], is to provide conditions on the density of a set $A \subset \mathbb{N}$ that ensure the existence of an infinite set $B \subset \mathbb{N}$ such that $B+B+t \subset A$, for some $t \in \{0,1\}$. These conditions turn out to be considerably more restrictive than those of Theorem C. With the following theorems we address [44, Conjecture 3.7] and provide optimal density thresholds for natural upper and lower densities for the unrestricted $B+B$ problem.

Theorem 1.3.1. *Let $A \subset \mathbb{N}$.*

1. *If $\overline{\mathrm{d}}(A) > 5/6$, there is an infinite set $B \subset \mathbb{N}$ such that $B+B \subset A$.*
2. *If $\underline{\mathrm{d}}(A) > 3/4$, there is an infinite set $B \subset \mathbb{N}$ such that $B+B \subset A$.*

When we allow for a shift, the density assumptions can be weakened.

Theorem 1.3.2. *Let $A \subset \mathbb{N}$.*

1. *If $\overline{\mathrm{d}}(A) > 2/3$, there is an infinite set $B \subset \mathbb{N}$ and $t \in \{0,1\}$ such that $B+B+t \subset A$.*
2. *If $\underline{\mathrm{d}}(A) > 1/2$, there is an infinite set $B \subset \mathbb{N}$ and $t \in \{0,1\}$ such that $B+B+t \subset A$.*

To prove Theorems 1.3.1 and 1.3.2 we use the tools developed in [41] and translate the problem of finding infinite configurations to a statement in ergodic theory. As was shown in [41] the containment $B \oplus B \subset A$, for some infinite $B \subset \mathbb{N}$, is related

to the existence of a particular system (X,T) and specific points $a, x_1, x_2 \in X$ such that for some increasing sequence $(n_k)_{k\in\mathbb{N}}$ we have $(T\times T)^{n_k}(a,x_1)\to(x_1,x_2)$ (see [41, Theorem 2.2]). In the present work, we also need that $T^{2n_k}a\to x_2$ in order to ensure that $2B:=\{2b : b\in B\}\subset A$. Given that we care simultaneously about the convergence under iterations of T and T^2, we develop a modified version of Furstenberg's correspondence principle, where instead of building an appropriated ergodic invariant measure on the full shift in two symbols, $(\{0,1\}^{\mathbb{Z}},T)$, we build an appropriated $(T^2\times T)$-invariant probability measure on $(\{0,1\}^{\mathbb{Z}}\times\{0,1\}^{\mathbb{Z}},T^2\times T)$. This is carried out in Lemmas 3.3.3 and 3.3.7 and the proofs of Theorems 1.3.1 and 1.3.2 are presented in Section 3.4.

In Section 3.5.1 we establish that the density bounds listed in Theorems 1.3.1 and 1.3.2 are sharp. As an illustrative example, we construct a set $A\subset\mathbb{N}$ with $\overline{\mathrm{d}}(A)=5/6$ which does not contain a sumset $B+B$, for any infinite $B\subset\mathbb{N}$.

1.3.2 Sumsets of the form $B+B$ in countable abelian groups

It is natural to ask whether Theorem C could be extended to other countable amenable (semi)groups. This was explored in [11], where Charamaras and Mountakis established an extension of Theorem C for a wide class of such groups. In particular, [11, Corollary 1.13] asserts that if $(G,+)$ is a countable abelian group with index $[G:2G]<\infty$ and $A\subset G$ has positive upper Banach density, there exist an infinite set $B\subset A$ and some $t\in G$ such that $B\oplus B+t\subset A$.

To define a density scheme for which any set with large enough density contains a sumset of the form $B+B+t$, we use the notion of *Følner sequences*. Recall that given an abelian group G, a sequence $\Phi=(\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of G is a Følner sequence if for any $g\in G$,

$$\lim_{N\to\infty}\frac{|\Phi_N\cap(g+\Phi_N)|}{|\Phi_N|}=1.$$

The *upper density* of a set $A\subset G$ with respect to Φ, is defined as the limit $\overline{\mathrm{d}}_\Phi(A)=\limsup_{N\to\infty}\frac{|A\cap\Phi_N|}{|\Phi_N|}$, and we speak of the density $\mathrm{d}_\Phi(A)$ when the limit exists.

We remark that for the results presented in Section 1.3.1, taking the density with respect to *the initial Følner sequence* $N\mapsto[1,N]$ in $\mathbb{N}$ is essential, as it is clear from [44, Example 2.3] that not all Følner sequences can be used to guarantee such density threshold values.

The main aim of the preceding discussion is to set the stage for the following – a posteriori natural – questions, which we initially address in Chapter 4: Let G be a countable abelian group, such that $[G:2G]<\infty$.

(a) Can we find a Følner sequence Φ and a constant $c = c(G, \Phi) > 0$ such that any set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) > c$ contains an unrestricted sumset of the form $B + B + t$ for some infinite set $B \subset G$ and some $t \in G$?

(b) Can we answer (a) in an optimal way?

This is a more precise formulation of *the unrestricted $B+B$ problem* in countable abelian groups. In Chapter 4 this exact question is addressed successfully. For brevity we do not present the main theorems of this chapter here, which are based on the joint work with Charamaras, Mountakis and Radić [10], but we opt to present the stronger results of Chapter 5 which are based on the solo-authored recent paper [39]. Nevertheless, we want to emphasise that the arguments in Chapter 5 are highly influenced by those developed in [10] and rely on many technical results established therein.

Throughout, G denotes a countable abelian group and D denotes the doubling map $D : G \to G, D(g) = 2g$. In addition, given a set $A \subset G$, we write $2A$ for the set $D(A)$, and $A/2$ for $D^{-1}(A)$. The main result of Chapter 5 reads as follows.

Theorem 1.3.3. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $A \subset G$ and Φ be any Følner sequence in G. Pass to a subsequence of Φ if necessary so that all densities exist. Then the following hold:*

(1) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$, there exist an infinite set $B \subset G$ and some $t \in G$ such that $B + B + t \subset A$.

(2) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > \frac{2\ell-1}{\ell}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

(3) If $\mathrm{d}_\Phi(A \cap 2G) + \mathrm{d}_{\Phi/2}(A \cap 2G) > \frac{1}{\ell}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

The need to pass to an appropriate subsequence in Theorem 1.3.3 is not a mere artifact of the proof, but rather we even have the following extreme behaviour if we consider densities along different subsequences of Φ and $\Phi/2$.

Proposition 1.3.4. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. There is a set $A \subset G$ and a Følner sequence Φ such that*

$$\overline{\mathrm{d}}_\Phi(A) + \overline{\mathrm{d}}_{\Phi/2}(A) = 2,$$

but there does not exist an infinite set $B \subset G$ and $t \in G$ with $B + B + t \subset A$.

We also show the optimality of the density threshold values in Theorem 1.3.3 via the following result.

Theorem 1.3.5. *Let $(G,+)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. There is a set $A \subset G$ such that*

$$\mathrm{d}_{\Phi}(A) + \mathrm{d}_{\Phi/2}(A) = 1,$$

for some Følner sequence Φ in G, but there does not exist an infinite set $B \subset G$ and $t \in G$ with $B + B + t \subset A$.

Finally, we want to remark that both the assumptions $[G : 2G] < \infty$ and $r = |\ker(D)| < \infty$ are necessary for the previous results to hold. Indeed, this is already the case for the weaker results of Chapter 4. In particular – even though it may seem surprising at first since the density thresholds in Theorem 1.3.3 do not depend on r – there are groups with $|\ker(D)| = \infty$ for which Theorem 1.3.3 fails dramatically (see Section 4.6.1).

1.3.3 Sumsets of the form $mB + \ell B$

Analogously, we are also interested in an unrestricted version of Theorem 1.2.1. Including the diagonal in the sumsets is an important first step in this direction. Building on the ideas involved in the proofs of Chapter 3 and the proof of Theorem 1.2.1, we are able to prove the following results.

Theorem 1.3.6. *Let $\ell, m \in \mathbb{N}$. For any $A \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) > 1/2$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A.$$

Theorem 1.3.7. *Let $\ell, m \in \mathbb{N}$ and $k = m/\ell$. For any $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 1 - 1/(k+2)$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A.$$

These theorems are proven in Chapter 6. We also prove similar results for the case of unshifted patterns (see, e.g., Theorem 6.2.2). Once again, we will show that both previous density threshold bounds are optimal. To demonstrate this, we show, for example, the following.

Proposition 1.3.8. *Let $\ell, m \in \mathbb{N}$ and $k = m/\ell$. Then, there exists a set $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = 1 - 1/(k+2)$ such that for any infinite set B and any integer $t \in \mathbb{N}$ we have that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \not\subset A$.*

Recall the discussion after Theorem 1.2.1. We try to mimic this approach for the latter results, but the problem gets more complicated – already at the level of the correspondence principle – if we also want to include the diagonal in the sumsets. More precisely, in the dynamical setting of that discussion, in order to guarantee that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A$, we additionally need to know that $(T^{j\ell})^{n_i} a \to x_2$, for some $j \in \mathbb{N}$ such that $j\ell = \ell + m$. However, this equation is only solvable if $k = m/\ell$ is an integer. In this special case, one could devise a modified version of the correspondence principle, by building an appropriated $(T^{(k+1)} \times T)$-invariant probability measure on $(\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, T^{(k+1)} \times T)$. For the case $\ell = m = 1$, this argument will be utilised in Chapter 3.

To handle the general case when the ratio $k = m/\ell$ is not an integer, we consider a $(T^{(\lceil k \rceil + 1)} \times T)$-invariant probability measure on $(\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, T^{(\lceil k \rceil + 1)} \times T)$, where $\lceil k \rceil$ denotes the ceiling of k, arising from a generic pair of points (a', a), where a corresponds to the indicator of A, and a' corresponds to the indicator of an auxiliary set $A' \subset \mathbb{N}$. Essentially, A' is such that the inclusion $(\ell(\lceil k \rceil + 1)B + t \subset A'$ also implies $(m + \ell)B + t \subset A$. In Section 6.3.2 we formulate this version of the correspondence principle as Lemma 6.3.8 and use it to deduce Theorem 1.3.7 from a dynamical statement, namely Theorem 6.3.6.

1.4 Outline of the following chapters

The proofs of our main results rely on arguments from ergodic theory and topological dynamics. Thus, for completeness, in Chapter 2 we introduce relevant terminology, and collect several necessary ingredients, as well as classical results that will be needed. At the end of the chapter we gather the notation that is used throughout this thesis.

In Chapter 3 we present the results of [40] some of which were laid out in Section 1.3.1. Then, in Chapter 4 we address the problem studied in Chapter 3 to the more general setting of countable abelian groups. The culmination of these first chapters is, in a sense, completed in Chapter 5, where we strengthen the results presented up to that point, as explained in Section 1.3.2.

Chapter 6 contains the results discussed in Sections 1.2.1 and 1.3.3 and is based on the work [38].

We conclude with Chapter 7 that comprises the results of [24], some of which

were presented in Section 1.2.2. We also remark that the ergodic-theoretic machinery developed and used in this chapter is the most sophisticated of all this thesis.

The appendices, Appendix A and Appendix B, consist, respectively, of useful general facts about countable abelian groups and Følner sequences, and some technical examples that justify our approach in Chapter 7.

Finally, we note that to facilitate selective reading, we have chosen to include detailed introductions of the main results to follow at the beginning of each chapter – potentially at the cost of slight repetitions of things that appeared in the general introduction of Chapter 1.

For more work related to this interesting and flourishing theory see also [21, 23, 25, 42, 48, 52, 60].

Chapter 2

Preliminaries

2.1 Densities and notions of largeness

Let $A \subset \mathbb{N}$. The upper and lower asymptotic densities of A are defined, respectively, by

$$\overline{\mathrm{d}}(A) := \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N} \text{ and } \underline{\mathrm{d}}(A) := \liminf_{N\to\infty} \frac{|A \cap [1,N]|}{N},$$

where $[1,N] = \{1,\dots,N\}$, for any $N \in \mathbb{N}$. If these two quantities are equal, their common value is called the natural density of A and is denoted by $\mathrm{d}(A)$.

More general notions of density measure the asymptotic proportion of a set's elements in other sequences of intervals. In particular, the upper Banach density of $A \subset \mathbb{N}$ is

$$\mathrm{d}^*(A) := \limsup_{N-M\to\infty} \frac{|A \cap \{M, M+1, \dots, N-1\}|}{N-M}.$$

and its lower Banach density is

$$\mathrm{d}_*(A) := \liminf_{N-M\to\infty} \frac{|A \cap \{M, M+1, \dots, N-1\}|}{N-M}.$$

One important feature of a density function is shift-invariance. This is guaranteed by averaging along asymptotically shift-invariant sequences of sets, which is captured by the notion of *Følner* sequences.

For the generality that we will be interested in throughout this thesis, let $(G,+)$ be a countable abelian group. Then, a Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ in G is a sequence of nonempty finite subsets of G which are asymptotically invariant under any shift, that is,

$$\lim_{N\to\infty} \frac{|\Phi_N \cap (\Phi_N + t)|}{|\Phi_N|} = 1,$$

for any $t \in G$. The upper density of a set $A \subset G$ along Φ is given by

$$\overline{\mathrm{d}}(A) := \limsup_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|}.$$

The lower density of A along Φ and the density, when it exists, are then defined analogously to how asymptotic natural density was defined in $\mathbb{N}$. For reference, the special sequence $N \mapsto [1, N]$ in $\mathbb{N}$ is typically called *the initial Følner sequence*.

For a countable abelian group $(G, +)$, the upper Banach density of a set $A \subset G$ is defined as

$$\mathrm{d}^*(A) := \sup\{\overline{\mathrm{d}}_\Phi(A) : \Phi \text{ is a Følner sequence in } G\}.$$

The tentative reader may observe that this definition seems to differ from the original one in $\mathbb{N}$. Nevertheless, it follows (see, e.g. [4, eq. (5)]) that in any abelian group $(G, +)$, for any set $A \subset G$ and any Følner sequence $\Phi = (\Phi_N)$, there exists a sequence (t_n) in G, such that

$$\mathrm{d}^*(A) = \mathrm{d}_{(\Phi_N + t_N)}(A).$$

Although $\mathbb{N}$ is technically only a semigroup, the same results hold for $\mathbb{N}$ also.

The following notions of largeness for subsets of $\mathbb{N}$ are very important and appear frequently in the intersection of Ramsey theory and topological dynamics or ergodic theory. For $A \subset \mathbb{N}$ and $m \in \mathbb{N}$ we let $A - m := \{n \in \mathbb{N} : n + m \in A\}$. A set $A \subset \mathbb{N}$ is

- *syndetic* if there exists $k \in \mathbb{N}$ such that $\bigcup_{j=1}^{k} A - j = \mathbb{N}$
- *thick* if for all $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\{m, m+1, \ldots, m+N\} \subset A$
- *piecewise syndetic* if there exists $k \in \mathbb{N}$ such that $\bigcup_{j=1}^{k} A - j$ is thick.

It is well-known and easy to deduce from the definitions that a set is thick if and only if it intersects every syndetic set, or equivalently, if its complement is not syndetic. The roles of thick and syndetic sets in the previous sentence can be interchanged. Moreover, a set is piecewise syndetic if and only if it is the intersection of a syndetic and a thick set. Note also that shifts of syndetic, thick or piecewise syndetic sets are syndetic, thick or piecewise syndetic, respectively. Finally, these families of sets are upward closed; for example, a set that contains a syndetic set is also syndetic, and similarly for the rest.

The same definitions, with obvious modifications, hold in $\mathbb{Z}$. These notions can also be defined analogously for countable abelian groups, but we will only pursue this on demand in Chapter 4.

We conclude this section with a simple combinatorial lemma about properties of the upper and lower density. The proof is elementary and so we omit it. This result will be particularly useful in Section 3.5.2.

Lemma 2.1.1. *Let $\mathcal{P}(\mathbb{N})$ denote the power-set of $\mathbb{N}$.*

(1) The function $\overline{\mathrm{d}}\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ is sub-additive. That is, for any sets $A, B \subset \mathbb{N}$, we have $\overline{\mathrm{d}}(A \cup B) \leq \overline{\mathrm{d}}(A) + \overline{\mathrm{d}}(B)$. In the special case that the sets are disjoint and the density of B exists we have $\overline{\mathrm{d}}(A \cup B) = \overline{\mathrm{d}}(A) + \mathrm{d}(B)$.

(2) The function $\underline{\mathrm{d}}\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ is super-additive. That is, for any disjoint sets $A, B \subset \mathbb{N}$, we have $\underline{\mathrm{d}}(A \cup B) \geq \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$. In the special case that the density of B exists we have $\underline{\mathrm{d}}(A \cup B) = \underline{\mathrm{d}}(A) + \mathrm{d}(B)$.

(3) If $A, B \subset \mathbb{N}$ are such that $B = A + t$, for some $t \in \mathbb{N}$ and $A \cap B = \emptyset$, we have $\overline{\mathrm{d}}(A \cup B) = 2 \cdot \overline{\mathrm{d}}(A)$ and $\underline{\mathrm{d}}(A \cup B) = 2 \cdot \underline{\mathrm{d}}(A)$.

2.2 Topological dynamical systems

Throughout, a topological (dynamical) system is a pair (X, T), where X is a compact metric space and $T\colon X \to X$ is a homeomorphsim.

A system (X, T) is *minimal* if every point x has a dense orbit, i.e., the set $\{T^n x : n \in \mathbb{Z}\}$ is dense in X. It is *transitive* if there exists one dense orbit.

A *factor map* between two systems (X, T) and (Y, S) is a continuous surjection $\pi : X \to Y$ satisfying $\pi \circ T = S \circ \pi$, meaning that $\pi(T(x)) = S(\pi(x))$, for all $x \in X$. In this case, we call (Y, S) a *factor* of (X, T) or the latter an *extension* of the former and we sometimes use the same letter for the transformations T and S, writing (Y, T) instead of (Y, S). If (Y_1, T) and (Y_2, T) are two factors of (X, T) with corresponding factor maps $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ and there is another factor map $\pi : Y_1 \to Y_2$ such that $\pi_2 = \pi \circ \pi_1$, we say that (Y_1, T) *contains* (Y_2, T).

To determine when two topological systems are essentially equivalent, we use the notion of topological isomorphism (or topological conjugacy). A *topological isomorphism* between two topological systems (X, T) and (Y, S) is a homeomorpshim $\phi : X \to Y$ satisfying $\phi \circ T = S \circ \phi$. Note that, ϕ is a factor map from (X, T) to (Y, S), but it also follows that $\phi^{-1} \circ S = T \circ \phi^{-1}$, and in particular, ϕ^{-1} is a factor map from (Y, S) to (X, T) as well. In this case, we say that (Y, S) and (X, T) are *topologically isomorphic*.

Let $(X_j, T_j)_{j\in\mathbb{N}}$ be a sequence of topological dynamical systems endowed with factor maps $\pi_{j,j+1} : X_{j+1} \to X_j$ for each $j \in \mathbb{N}$. An *inverse limit* of this sequence of systems is the unique system (X, T) (up to topological isomorphism) endowed with factor maps $\pi_j : X \to X_j$, $j \in \mathbb{N}$, such that

(i) $\pi_{j+1} = \pi_{j,j+1} \circ \pi_j$, for every $j \in \mathbb{N}$ and

(ii) $\bigcup_{i\in\mathbb{N}}\{f \circ \pi_i : f \in C(X_i)\}$ is dense in $C(X)$.

2.3 Measure-preserving systems

A *measure-preserving system* (X, μ, T) is a triple where (X, T) is a topological dynamical system and μ is a T-invariant Borel probability measure, namely $T\mu = \mu$, where $T\mu$ denotes the pushforward of μ by T. We stress that, throughout this thesis, all our measure-preserving systems have an underlying topological stucture.

The system (X, μ, T) is *ergodic* if $T^{-1}A = A$ implies that $\mu(A) \in \{0, 1\}$ for any Borel set A. The system (X, T) is *uniquely ergodic* if there exists only one T-invariant measure μ, which makes the system (X, μ, T) automatically ergodic.

If (X, μ, T) is measure-preserving, the *support of* μ is defined as the smallest closed, full measure subset of X, or equivalently, as

$$\texttt{supp}(\mu) = \{x \in X : \text{for each open neighbourhood U of } x,\ \mu(U) > 0\}.$$

We recall that a measurable map $\pi\colon X \to Y$ between two systems (X, μ, T) and (Y, ν, S) is called a *factor map* if $\pi\mu = \nu$, where $\pi\mu$ denotes the push forward of μ by π, and

$$\pi \circ T = S \circ \pi \quad \mu\text{-almost everywhere.} \tag{2.3.1}$$

In turn, (Y, ν, S) is called a *factor* of (X, μ, T) and (X, μ, T) is called an *extension* of (Y, ν, S). If, in addition, π is continuous, surjective and (2.3.1) holds everywhere we call π a *continuous factor map*. Again, we use the same letter for the transformation of a factor and its extension, that is, we write (Y, ν, T).

Given a factor map $\pi : X \to Y$ and a function $f \in L^2(\mu)$, we consider the conditional expectation $\mathbb{E}(f|\pi^{-1}\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra on Y. We write $\mathbb{E}(f|\mathcal{B}(Y))$ or $\mathbb{E}(f|Y)$ for the $\mathcal{B}(Y)$-measurable function that satisfies $\mathbb{E}(f|\mathcal{B}(Y)) \circ \pi = \mathbb{E}(f|\pi^{-1}\mathcal{B}(Y))$.

We remark that a factor of an ergodic system is also ergodic. Of special interest are factors with the structure of group rotations. More precisely, a *group rotation* is a system (Z, m, R) where Z is a compact abelian group with its normalized Haar

measure m and $R\colon Z \to Z$ is the rotation map given by $R(z) = z+\alpha$, for some $\alpha \in Z$. Every ergodic system has a maximal group rotation factor, called the Kronecker factor (see, for example, [13, Theorem 6.10]), and while in general the factor map from an ergodic system (X, μ, T) to its Kronecker (Z, ν, R) is only measurable, for our purposes we may assume that it is also a continuous surjection. We explain this in Proposition 3.4.1.

Let $(X_j, \mu_j, T_j)_{j\in\mathbb{N}}$ be a sequence of measure-preserving systems endowed with factor maps $\pi_{j,j+1} : X_{j+1} \to X_j$ for each $j \in \mathbb{N}$. An *inverse limit* of this sequence of systems is the unique system (X, μ, T) (up to isomorphism) endowed with the factor maps $\pi_j : X \to X_j$, $j \in \mathbb{N}$ such that

(i) $\pi_{j+1} = \pi_{j,j+1} \circ \pi_j$, for every $j \in \mathbb{N}$ and

(ii) For $1 \leq p < \infty$, $\bigcup_{i\in\mathbb{N}}\{f \circ \pi_i : f \in L^p(\mu_i)\}$ is dense in $L^p(\mu)$.

Given a system (X, μ, T), a point $a \in X$ is *generic* for μ along a Følner sequence Φ, written as $a \in \mathbf{gen}(\mu, \Phi)$, if

$$\mu = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^n a},$$

where δ_x is the Dirac mass at $x \in X$ and the limit is in the weak* topology. A well-known consequence of the mean ergodic theorem [51] is that for an ergodic system (X, μ, T) and any Følner sequence, there is a subsequence Φ such that μ-almost every point $x \in X$ is generic along Φ (see, e.g. [43, Corollary 2.6]).

Finally, given a measure-preserving system (X, μ, T), a function $f \in L^2(X)$ is called *weak-mixing* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X T^n f \cdot \bar{f}\, d\mu \right| = 0.$$

2.4 Disintegrations over factors

Ergodic systems are fundamental as building blocks of all systems in the sense of the ergodic decomposition theorem (see, for example, [13, Theorem 6.2]).

Theorem 2.4.1 (**Ergodic decomposition**). *Let (X, μ, T) be a measure-preserving system and let $M_T(X)$ denote the set of T-invariant Borel probability measures on X. Then there exist a Borel probability space $(Y, \mathcal{B}(Y), \nu)$ and measures $\mu_y \in M_T(X)$ such that,*

(i) $y \mapsto \mu_y$ *is measurable, that is,* $y \mapsto \int_X f \, d\mu_y$ *is measurable for every* $f \in L^1(\mu)$

(ii) μ_y *is ergodic for* ν*-almost every* $y \in Y$*, and*

(iii) $\mu = \int_Y \mu_y \, d\nu(y)$*, which means that for every* $f \in L^1(\mu)$ *we have*

$$\int_X f(x) \, d\mu(x) = \int_Y \left(\int_X f(x) \, d\mu_y(x) \right) \, d\nu(y).$$

Remark. Indeed, $(Y, \mathcal{B}(Y), \nu)$ can be chosen to be $(X, \mathcal{B}(X), \mu)$. To be precise, the theorem's conclusion means that for any function $f \in L^1(\mu)$, the integral $\int_X f \, d\mu_y$ exists for ν-almost all y, and that on this full-measure set,

$$y \mapsto \int_X f \, d\mu_y$$

is measurable and

$$\int_X f(x) \, d\mu(x) = \int_Y \left(\int_X f(x) \, d\mu_y(x) \right) \, d\nu(y).$$

Another way to describe the ergodic decomposition is via disintegrations over certain factor maps, and we explain this more general approach as well since it will be useful, particularly in Chapter 6.

Let (X, μ, T) be a measure-preserving system and (Y, ν) a probability space. Then, a map $y \mapsto \mu_y$ from $Y \to M(X)$, the space of Borel probability measures on X, with the properties

- $y \mapsto \mu_y(F)$ is measurable for every Borel set $F \subset X$
- $\int_Y f \, d\mu = \int_Y \int_X f \, d\mu_y \, d\nu(y)$, for all measurable and bounded $f : X \to \mathbb{C}$,

is called a disintegration of μ.

For an exposition on the following theorem about disintegrations over factor maps, see for example, [13, Chapter 5].

Theorem 2.4.2. *Let* $\pi : (X, \mu, T) \to (Y, \nu, S)$ *be a factor map between measure-preserving systems. Then, there exists a disintegration* $y \mapsto \mu_y$ *of* μ *defined on* (Y, ν) *satisfying*

$$\mathbb{E}(f|Y) = \int_X f \, d\mu_y,$$

for ν*-a.e.* $y \in Y$*, whenever* $f : X \to \mathbb{C}$ *is measurable and bounded. Moreover, if* $y \mapsto \eta_y$ *is another disintegration with the same property, then* $\eta_y = \mu_y$ *for* ν*-a.e.* $y \in Y$*.*

A fruitful idea in ergodic theory, particularly when structural questions are considered, is that a measurable factor of a measure-preserving system determines uniquely (up to isomorphism), and is conversely, uniquely determined by, an invariant sub-σ-algebra. For a measure-preserving system $(X, \mathcal{B}, \mu, T)$, an invariant sub-σ-algebra is a sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ with $T^{-1}\mathcal{A} = \mathcal{A}$ modulo μ). For the details see [13, Lemma 6.4 and Theorem 6.5].

Using Theorem 2.4.2 one can show that for a measure-preserving system (X, μ, T), an ergodic decomposition of μ is given via the disintegration over the invariant factor, which is the factor identified with the invariant σ-algebra, namely $\mathcal{I}(T) = \{A \in \mathcal{B}(X) : T^{-1}A = A \pmod{\mu}\}$.

2.5 Dynamics of group actions

Throughout this section, let G be a countable abelian group. Given a compact metric space $X = (X, d_X)$, a *continuous action* $T = (T_g)_{g \in G}$ of G on X is a collection of continuous functions $T_g : X \to X$ such that for any $g_1, g_2 \in G$, $T_{g_1} \circ T_{g_2} = T_{g_1+g_2}$. Given such an action, we call the pair (X, T) a *topological G-system.*

Fix a topological G-system (X, T). A measure μ in the space of Borel probability measures on X is said to be T-invariant, if it is invariant under T_g for all $g \in G$. The Borel σ-algebra on X is denoted by $\mathscr{B}(X)$ or just $\mathscr{B}$, if no confusion may arise. The action T on the Borel probability space (X, μ) is called a *measure-preserving G-action* and (X, μ, T) is called a *measure-preserving G-system.* For simplicity, we refer to the above as G-actions, and G-systems, respectively. A G-system (X, μ, T) is called *ergodic* if for any measurable set A the following holds:

$$T_g^{-1}A = A \text{ for all } g \in G \implies \mu(A) = 0 \text{ or } \mu(A) = 1.$$

Given a G-system (X, μ, T) and a Følner sequence Φ, a point $a \in X$ is called *generic with respect to μ along* Φ, denoted by $a \in \mathtt{gen}(\mu, \Phi)$, if

$$\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \delta_{T_g a} = \mu,$$

where δ_x is the Dirac mass at $x \in X$ and the limit is in the weak* topology.

Given two G-systems (X, μ, T) and (Y, ν, S), we say that (Y, ν, S) is a *factor* of (X, μ, T) if there exists a measurable map $\pi : X \to Y$, which we call *factor map*, such that $\mu(\pi^{-1}E) = \nu(E)$ for any measurable $E \subset Y$, and for any $g \in G$, $\pi \circ T_g = S_g \circ \pi$ holds μ-almost everywhere on X. When, additionally, the factor map

π is continuous and $\pi \circ T_g = S_g \circ \pi$ holds everywhere on X for any $g \in G$, we say that π is a *continuous factor map* and (Y, ν, S) is a *continuous factor* of (X, μ, T).

An important class of factors is those with the structure of group rotations. In particular, for any ergodic G-system (X, μ, T) we utilize its *Kronecker factor*, which is the maximal factor of the system that is isomorphic to an abelian group rotation (see [47, Theorem 1]): There exists a compact abelian group Z and a group homomorphism $\theta : G \to Z$ with dense image, such that the Kronecker factor of (X, μ, T) is measurably isomorphic to (Z, m, R), where m is the normalized Haar measure on Z and R is the rotation by θ, i.e. for all $z \in Z$ and $g \in G$,

$$R_g(z) = \theta(g) + z. \tag{2.5.1}$$

In this setting, the acting group G is countable, but not necessarily Z also.

2.6 Nilsystems and uniformity seminorms

A k-step nilsystem $(X, T) = (G/\Gamma, T)$ is a dynamical system given by a k-step nilpotent Lie group G, with Γ being a co-compact discrete subgroup, $T\colon X \to X$ denoting the left translation by a fixed element of G. If μ denotes the Haar measure of X, then $(X, \mu, T) = (G/\Gamma, \mu, T)$ is a measure-preserving system. For the k-step nilsystem (X, μ, T), the properties of transitivity, minimality, ergodicity and unique ergodicity are equivalent. Moreover, if $x \in X$ is any point, the orbit closure $\overline{\{T^n x\colon n \in \mathbb{Z}\}}$ is a minimal and uniquely ergodic k-step subnilsystem. An inverse limit (in either the topological or the measure-theoretic sense) of k-step nilsystems is called a k-step pronilsystem. An *infinite-step pronilsystem* is an inverse limit of a sequence of finite-step pronilsystems of increasing and unbounded step.

All the above properties are preserved under inverse limits of k-step nilsystems, so they also hold for k-step pronilsystems (e.g. the equivalence between minimality, ergodicity and unique ergodicity). For these classical facts and other details we refer the reader to [31, Chapter 11 and Chapter 13].

Let (X, μ, T) be an ergodic measure-preserving system. Then, for every $k \in \mathbb{N}$, (X, μ, T) has a maximal factor isomorphic to a k-step pronilsystem, called the *k-step pronilfactor*. We shall denote this factor by (Z_k, m_k, T). For $k = 1$ this factor corresponds to the Kronecker factor. The inverse limit of $((Z_k, m_k, T))_{k\in\mathbb{N}}$ is the *infinite-step pronilfactor of X* and we denote it by (Z_∞, m, T). The infinite-step pronilfactor of X coincides with the maximal factor of X isomorphic to an infinite-step pronilsystem.

Infinite-step pronilsystems share many properties with finite-step pronilsystems;

some of those, which are exploited in this thesis, are listed in the end of this subsection.

In their seminal paper [32], Host and Kra showed that it is possible to describe an ergodic system's k-step pronilfactor by studying uniformity seminorms, classically referred to as Gowers-Host-Kra seminorms. For an alternative definition of these factors, see [62], and for the equivalence between them, see [46].

Definition 2.6.1. Let (X, μ, T) be an ergodic system and $f \in L^\infty(\mu)$. The k-uniformity seminorms of f, $\|f\|_{U^k(X,\mu,T)}$, are defined inductively as follows:

$$\|f\|_{U^0(X,\mu,T)} = \int f \, \mathrm{d}\mu$$

and

$$\|f\|^{2^{k+1}}_{U^{k+1}(X,\mu,T)} = \lim_{H\to\infty} \frac{1}{H} \sum_{h\leq H} \left\| f \cdot \overline{T^h f} \right\|^{2^k}_{U^k(X,\mu,T)}. \tag{2.6.1}$$

Remark. The function $\|\cdot\|_{U^k(X,\mu,T)}$ defines a seminorm when $k \geq 1$. The fact that the limit in (2.6.1) exists is proved in [32] by iterative applications of the mean ergodic theorem. In the same paper, it is also shown that $\|\cdot\|_{U^{k+1}(X,\mu,T)}$ is actually a norm for k-step pronilsystems.

The main theorem in [32] shows that the k-step pronilfactor, Z_k, of an ergodic system (X, μ, T) is characterized by the property

$$\|f\|_{U^{k+1}(X,\mu,T)} = 0 \iff \mathbb{E}(f \mid Z_k) = 0,$$

for any $f \in L^\infty(\mu)$. Similarly, for Z_∞ we have the following characterization.

Proposition 2.6.2. Let (X, μ, T) be an ergodic system and (Z_∞, m, T) denote its infinite-step pronilfactor. For any $f \in L^\infty(\mu)$ we have that

$$\|f - \mathbb{E}(f \mid Z_\infty)\|_{U^k(X,\mu,T)} = 0 \quad \text{for all } k \in \mathbb{N}.$$

Conversely, if $h \in L^\infty(\mu)$ satisfies $\|h\|_{U^k(X,\mu,T)} = 0$ for all $k \in \mathbb{N}$, then $\mathbb{E}(h|Z_\infty) = 0$.

Proof. By the definition of Z_∞, we have $\mathbb{E}(f \mid Z_\infty) = \lim_{s\to\infty} \mathbb{E}(f \mid Z_s)$ in $L^2(\mu)$. For $k \in \mathbb{N}$ fixed,

$$\|f - \mathbb{E}(f \mid Z_s)\|_{U^k(X,\mu,T)} \leq \|f - \mathbb{E}(f \mid Z_s)\|_{U^{s+1}(X,\mu,T)} = 0 \quad \text{for all } s \geq k.$$

Thus $\|f - \mathbb{E}(f \mid Z_\infty)\|_{U^k(X,\mu,T)} = \lim_{s\to\infty} \|f - \mathbb{E}(f \mid Z_s)\|_{U^k(X,\mu,T)} = 0$.

For the converse, assume $\mathbb{E}(h|Z_\infty) \neq 0$ in $L^2(\mu)$. Then, since $\mathbb{E}(h|Z_\infty) =$

$\lim_{s\to\infty}\mathbb{E}(h|Z_s)$ in $L^2(\mu)$, there must exist $k\in\mathbb{N}$ so that $\mathbb{E}(h|Z_k)\neq 0$ in $L^2(\mu)$. However this contradicts the assumption that $\|h\|_{U^{k+1}(X,\mu,T)}=0$. □

In [33] and [57], the theory of regionally proximal relations for pairs of points in a topological system (X,T) was developed, ultimately leading to a topological analogue of the pronilfactors (Z_k,m_k,T). Specifically, it was shown that for every $k\in\mathbb{N}$, every minimal system (X,T) admits a (unique) factor that is a k-step pronilsystem and which contains all k-step nilfactors of (X,T) as factors. We denote this factor by (X_k,T) and it is called the maximal k-step pronilfactor of (X,T). The (topological) inverse limit of the sequence $((X_k,T))_{k\in\mathbb{N}}$, denoted by (X_∞,T), is the maximal infinite-step pronilfactor of (X,T).

2.7 Notation

By $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote, respectively, the integers, nonnegative integers, positive integers (natural numbers), rationals, reals and the complex numbers. In addition, we use the symbol $\sqcup$ to denote unions of pairwise disjoint sets. The one-dimensional torus is denoted by $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

The groups under consideration will always be abelian and group operations are written additively. If G is an abelian group and $A,B\subset G$, then

$$A+B:=\{a+b:a\in A,\ b\in B\}.$$

For $g\in G$, we write

$$A-g:=\{h\in G:h+g\in A\}.$$

If $D:G\to G$ is the doubling map $D(g)=2g$, then we write

$$2A:=D(A)=\{2a:a\in A\}\ \text{ and }\ A/2:=D^{-1}(A)=\{g\in G:2g\in A\}.$$

Given a group G, we denote by e_G or 0 the identity element of the group. Finally, we write $o_{k\to\infty}(1)$ to denote an error term that goes to 0 as k grows to infinity.

For a finite set F, $|F|$ denotes its cardinality. If X is a topological space and μ is a Borel probability measure on X, then $\texttt{supp}(\mu)$ denotes the support of μ.

Given a measure space $(X,\mathcal{B},\mu)$, a measurable space $(Y,\mathcal{C})$ and a measurable function $F:X\to Y$, we denote by $F\mu$ the push-forward of μ under F. That is, $F\mu(C)=\mu(F^{-1}C)$, for any $C\in\mathcal{C}$. Moreover, given a measurable transformation $T:X\to X$ and some integer $n\in\mathbb{N}$, we let T^n denote the composition $T\circ T\circ\cdots\circ T$ taken n-times and we sometimes write T^nf instead of $f\circ T^n$, where $f:X\to\mathbb{C}$ is a

measurable map. For $k \in \mathbb{N}$ we write $\tilde{T}$ for the map $T \times T^2 \times \cdots \times T^k$, and T_Δ for the map $T^{\otimes k} = T \times T \times \cdots \times T$, where the product is taken k-times, omitting the dependence on k which is clear from the context.

If $S = \emptyset$ we let $T^{-S}E = X$. Also, if $S \subset \mathbb{N}$ is nonempty and $E \subset X$, we denote

$$T^{-S}E = \bigcap_{n \in S} T^{-n}E.$$

In addition, for a finite set $F \subset \mathbb{N}$ we define $F^{\oplus 0} = \{0\}$, and for $i \in \mathbb{N}$

$$F^{\oplus i} = \begin{cases} \{f_1 + \cdots + f_i : f_1, \ldots, f_i \in F \text{ distinct}\} & \text{if } i \leq |F| \\ \emptyset & \text{otherwise} \end{cases}.$$

Notice that, under this convention, $\emptyset^{\oplus 0} = \{0\}$. The motivation for introducing such notation is that if $A \subset \mathbb{N}$, $a \in X$ and $E \subset X$ are as in Proposition 7.4.1 in Chapter 7, then for any nonempty set $S \subset \mathbb{N}$, $a \in T^{-S}E$ if and only if $S \subset A$.

Chapter 3

Unrestricted $B + B$ sumsets in the integers

3.1 Overview

This chapter is based on the results of [40]. The goal is to understand what density conditions are sufficient for a set $A \subset \mathbb{N}$ to contain unrestricted infinite sumsets of the form

$$B + B + t = \{b_1 + b_2 + t : b_1, b_2 \in B\}$$

for some infinite $B \subset \mathbb{N}$, and a shift $t \in \mathbb{N}_0$.

The starting point, and indeed an ingredient in our proofs, is Theorem C of Kra, Moreira, Richter and Robertson, which states that any set with positive upper Banach density contains restricted infinite sumsets

$$B \oplus B + t = \{b_1 + b_2 : b_1, b_2 + t \in B,\ b_1 \neq b_2\}.$$

Removing the restriction $b_1 \neq b_2$ makes the analogous statement false, and this raises the natural question of whether stronger density assumptions can force the diagonal into the set as well.

The results of this chapter give a complete answer in terms of the upper and lower natural asymptotic density in the integers. For a set $A \subset \mathbb{N}$ we characterize the existence of an infinite set $B \subset \mathbb{N}$ and $t \in \{0, 1\}$ such that $B + B + t \subset A$, where $B + B = \{b_1 + b_2 \colon b_1, b_2 \in B\}$, in terms of the density of the set A. Specifically, when the lower density $\underline{d}(A) > 1/2$ or the upper density $\overline{d}(A) > 2/3$, the existence of such a set $B \subset \mathbb{N}$ and $t \in \{0, 1\}$ is assured.

Furthermore, whenever $\underline{d}(A) > 3/4$ or $\overline{d}(A) > 5/6$, we show that the shift t is

unnecessary and we also provide examples to show that these bounds are sharp. Finally, we construct a syndetic three-colouring of the natural numbers that does not contain a monochromatic $B + B + t$ for any infinite set $B \subset \mathbb{N}$ and number $t \in \mathbb{N}$.

3.2 Introduction

We recall Theorem C due to Kra, Moreira, Richter and Robertson from [41], mentioned in the introduction, which resolved a longstanding conjecture of Erdős (see for example [14, 15, 16]).

Theorem C. *For any $A \subset \mathbb{N}$ with positive upper (Banach) density there exists some infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$ such that*

$$B \oplus B = \{b_1 + b_2 \colon b_1, b_2 \in B,\ b_1 \neq b_2\} \subset A - t.$$

As we explained in the introcution, the restriction $b_1 \neq b_2$ in the sumset is a necessary one. Indeed, one can easily construct a set $A \subset \mathbb{N}$ of full upper Banach density and positive lower density that does not contain a set of the form $B+B+t$, for any infinite set $B \subset \mathbb{N}$ and any $t \in \mathbb{N}$ (see [44, Example 2.3]).

Relevant to this, it turns out that the structure $B + B$ for infinite $B \subset \mathbb{N}$ is not partition regular. This was first shown by Hindman in [27], who produced a three-colouring of $\mathbb{N}$ that does not contain monochromatic $B + B$ for any infinite set $B \subset \mathbb{N}$. The analogous problem for a two-colouring of $\mathbb{N}$ (see [36, Problem E2494]) has only been resolved very recently in [34], but was still open when [40] was published.

The focus of this chapter is to provide conditions on the density of a set $A \subset \mathbb{N}$, in order to ensure the existence of an infinite set $B \subset \mathbb{N}$ such that $B + B + t \subset A$, for some $t \in \mathbb{N}$, where we emphasize that repeated entries are permitted. Naturally, these conditions are much stricter than those of Theorem C.

The known examples of integer sets not containing infinite sumsets of the form $B + B$ were sets of upper density less than $2/3$ and lower density less than $1/2$, but arbitrarily close to these values (see [44, Example 3.6]). This led Kra, Moreira, Richter and Robertson to conjecture that these bounds were sharp [44, Conjecture 3.7]. In Theorem 1.3.2 we show that this holds once we allow a potential shift $t \geq 0$, which can actually be restricted to $t \in \{0, 1\}$. We are able to remove this shift, as shown in Theorem 1.3.1, once we increase the lower bounds on the densities and in Section 3.5 we show that this increase is necessary and our bounds are optimal.

Our first main result is the following theorem.

Theorem 1.3.1. *Let $A \subset \mathbb{N}$.*

1. *If $\overline{\mathrm{d}}(A) > 5/6$, there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$.*

2. *If $\underline{\mathrm{d}}(A) > 3/4$, there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$.*

When we allow for a shift $t \geq 0$, we can weaken the assumptions on the density.

Theorem 1.3.2. *Let $A \subset \mathbb{N}$.*

(1) If $\overline{\mathrm{d}}(A) > 2/3$, there is an infinite set $B \subset \mathbb{N}$ and $t \in \{0, 1\}$ such that $B+B+t \subset A$.

(2) If $\underline{\mathrm{d}}(A) > 1/2$, there is an infinite set $B \subset \mathbb{N}$ and $t \in \{0, 1\}$ such that $B+B+t \subset A$.

Our approach in order to prove these results is ergodic-theoretic and is based on the tools developed in [41]. We first translate the problem of finding the desired infinite configurations to a statement in ergodic theory. In our case, we need to find a particular system (X, T) and specific points $a, x_1, x_2 \in X$ such that for some increasing sequence $(n_k)_{k\in\mathbb{N}}$ we have $(T \times T)^{n_k}(a, x_1) \to (x_1, x_2)$, in order to ensure that $B \oplus B \subset A$, and $T^{2n_k}a \to x_2$, in order to ensure that $2B = \{2b : b \in B\} \subset A$. This leads to consider an appropriated $(T^2 \times T)$-invariant probability measure in $(\{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}, T^2 \times T)$ and devise a new variance of Furstenberg's correspondence principle, according to Lemmas 3.3.3 and 3.3.7.

The phenomena captured by Theorems 1.3.1 and 1.3.2 are directly related to the distribution of a set A along the even numbers or its distribution along the odd numbers. More precisely, we show the following corollary of Theorem 1.3.1.

Corollary 3.2.1. *Let $A \subset \mathbb{N}$. If one of the following holds:*

(1) $\overline{\mathrm{d}}(A \cap 2\mathbb{N}) > 1/3$ or

(2) $\underline{\mathrm{d}}(A \cap 2\mathbb{N}) > 1/4$,

then there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$. Similarly, if one of the following holds:

(3) $\overline{\mathrm{d}}(A \cap (2\mathbb{N} + 1)) > 1/3$ or

(4) $\underline{\mathrm{d}}(A \cap (2\mathbb{N} + 1)) > 1/4$,

then there is an infinite set $B \subset \mathbb{N}$ such that $B + B + 1 \subset A$.

We prove Corollary 3.2.1 in Section 3.5.2. In Section 3.5.1 we establish that the density bounds listed in Theorems 1.3.1 and 1.3.2 are sharp. Specifically, we construct a set $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = 5/6$ which does not contain a sumset $B + B$, for any infinite $B \subset \mathbb{N}$. Moreover, we construct a set $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = 2/3$, which does not contain any shifted infinite sumset $B + B + t$, for any $t \in \mathbb{N}$. To complete the picture, we give analogous constructions of sets $A, A' \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) = 3/4$ and $\underline{\mathrm{d}}(A') = 1/2$, that do not contain infinite sumsets of the form $B + B$ and $B + B + t$, respectively. We note that for the latter two sets, A and A', their lower densities are in fact natural densities.

The aforementioned examples include *thick* sets, i.e. sets which contain arbitrarily large discrete intervals. This lead us to the natural question whether for any *syndetic* set $A \subset \mathbb{N}$ there exists an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that $B + B + t \subset A$. We recall that a set $A \subset \mathbb{N}$ is called syndetic if its complement is not thick or, equivalently, if there is a number $d \in \mathbb{N}$ such that $\mathbb{N} \subset A \cup (A - 1) \cup \cdots \cup (A - d)$, where d is called the gap of A. We answer this question negatively and, consequently, provide an alternative proof of Hindman's result [27] on three-colourings. We note that, since all colours are syndetic, this is a sharpening of Hindman's construction.

Proposition 3.2.2. *There exists a colouring $\mathbb{N} = C_1 \cup C_2 \cup C_3$ where the sets C_1, C_2, C_3 are all syndetic and there is no monochromatic sumset $B + B$ for any infinite set $B \subset \mathbb{N}$.*

The colouring of Proposition 3.2.2 is constructed in Section 3.6.

3.3 Ergodic-theoretic terminology and dynamical statements

The proofs of Theorems 1.3.1 and 1.3.2 are ergodic-theoretic in nature and depend on manipulations of the new dynamical methods developed in [43] and extended in [41] in order to find infinite patterns in sets of positive density. Indeed, by handling this new technology and by extrapolating crucial combinatorial information given by the densities of the sets we work with, we were able to minimize the technical details of both proofs.

Our first main dynamical result is the following theorem. In this section we will show that it implies Theorem 1.3.1.

Theorem 3.3.1. *Let (X, μ, T) be an ergodic system, let $a \in \boldsymbol{gen}(\mu, \Phi)$ for some Følner sequence Φ and $E_1, E_2 \subset X$ be open sets satisfying*

$$2\mu(E_2) + \mu(E_1) > 2. \tag{3.3.1}$$

Then, there exist $x_1, x_2 \in X$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers, such that $(T \times T)^{n_k}(a, x_1) \to (x_1, x_2)$ as $k \to \infty$ and $(x_1, x_2) \in E_1 \times E_2$.

Recalling terminology from [43], given a topological system (X, T), a point $(x_0, x_1, x_2) \in X^3$ will be called a (three-term) *Erdős progression* if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, such that $(T \times T)^{n_k}(x_0, x_1) \to (x_1, x_2)$. We are going to use this concept for the systems arising from Lemma 3.3.3 below. An obvious, yet useful, observation is that Theorem 3.3.1 can be restated using this terminology. Indeed, Theorem 3.3.1 states that when (3.3.1) holds, there always exists an Erdős progression of the form $(a, x_1, x_2) \in \{a\} \times E_1 \times E_2$.

The next result, proven in [43], facilitates the translation of the existence of Erdős progressions in topological systems to a combinatorial statement about infinite sumsets in return-times sets related to those systems.

Lemma 3.3.2 ([43, Theorem 2.2])**.** *Fix a topological system (X, T) and let $U, V \subset X$ be open. If there exists an Erdős progression $(x_0, x_1, x_2) \in X^3$ with $x_1 \in U$ and $x_2 \in V$, then there exists some infinite set $B \subset \{n \in \mathbb{N} \colon T^n x_0 \in U\}$ such that $\{b_1 + b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset \{n \in \mathbb{N} \colon T^n x_0 \in V\}$.*

In order to prove that Theorem 3.3.1 implies Theorem 1.3.1 we will make use of a new variant of Furstenberg's correspondence principle (cf. Lemma B) tailored to our purposes. For the rest of this section Σ denotes the space $\{0, 1\}^{\mathbb{Z}}$ and is endowed with the product topology, so that it becomes a compact metrizable space. We also consider the shift transformation $S \colon \Sigma \to \Sigma$ given by $S(x(n)) = x(n + 1)$, for any $n \in \mathbb{Z}$, $x = (x(n))_{n\in\mathbb{Z}} \in \Sigma$.

Lemma 3.3.3. *Let $A \subset \mathbb{N}$.*

(1) There exist an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ, such that $(a, a) \in \boldsymbol{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq 3\left(2 \cdot \overline{\mathrm{d}}(A) - 1\right).$$

(2) There exist an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ, such that $(a, a) \in \boldsymbol{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq 2 \cdot \underline{\mathrm{d}}(A) + 2 \cdot \overline{\mathrm{d}}(A) - 1.$$

In both cases we have that $A = \{n \in \mathbb{N} : S^n a \in E\}$.

Proof. Let $A \subset \mathbb{N}$. By definition there is a sequence of positive integers (N_k) with

$$\overline{\mathrm{d}}(A) = \lim_{k\to\infty} \frac{|A \cap [1, N_k]|}{N_k}.$$

Associate to the set A a point $a \in \Sigma = \{0,1\}^{\mathbb{Z}}$ via

$$a(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

Define the clopen set $E = \{x \in \Sigma \colon x(0) = 1\}$ and observe that, by construction, $A = \{n \in \mathbb{N} \colon S^n a \in E\}$. Now let $N'_k = \lfloor N_k/2 \rfloor$, for all $k \in \mathbb{N}$ and consider the sequence of Borel probabilities (μ_k) on $\Sigma \times \Sigma$ given by

$$\mu_k = \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{(S^2 \times S)^n (a,a)}.$$

We let μ' be a weak* accumulation point of (μ_k), and then it is well-known that μ' is an $(S^2 \times S)$-invariant measure (see, e.g. [9]). It follows by definition that for all $k \in \mathbb{N}$ we have

$$\mu_k(\Sigma \times E) = \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{S^n a}(E) = \frac{|A \cap [1, N'_k]|}{N'_k}. \tag{3.3.2}$$

For any $k \in \mathbb{N}$ we also have

$$\begin{aligned} \mu_k(\Sigma \times E) = \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{S^n a}(E) &= \frac{1}{N'_k} \left(\sum_{n=1}^{N_k} \delta_{S^n a}(E) - \sum_{n=N'_k+1}^{N_k} \delta_{S^n a}(E) \right) \\ &\geq \frac{2}{N_k} (\sum_{n=1}^{N_k} \delta_{S^n a}(E) - \frac{1}{2} N_k) = 2 \frac{|A \cap [1, N_k]|}{N_k} - 1 \end{aligned} \tag{3.3.3}$$

and similarly

$$\begin{aligned} \mu_k(E \times \Sigma) = \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{S^{2n} a}(E) &\geq \frac{1}{N'_k} \left(\sum_{n=2}^{N_k - 1} \delta_{S^n a}(E) - \sum_{n=1}^{N'_k} \delta_{S^{2n+1} a}(E) \right) \\ &\geq \frac{2}{N_k} \left(\sum_{n=1}^{N_k} \delta_{S^n a}(E) - 2 - \frac{N_k}{2} \right) = 2 \frac{|A \cap [1, N_k]|}{N_k} - \frac{4}{N_k} - 1. \end{aligned} \tag{3.3.4}$$

Taking limits as $k \to \infty$ in (3.3.2), (3.3.3) and (3.3.4) we have, by the definition of

μ', the fact that $E \subset X$ is clopen and the choice of (N_k) that

$$\mu'(\Sigma \times E) \geq \underline{\mathrm{d}}(A), \tag{3.3.5}$$

$$\mu'(\Sigma \times E) \geq 2 \cdot \overline{\mathrm{d}}(A) - 1, \tag{3.3.6}$$

$$\mu'(E \times \Sigma) \geq 2 \cdot \overline{\mathrm{d}}(A) - 1, \tag{3.3.7}$$

respectively. Combining (3.3.5) and (3.3.7) we have that

$$2\mu'(\Sigma \times E) + \mu'(E \times \Sigma) \geq 2 \cdot \underline{\mathrm{d}}(A) + 2 \cdot \overline{\mathrm{d}}(A) - 1.$$

Although μ' is not necessarily ergodic, we can use its ergodic decomposition to find an $(S^2 \times S)$-ergodic component of it, call it μ, such that

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq 2 \cdot \underline{\mathrm{d}}(A) + 2 \cdot \overline{\mathrm{d}}(A) - 1.$$

Without loss of generality we may assume that μ is supported on the orbit closure of (a, a), since this holds for μ' by construction. Then by a standard argument (see [19, Proposition 3.9]) we see there is a Følner sequence Φ in $\mathbb{N}$, such that $(a, a) \in \mathbf{gen}(\mu, \Phi)$. This completes the proof of (2).

In a similar fashion, combining (3.3.6) and (3.3.7) and choosing the ergodic component μ accordingly, we recover (1). □

We now use the above construction, the information provided by the density of our sets and Lemma 3.3.2 to deduce Theorem 1.3.1 from Theorem 3.3.1.

Proof that Theorem 3.3.1 implies Theorem 1.3.1. (1) : Let $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 5/6$. By part (1) of Lemma 3.3.3 there exists an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ with $A = \{n \in \mathbb{N} : S^n a \in E\}$ and a Følner sequence Φ, such that $(a, a) \in \mathbf{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq 3\left(2 \cdot \overline{\mathrm{d}}(A) - 1\right) > 2.$$

We now apply Theorem 3.3.1 for the ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, the generic point (a, a) and the open sets $E_1 = E \times \Sigma$, $E_2 = \Sigma \times E$ to recover an Erdős progression $((a, a), (x_{10}, x_{11}), (x_{20}, x_{21})) \in \Sigma^6$, with

$$(x_{10}, x_{11}, x_{20}, x_{21}) \in E_1 \times E_2 = E \times \Sigma \times \Sigma \times E.$$

Setting $U = E_1$, $V = E_2$ and applying Lemma 3.3.2, we obtain an infinite set $B \subset \mathbb{N}$

such that

$$B \subset \{n \in \mathbb{N} \colon (S^2 \times S)^n (a,a) \in E \times \Sigma\} = \{n \in \mathbb{N} \colon S^{2n} a \in E\}$$

and

$$B \oplus B \subset \{n \in \mathbb{N} \colon (S^2 \times S)^n (a,a) \in \Sigma \times E\} = \{n \in \mathbb{N} \colon S^n a \in E\} = A.$$

From the former it follows that $2B = \{b + b : b \in B\} \subset A$, because

$$S^{2n} a \in E \iff a(2n) = 1 \iff 2n \in A.$$

This fact together with $B \oplus B \subset A$ imply that $B + B \subset A$, by the simple observation that $B + B = 2B \cup (B \oplus B)$.

(2) : Let $A \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) > 3/4$. By part (2) of Lemma 3.3.3 there is an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ with $A = \{n \in \mathbb{N} : S^n a \in E\}$ and a Følner sequence Φ, such that $(a,a) \in \mathbf{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq 2 \cdot \underline{\mathrm{d}}(A) + 2 \cdot \overline{\mathrm{d}}(A) - 1 \geq 4 \cdot \underline{\mathrm{d}}(A) - 1 > 2.$$

Repeating the argument given at the end of the previous proof, in view of Theorem 3.3.1 and then Lemma 3.3.2 we find an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$. $\square$

Remark 3.3.4. Notice that in our proof that Theorem 3.3.1 implies (2) of Theorem 1.3.1 we did not really need the condition

$$4 \cdot \underline{\mathrm{d}}(A) - 1 > 2,$$

but only the weaker condition that

$$2 \cdot \underline{\mathrm{d}}(A) + 2 \cdot \overline{\mathrm{d}}(A) - 1 > 2,$$

which can be rewritten as

$$\underline{\mathrm{d}}(A) + \overline{\mathrm{d}}(A) > 3/2.$$

In light of the simple observation made in Remark 3.3.4, we see how Theorem 3.3.1 also implies the following result, which directly implies part (2) of Theorem 1.3.1, and may be of independent interest.

Theorem 3.3.5. *Let $A \subset \mathbb{N}$. If $\overline{\mathrm{d}}(A) + \underline{\mathrm{d}}(A) > 3/2$, then there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$.*

We now state our second main dynamical result, which implies Theorem 1.3.2.

Theorem 3.3.6. *Let (X, μ, T) be an ergodic system, let $a \in \boldsymbol{gen}(\mu, \Phi)$, for some Følner sequence Φ and $E_1, E_2, F_1, F_2 \subset X$ be open sets with $F_2 = T^{-1}E_2$ and*

$$2\mu(E_2) + \mu(E_1) + \mu(F_1) > 2. \tag{3.3.8}$$

Then, there exist $x_1, x_2 \in X$ such that $(a, x_1, x_2) \in X^3$ is an Erdős progression with either $(x_1, x_2) \in E_1 \times E_2$ or $(x_1, x_2) \in F_1 \times F_2$.

Note that, by the μ-invariance of T and the definition of F_2, (3.3.8) can be rewritten as

$$\mu(E_1) + \mu(E_2) + \mu(F_1) + \mu(F_2) > 2.$$

The next instance of a correspondence principle is the analogue of Lemma 3.3.3 which allows for the transition from the dynamics in Theorem 3.3.6 to the combinatorics in Theorem 1.3.2.

Lemma 3.3.7. *Let $A \subset \mathbb{N}$.*

(1) There exists an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ, such that $(a, a) \in \boldsymbol{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) + \mu(S^{-1}E \times \Sigma) \geq 2\left(3 \cdot \overline{\mathrm{d}}(A) - 1\right).$$

(2) There exists an ergodic system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ, such that $(a, a) \in \boldsymbol{gen}(\mu, \Phi)$ and

$$2\mu(\Sigma \times E) + \mu(E \times \Sigma) + \mu(S^{-1}E \times \Sigma) \geq 2\left(\underline{\mathrm{d}}(A) + \overline{\mathrm{d}}(A)\right).$$

In both cases we have that $A = \{n \in \mathbb{N} : S^n a \in E\}$.

The proof of the above is almost identical to that of Lemma 3.3.3, so we omit it. We merely point out that in the context and notation of Lemma 3.3.3, we have

that

$$\mu_k(E \times \Sigma) + \mu_k(S^{-1}E \times \Sigma) = \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{S^{2n}a}(E) + \frac{1}{N'_k} \sum_{n=1}^{N'_k} \delta_{S^{2n+1}a}(E)$$
$$\geq \frac{1}{N'_k} \sum_{n=2}^{N_k} \delta_{S^n a}(E) \geq 2\frac{|A \cap [1, N_k]|}{N_k} - \frac{2}{N_k},$$

for all $k \in \mathbb{N}$. Taking limits as $k \to \infty$ this yields

$$\mu'(E \times \Sigma) + \mu'(S^{-1}E \times \Sigma) \geq 2 \cdot \overline{\mathrm{d}}(A),$$

which is now used instead of (3.3.7).

We also remark that in order to prove Theorem 1.3.2 from Theorem 3.3.6 one would use an argument similar to that of recovering Theorem 1.3.1 from Theorem 3.3.1. This time, we need Lemma 3.3.7 instead of Lemma 3.3.3 and the sets $E_1 = E \times \Sigma$, $E_2 = \Sigma \times E$, $F_1 = S^{-1}E \times \Sigma$ and $F_2 = \Sigma \times S^{-1}E$. We omit the repetitive details.

Moreover, by an observation analogous to Remark 3.3.4, we have that Theorem 3.3.6 implies the following result, which in turn implies part (2) of Theorem 1.3.2.

Theorem 3.3.8. *Let $A \subset \mathbb{N}$. If $\overline{\mathrm{d}}(A) + \underline{\mathrm{d}}(A) > 1$, then there is an infinite set $B \subset \mathbb{N}$ and a number $t \in \{0, 1\}$ such that $B + B + t \subset A$.*

In Section 3.7 we give an application of Theorem 3.3.8 to the question of two-colourings [36, Problem E2494] mentioned in the introduction. In Chapter 5 we prove a more general result, namely Theorem 1.3.3, of which Theorem 3.3.8 is a special case.

3.4 Proofs of the dynamical statements

3.4.1 Tools from [43]

As was mentioned in the introduction, the proofs of Theorems 3.3.1 and 3.3.6 follow a scheme similar to that of [43]. We therefore need some of the tools introduced there. In particular, we use some properties of the auxiliary measures defined in (3.4.1) and (3.4.2) below. For our purposes, we only need the relevant results listed in Proposition 3.4.2 and Proposition 3.4.3, but we also include the definitions of these measures for completeness.

Every ergodic system has a maximal group rotation factor, called the *Kronecker factor* and, even though the factor map from an ergodic system (X, μ, T) to its Kronecker factor (Z, m, R) is a priori only measurable, for our purposes we may assume that it is also a continuous surjection. Indeed, the following result shows that any ergodic system has a continuous extension, which has a continuous Kronecker factor map, and this allows us to reduce matters to this case. The proof of this reduction is essentially carried out in the proofs of Theorems 3.3.1 and 3.3.6 from Theorems 3.4.4 and 3.4.5, respectively.

Proposition 3.4.1 ([41, Proposition 3.20])**.** *Let (X, μ, T) be an ergodic system and let $a \in \boldsymbol{gen}(\mu, \Phi)$ for some Følner sequence Φ. Then there is an ergodic system $(\tilde{X}, \tilde{\mu}, \tilde{T})$, a Følner sequence Ψ, a point $\tilde{a} \in \tilde{X}$ and a continuous factor $\tilde{\pi}\colon \tilde{X} \to X$, such that $\tilde{\pi}(\tilde{a}) = a$, $\tilde{a} \in \boldsymbol{gen}(\tilde{\mu}, \Psi)$ and $(\tilde{X}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor.*

For the rest of this subsection, (X, μ, T) denotes an ergodic system with continuous factor map $\pi : X \to Z$ to its Kronecker factor (Z, m, R). Let $z \mapsto \eta_z$ denote the disintegration of μ over the factor map π (see Theorem 2.4.2). Then, following [41], for every $(x_1, x_2) \in X \times X$, we define the measures

$$\lambda_{(x_1,x_2)} = \int_Z \eta_{z+\pi(x_1)} \times \eta_{z+\pi(x_2)}\, dm(z) \tag{3.4.1}$$

on $X \times X$. As it was shown in [41, Proposition 3.11], $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a *continuous ergodic decomposition* of $\mu \times \mu$, that is, a disintegration of $\mu \times \mu$ where the measures $\lambda_{(x_1,x_2)}$ are ergodic for $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$ and the map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is continuous in the weak* topology. To find Erdős progressions with specified first coordinate, a predetermined point $a \in X$, the authors in [43] also introduced a measure σ_a on $X \times X$ given by

$$\sigma_a = \int_Z \eta_z \times \eta_{2z-\pi(a)}\, dm(z) = \int_Z \eta_{\pi(a)+z} \times \eta_{\pi(a)+2z}\, dm(z). \tag{3.4.2}$$

Let us denote by $\pi_1\colon X \times X$ the projection $(x_1, x_2) \mapsto x_1$ onto the first coordinate, by $\pi_2\colon X \times X \to X$ the projection onto the second coordinate and by $\pi_i\sigma_a$ the pushforward of σ_a under π_i, $i = 1, 2$. In addition to the standing assumptions of this subsection, we let $a \in \mathbf{gen}(\mu, \Phi)$ for some Følner sequence Φ. We collect various results from [43, Section 3] in the following proposition.

Proposition 3.4.2. *Let $\lambda_{(x_1,x_2)}$, for $(x_1, x_2) \in X \times X$, and σ_a be the measures on $X \times X$ defined in* (3.4.1) *and* (3.4.2)*, respectively. Then*

(1) $\pi_1\sigma_a = \mu$ *and* $\pi_2\sigma_a + T\pi_2\sigma_a = 2\mu$.

(2) For σ_a*-almost every* $(x_1, x_2) \in X \times X$ *we have that* $(x_1, x_2) \in \boldsymbol{supp}(\lambda_{(a,x_1)})$.

(3) There exists a Følner sequence Ψ*, such that for* μ*-almost every* $x_1 \in X$ *the point* (a, x_1) *belongs to* $\boldsymbol{gen}(\lambda_{(a,x_1)}, \Psi)$.

Using these facts we can guarantee many Erdős progressions.

Proposition 3.4.3. *Let* (X, μ, T) *be an ergodic system and assume there is a continuous factor map* $\pi\colon X \to Z$ *to its Kronecker factor. Let* $a \in \boldsymbol{gen}(\mu, \Phi)$*, for some Følner sequence* Φ*. Then for* σ_a*-almost every* $(x_1, x_2) \in X \times X$*, the point* (a, x_1, x_2) *is an Erdős progression.*

Proof. This proposition is implicit in [43] and the proof is essentially contained in the first paragraph of the proof of Theorem 3.2 there. Let Ψ be the Følner sequence and $G \subset X$ be the full μ-measure set of $x_1 \in X$ such that (a, x_1) belongs to $\mathbf{gen}(\lambda_{(a,x_1)}, \Psi)$, arising from part (3) of Proposition 3.4.2. By Proposition 3.4.2, part (1), we have that $\sigma_a(G \times X) = \mu(G) = 1$ and so, for σ_a-almost every $(x_1, x_2) \in X \times X$ we have that $(a, x_1) \in \mathbf{gen}(\lambda_{(a,x_1)}, \Psi)$. In view of Proposition 3.4.2, part (2), it follows that for σ_a-almost every $(x_1, x_2) \in X \times X$,

(i) $(a, x_1) \in \mathbf{gen}(\lambda_{(a,x_1)}, \Psi)$ and

(ii) $(x_1, x_2) \in \mathbf{supp}(\lambda_{(a,x_1)})$.

Thus, applying [41, Lemma 2.4], we have that for σ_a-almost every $(x_1, x_2) \in X \times X$ the point (a, x_1, x_2) is indeed an Erdős progression. □

3.4.2 Proofs of the main results

We are now in the position to prove the following special case of Theorem 3.3.1.

Theorem 3.4.4. *Let* (X, μ, T) *be an ergodic system and assume there is a continuous factor map* π *to its Kronecker factor. Let* $a \in \boldsymbol{gen}(\mu, \Phi)$*, for some Følner sequence* Φ *and* $E_1, E_2 \subset X$ *be open sets with*

$$2\mu(E_2) + \mu(E_1) > 2. \tag{3.4.3}$$

Then, there is an Erdős progression (a, x_1, x_2) *such that* $(x_1, x_2) \in E_1 \times E_2$.

Proof. We need to find an Erdős progression of the form $(a, x_1, x_2) \in X^3$ with $(x_1, x_2) \in E_1 \times E_2$. By Proposition 3.4.3 it suffices to verify that $\sigma_a(E_1 \times E_2) > 0$. For this, it is enough to show that

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) > 1, \tag{3.4.4}$$

because

$$E_1 \times E_2 = (E_1 \times X) \cap (X \times E_2)$$

and σ_a is a probability measure. To see why (3.4.4) holds, observe that

$$\begin{aligned}
\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) &= \pi_1\sigma_a(E_1) + \pi_2\sigma_a(E_2) \\
&= \mu(E_1) + 2\mu(E_2) - \pi_2\sigma_a(T^{-1}E_2) \qquad \text{by Prop. 3.4.2 (1)} \\
&\geq \mu(E_1) + 2\mu(E_2) - 1 > 1 \qquad \text{by (3.4.3)},
\end{aligned}$$

which concludes the proof. □

In light of Proposition 3.4.1, it turns out that Theorem 3.3.1 reduces to Theorem 3.4.4.

Proof of Theorem 3.3.1. Let (X, μ, T), $a \in X$ and $(\tilde{X}, \tilde{\mu}, \tilde{T})$, $\tilde{\pi}$ and $\tilde{a} \in \tilde{X}$ be as in Proposition 3.4.1. As in Theorem 3.3.1, let $E_1, E_2 \subset X$ be open sets satisfying

$$2\mu(E_2) + \mu(E_1) > 2.$$

Then, we define $\tilde{E_1} = \tilde{\pi}^{-1}(E_1)$ and $\tilde{E_2} = \tilde{\pi}^{-1}(E_2)$. By the definition of a factor map we know that $\tilde{\pi}\tilde{\mu} = \mu$ and so it follows that

$$2\tilde{\mu}(\tilde{E_2}) + \tilde{\mu}(\tilde{E_1}) > 2.$$

By Theorem 3.4.4 we can find an Erdős progression $(\tilde{a}, \tilde{x}_1, \tilde{x}_2) \in \tilde{X}^3$, such that $(\tilde{x}_1, \tilde{x}_2) \in \tilde{E_1} \times \tilde{E_2}$. The continuity of $\tilde{\pi}$ allows us to conclude that the triple $(a, x_1, x_2) = (\tilde{\pi}(\tilde{a}), \tilde{\pi}(\tilde{x}_1), \tilde{\pi}(\tilde{x}_2)) \in X^3$ is also an Erdős progression in (X, T) and clearly, by definition, $(x_1, x_2) \in E_1 \times E_2$. □

Likewise, we can deduce Theorem 3.3.6 from the following theorem.

Theorem 3.4.5. *Let (X, μ, T) be an ergodic system and assume there is a continuous factor map π to its Kronecker factor. Let $a \in \boldsymbol{gen}(\mu, \Phi)$, for some Følner*

sequence Φ *and* $E_1, E_2, E_3 \subset X$ *be open sets with* $F_2 = T^{-1}E_2$ *and*

$$2\mu(E_2) + \mu(E_1) + \mu(F_1) > 2. \tag{3.4.5}$$

Then, there is an Erdős progression (a, x_1, x_2) *and either* $(x_1, x_2) \in E_1 \times E_2$ *or* $(x_1, x_2) \in F_1 \times F_2$.

Proof. This time we want to find an Erdős progression of the form $(a, x_1, x_2) \in X^3$ with either $(x_1, x_2) \in E_1 \times E_2$ or $(x_1, x_2) \in F_1 \times F_2$. Again, in view of Proposition 3.4.3 it suffices to show that either $\sigma_a(E_1 \times E_2) > 0$ or $\sigma_a(F_1 \times F_2) > 0$. Similarly to the proof of Theorem 3.4.4, the latter follows from $\sigma_a(F_1 \times X) + \sigma_a(X \times F_2) > 1$. Else, we have that

$$\sigma_a(F_1 \times X) + \sigma_a(X \times F_2) \leq 1,$$

which, using Proposition 3.4.2 (1), we rewrite as

$$\pi_2\sigma_a(F_2) \leq 1 - \pi_1\sigma_a(F_1) = 1 - \mu(F_1). \tag{3.4.6}$$

In this case we also have that

$$\begin{aligned}
\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) &= \pi_1\sigma_a(E_1) + \pi_2\sigma_a(E_2) \\
&= \mu(E_1) + 2\mu(E_2) - T\pi_2\sigma_a(E_2) \qquad \text{by Prop. 3.4.2 (1)} \\
&= \mu(E_1) + 2\mu(E_2) - \pi_2\sigma_a(F_2) \qquad \text{since } T^{-1}E_2 = F_2.
\end{aligned}$$

Using (3.4.6) we obtain that

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) \geq \mu(E_1) + 2\mu(E_2) + \mu(F_1) - 1.$$

By (3.4.5) it finally follows that

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) > 1$$

and as before we conclude that $\sigma_a(E_1 \times E_2) > 0$. □

The proof of Theorem 3.3.6 is similar to the one of Theorem 3.3.1, and we simply need to use Theorem 3.4.5 instead of Theorem 3.4.4.

3.5 Sharpness of the bounds and the role of even numbers

In this section we show that the lower bounds in Theorem 1.3.1 and Theorem 1.3.2 are sharp with illustrative examples. Here the proportion of even and odd numbers in the sets plays a crucial role. That role becomes clear in the constructions and is even more explicit in Corollary 3.2.1, which we prove below. Finally we provide several generalizations of Corollary 3.2.1.

3.5.1 Sets of large densities not containing infinite sumsets B+B

The following proposition shows that the bounds established in parts (1) of Theorems 1.3.1 and 1.3.2 are optimal.

Proposition 3.5.1. *There exist sets $A, A' \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = 2/3$ and $\overline{\mathrm{d}}(A') = 5/6$ and such that for all infinite sets $B \subset \mathbb{N}$ and numbers $t \in \mathbb{N}$ we have $B + B + t \not\subset A$ and $B + B \not\subset A'$.*

Proof. Let A be the subset of $\mathbb{N}$ defined by

$$A = \mathbb{N} \cap \bigcup_{n\in\mathbb{N}} [4^n, (2 - 1/n) \cdot 4^n).$$

It is clear by the definition of A that $\overline{\mathrm{d}}(A) = \mathrm{d}_\Phi(A)$, where (Φ_N) is the Følner sequence given by $N \mapsto [1, (2 - 1/N)4^N) \cap \mathbb{N}$. By standard calculations we see that $\overline{\mathrm{d}}(A) = 2/3$.

Assume there is an infinite $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ with $B + B + t \subset A$. Then, in particular, for any $b' \in B$ fixed there is $b \in B$ arbitrarily large such that $\{2b + t, b + b' + t\} \subset A$. Therefore, $2b \in [4^{n+1}, (2 - \frac{1}{n+1})4^{n+1})) - t$, for some $n \in \mathbb{N}$ and diving by 2 we have

$$2 \cdot 4^n + t/2 \leq b + t < 2\left(2 - \frac{1}{n+1}\right) 4^n + t/2.$$

Since t is fixed and b' is negligible with respect to b (and hence with respect to n), we see that $b + b' + t \in [2 \cdot 4^n,\ 4^{n+1}) \subset \mathbb{N} \setminus A$, reaching a contradiction. To see this, observe that $4^{n+1} - 2\left(2 - \dfrac{1}{n+1}\right) 4^n = 2 \cdot \dfrac{4^n}{n+1} \xrightarrow{n\to\infty} \infty$. This completes the first construction.

Keeping A as defined before we now set $A' = A \cup (2\mathbb{N} + 1)$. In other words, $A' = A \cup ((\mathbb{N} \setminus A) \cap (2\mathbb{N} + 1))$. By the definition of A we have that $\mathbb{N} \setminus A$ is a union

of discrete intervals of lower density $1/3$ and so $\underline{\mathrm{d}}((\mathbb{N} \setminus A) \cap 2\mathbb{N}) = 1/6$. Since the complement of A' is precisely $(\mathbb{N} \setminus A) \cap 2\mathbb{N}$, it follows that $\overline{\mathrm{d}}(A') = 5/6$.

We now show there is no infinite set $B \subset \mathbb{N}$ with $B + B \subset A'$. Indeed, if there were such a set, we could consider an infinite subset $B' \subset B$ consisting of integers with the same parity and so we would have that $B' + B' \subset A' \cap 2\mathbb{N} \subset A$. This is a contradiction to the first part of this proof. □

Analogously to Proposition 3.5.1, the next result shows that the bounds established in parts (2) of Theorems 1.3.1 and 1.3.2 are optimal as well.

Proposition 3.5.2. *There exist sets $A, A' \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) = 1/2$ and $\underline{\mathrm{d}}(A') = 3/4$ and such that for all infinite sets $B \subset \mathbb{N}$ and numbers $t \in \mathbb{N}$ we have $B + B + t \not\subset A$ and $B + B \not\subset A'$.*

Proof. Let $A_1, A_2 \subset \mathbb{N}$ be defined by

$$A_1 = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [4^n, (2 - 1/n) \cdot 4^n)$$

and

$$A_2 = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(2 + 1/n) \cdot 4^n, 4^{n+1}).$$

It was shown in Proposition 3.5.1 that there is no infinite $B \subset \mathbb{N}$ and $t \in \mathbb{N}$ so that $B + B + t \subset A_1$. By symmetry, the same argument shows that there is no infinite $B \subset \mathbb{N}$ and $t \in \mathbb{N}$ so that $B + B + t \subset A_2$ either. We remark that the set

$$\mathbb{N} \setminus (A_1 \cup A_2) = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(2 - 1/n) \cdot 4^n, (2 + 1/n) \cdot 4^n]$$

has zero density. This boils down to the fact that

$$\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n \cdot 4^{N-n}} = 0.$$

To see this, note that $n \mapsto 1/\left(n \cdot 4^{N-n}\right)$ is increasing in $[1, N]$ and so

$$\lim_{\theta \nearrow 1} \lim_{N \to \infty} \sum_{n=1}^{\lfloor \theta N \rfloor} \frac{1}{n \cdot 4^{N-n}} + \sum_{n > \lfloor \theta N \rfloor}^{N} \frac{1}{n \cdot 4^{N-n}} \leq \lim_{\theta \nearrow 1} \lim_{N \to \infty} \frac{1}{4^{(1-\theta)N}} + \frac{(1-\theta)N}{N} = 0.$$

We now consider the set $A \subset \mathbb{N}$ defined by

$$A = (A_1 \cap 2\mathbb{N}) \cup (A_2 \cap (2\mathbb{N} + 1)).$$

It follows that $\mathrm{d}(A) = 1/2$ and so, in particular, $\underline{\mathrm{d}}(A) = 1/2$. Moreover, we claim that there is no infinite set $B \subset \mathbb{N}$ and no $t \in \mathbb{N}$ such that $B + B + t \subset A$. Indeed, if this were the case we could consider an infinite subset of B, call it B', with elements of the same parity. Then, if $t \in 2\mathbb{N}$ we should have that $B' + B' + t \subset A \cap 2\mathbb{N} \subset A_1$ and if $t + 1 \in 2\mathbb{N}$ we should have $B' + B' + t \subset A \cap (2\mathbb{N} + 1) \subset A_2$, a contradiction.

For the second part of the construction we keep A_1 and A_2 as they were defined above. Then, we let $A' \subset \mathbb{N}$ be defined by

$$A' = (A_1 \setminus 4\mathbb{N}) \cup (A_2 \setminus (4\mathbb{N} + 2)) .$$

As we showed before $\mathrm{d}(A_1 \cup A_2) = 1$ and from this it easily follows that $\mathrm{d}(A') = 3/4$, because we are only removing from $A_1 \cup A_2$ a set of density $1/4$. Finally, we claim there is no infinite set $B \subset \mathbb{N}$ satisfying $B + B \subset A'$. Indeed, given an infinite set B we can consider an infinite subset $B' \subset B$ with all its elements equivalent mod 4. That is, there exists $j \in \{0, 1, 2, 3\}$ so that $b' \equiv j \pmod 4$, for any $b' \in B'$. Note that if $j \in \{0, 2\}$, then $B' + B' \subset 4\mathbb{N}$ and if $j \in \{1, 3\}$, then $B' + B' \subset 4\mathbb{N} + 2$. In the former case we must have $B' + B' \subset A_2$ and in the latter that $B' + B' \subset A_1$, which both contradict the results established in the first part. □

Remark. In the first part of the previous proof, we could instead have considered the set $\tilde{A} \subset \mathbb{N}$ defined as $\tilde{A} = (A_1 \cap (2\mathbb{N} + 1)) \cup (A_2 \cap 2\mathbb{N})$. Notice that as before, $\tilde{A}$ does not contain an infinite sumset of the form $B + B + t$ either. The set $A \cup \tilde{A}$ has density 1, so with this construction we find a 2-colouring of a set with full density that does not have a monochromatic infinite sumset $B + B + t$.

Notice that the sets constructed in Proposition 3.5.2 are both sets whose natural densities exist. Thus, considering this and Theorems 1.3.1 and 1.3.2 we see that information on the value of the natural density of a set, when it exists, is no more powerful than information on the value of the lower density of a set, which always exists. This was also to be expected from Theorems 3.3.5 and 3.3.8. For example, from 3.3.8 we already knew that if there existed a set $A \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) = 1/2$ and such that no sumset $B + B + t$ was contained in A, it would have to be the case that $\overline{\mathrm{d}}(A) = 1/2$ too, hence A would have a natural density $\mathrm{d}(A) = 1/2$.

3.5.2 Density over the even numbers

We begin by stating and proving a seemingly more general result than Corollary 3.2.1.

Proposition 3.5.3. *Let $A, D \subset \mathbb{N}$. If one of the following holds:*

(1) $\overline{\mathrm{d}}(A \cap 2D) > 1/3$ *or*

(2) $\underline{\mathrm{d}}(A \cap 2D) > 1/4$,

then there is an infinite set $B \subset D$ *such that* $B + B \subset A$. *Likewise, if one of the following holds:*

(3) $\overline{\mathrm{d}}\,(A \cap (2D+1)) > 1/3$ *or*

(4) $\underline{\mathrm{d}}\,(A \cap (2D+1)) > 1/4$,

then there is an infinite set $B \subset D$ *such that* $B + B + 1 \subset A$.

Proof. (1) : For $A, D \subset \mathbb{N}$ such that $\overline{\mathrm{d}}(A \cap 2D) > 1/3$ we let $\tilde{A} = (A \cap 2D) \cup (2\mathbb{N}+1)$. Then by part (1) of Lemma 2.1.1 we see that $\overline{\mathrm{d}}(\tilde{A}) > 5/6$ and so by Theorem 1.3.1 there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset \tilde{A}$. Without loss of generality we can assume that the elements of B have the same parity, which means that the set $B + B$ consists solely of even numbers. Therefore, $B + B \subset \tilde{A} \cap 2\mathbb{N} = A \cap 2D$. In particular, $B + B \subset A$ and $2B \subset 2D$, which also implies that $B \subset D$.

(3) : For $A, D \subset \mathbb{N}$ such that $\overline{\mathrm{d}}\,(A \cap (2D+1)) > 1/3$ we can shift to obtain that $\overline{\mathrm{d}}\,((A-1) \cap 2D) > 1/3$. Thus, by (1) there is $B \subset D$ infinite with $B + B \subset A - 1$ which implies that $B + B + 1 \subset A$.

(2) : For $A, D \subset \mathbb{N}$ such that $\underline{\mathrm{d}}(A \cap 2D) > 1/4$ we define $\tilde{A} = (A \cap 2D) \cup (2\mathbb{N}+1)$. Using part (2) of Lemma 2.1.1 we have that $\underline{\mathrm{d}}(\tilde{A}) > 3/4$, so that by Theorem 1.3.1 there is an infinite set $B \subset \mathbb{N}$ such that $B + B \subset \tilde{A}$, and repeating the parity argument in (1) we conclude.

(4) : This follows from (2) by shifting, exactly like (3) follows from (1). □

A special case of interest of Proposition 3.5.3 is when $D = A$, because it gives a criterion to ensure the existence of an infinite set B such that $B \cup (B + B) \subset A$. Moreover, as was mentioned before, Corollary 3.2.1 is a consequence of Proposition 3.5.3. This is clear for (1) and (2) in Corollary 3.2.1 by replacing the set D by $\mathbb{N}$. Now notice that $\overline{\mathrm{d}}(A \cap 2\mathbb{N}) > 1/3$ implies the existence of an infinite set $B \subset \mathbb{N}$ such that $B + B \subset A$ if and only if $\overline{\mathrm{d}}(A' \cap (2\mathbb{N}+1)) > 1/3$ implies $B + B + 1 \subset A'$ (where $A' = A + 1$). That means (1) and (3) of Corollary 3.2.1 are equivalent. The same can be done to prove that (2) is equivalent to (4).

We would also like to point out that using the sub-additivity of the upper density one can also use the equivalent parts (1) and (3) of Corollary 3.2.1 to deduce Theorem 1.3.2 (1) and Theorem 1.3.1 (1). We only show the first implication as the other one is very similar.

Proof that Corollary 3.2.1 implies Theorem 1.3.2 (1). If $A \subset \mathbb{N}$ has $\overline{\mathrm{d}}(A) > 2/3$, then writing $A = (A \cap 2\mathbb{N}) \cup (A \cap 2\mathbb{N}+1)$, we have by Lemma 2.1.1 (1) that either $\overline{\mathrm{d}}(A \cap 2\mathbb{N}) > 1/3$ or $\overline{\mathrm{d}}(A \cap 2\mathbb{N}+1) > 1/3$. In the former we see by (1) of Corollary 3.2.1 that there exists $B \subset \mathbb{N}$ infinite and such that $B+B \subset A$ and in the latter we apply (3) of the same corollary, to get an infinite $B \subset \mathbb{N}$ such that $B+B+1 \subset A$. $\square$

We finish this section by showing the special role that the powers of 2 play in the existence of the set B. The next proposition is a generalization of Corollary 3.2.1, but they share a similar proof.

Proposition 3.5.4. *Let $A \subset \mathbb{N}$, $n \geq 1$ and $0 \leq \ell < 2^{n-1}$. If one of the following holds:*

(1) $\overline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell)) > 1/(3 \cdot 2^{n-1})$ *or*

(2) $\underline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell)) > 1/2^{n+1}$,

then there is an infinite set $B \subset \mathbb{N}$ such that $B+B \subset A$. Similarly, if one of the following holds:

(3) $\overline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell + 1)) > 1/(3 \cdot 2^{n-1})$ *or*

(4) $\underline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell + 1)) > 1/2^{n+1}$,

then there is an infinite set $B \subset \mathbb{N}$ such that $B+B+1 \subset A$.

Proof. It is clear that (1) implies (3), because if $\overline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell + 1)) > 1/(3 \cdot 2^{n-1})$ then $\overline{\mathrm{d}}((A-1) \cap (2^n\mathbb{N} + 2\ell)) > 1/(3 \cdot 2^{n-1})$ and then by (1) there is an infinite set $B \subset \mathbb{N}$ such that $B+B \subset A-1$. A similar observation allows us to conclude that (2) implies (4).

It is also clear that it is enough to prove (1) for the case $\ell = 0$. Indeed, if $\overline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell)) > 1/(3 \cdot 2^{n-1})$, then also $\overline{\mathrm{d}}((A - 2\ell) \cap 2^n\mathbb{N}) > 1/(3 \cdot 2^{n-1})$ and so there is an infinite set B' such that $B'+B' \subset A - 2\ell$, therefore taking $B = B' + \ell$ we conclude that $B+B \subset A$.

We will thus prove by induction in n that whenever $\overline{\mathrm{d}}(A \cap 2^n\mathbb{N}) > 1/(3 \cdot 2^{n-1})$, there exists $B \subset \mathbb{N}$ infinite so that $B+B \subset A$. Notice that the base case is given by Corollary 3.2.1.

Assuming the result for $n \in \mathbb{N}$, we let $A \subset \mathbb{N}$ satisfy $\overline{\mathrm{d}}(A \cap 2^{n+1}\mathbb{N}) > 1/(3 \cdot 2^n)$. We define $\tilde{A} = (A \cap 2^{n+1}\mathbb{N}) \cup \big((A \cap 2^{n+1}\mathbb{N}) + 2^n\big)$. Then, $\tilde{A} = \tilde{A} \cap 2^n\mathbb{N}$ and by part (3) of Lemma 2.1.1 we have

$$\overline{\mathrm{d}}(\tilde{A} \cap 2^n\mathbb{N}) = 2 \cdot \overline{\mathrm{d}}(A \cap 2^{n+1}\mathbb{N}) > 2/(3 \cdot 2^n) = 1/(3 \cdot 2^{n-1}).$$

Thus by the inductive hypothesis, there exists an infinite set $B' \subset \mathbb{N}$ such that $B' + B' \subset \tilde{A}$. By the pigeonhole principle, if $B'_i = \{b \in B' \colon b \equiv i \pmod{2^{n+1}}\}$, $i \in \{0, \dots, 2^{n+1} - 1\}$, then one of those B'_i is infinite. Let B'_{i_0} be such a set and denote it by B''. As $B'' + B'' \subset 2^{n+1}\mathbb{N} + 2i_0$ and $B'' + B'' \subset \tilde{A} \subset 2^n\mathbb{N}$, it must be the case that $i_0 \in \{\ 0,\ 2^{n-1},\ 2^n,\ 2^{n-1} + 2^n\ \}$.

If $i_0 = 0$ then $B'' + B'' \subset \tilde{A} \cap 2^{n+1}\mathbb{N} = A \cap 2^{n+1}\mathbb{N}$ and we are done. Similarly, if $i_0 = 2^n$ then $B'' + B'' \subset \tilde{A} \cap (2^{n+1}\mathbb{N} + 2^{n+1}) \subset A \cap 2^{n+1}\mathbb{N}$ and we are also done. Alternatively, if $i_0 = 2^{n-1}$ then $B'' + B'' \subset \tilde{A} \cap (2^{n+1}\mathbb{N} + 2^n) = \big(A \cap 2^{n+1}\mathbb{N}\big) + 2^n$ and likewise, for $i_0 = 2^{n-1} + 2^n$, we also get $B'' + B'' \subset \big(A \cap 2^{n+1}\mathbb{N}\big) + 2^n$. In both cases, setting $B = B'' - 2^{n-1}$ we deduce that $B + B \subset A \cap 2^{n+1}\mathbb{N}$.

The proof of (2) is completely analogous. □

3.6 B+B in syndetic sets and partition regularity

Theorem 1.3.1, Theorem 1.3.2 and the examples constructed in Section 3.5 offer a characterization of sets $A \subset \mathbb{N}$ containing $B + B$ or $B + B + 1$ for some infinite $B \subset \mathbb{N}$ in terms of information provided by their upper and lower density. It is then natural to ask for a similar characterization in terms of upper or lower Banach density.

It turns out that no conclusions can be drawn from information on the upper Banach density, because the first example given in Section 3.5.1 has full upper Banach density and does not contain an infinite sumset $B + B + t \subset A$, for any shift $t \in \mathbb{N}$.

So far, these results do not offer any insight on the relation between the size of a set $A \subset \mathbb{N}$ captured by its lower Banach density and whether it contains $B+B+t$ for infinite $B \subset \mathbb{N}$ and a number $t \geq 0$, apart from the obvious ones following from the results on lower density (for example, since for $A \subset \mathbb{N}$ one always has $\underline{d}(A) \geq d_*(A)$, we know that $d_*(A) > 1/2$ guarantees $B + B + t \subset A$, some $t \in \{0, 1\}$).

The question indirectly raised in the last paragraph is also interesting and very natural for another reason: sets with positive lower Banach density are precisely the very well−studied syndetic sets. We have the following proposition.

Proposition 3.6.1. *There is a syndetic set $A \subset \mathbb{N}$ with gaps of length two such that A does not contain any set $B + B + t$ for $B \subset \mathbb{N}$ infinite and $t \in \mathbb{N}$.*

Proof. Let $\mathbb{T}$ be the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$ and for a real number x, $\{x\}$ denotes its fractional part, that is $\{x\} = x - \lfloor x \rfloor$. Let $A \subset \mathbb{N}$ be given by

$$A = \{m \in \mathbb{N} \colon \{\theta \log_2(m)\} \in U\} \tag{3.6.1}$$

where $\theta > 0$ and the sets U and $U + \theta$ (taken mod 1) are separated by a positive distance as subsets of the torus $\mathbb{T}$. As remarked in [44, Example 3.6], A does not contain any $B + B + t$ for any infinite set $B \subset \mathbb{N}$ and shift $t \geq 0$. Indeed, for any infinite set $B \subset \mathbb{N}$ and $t \geq 0$, we can take $b' \in B$ arbitrary and then choose $b \in B$ sufficiently large so that the difference

$$\left| \log_2 \left(\frac{2b + t}{b + b' + t} \right) - 1 \right|$$

is less than the distance $d(U, U + \theta)/\theta$. Multiplying by θ this implies that

$$\left| \theta \log_2(2b + t) - \theta \log_2(b + b' + t) - \theta \mod 1 \right| < d(U, U + \theta).$$

Thus, if $\{\theta \log_2(b + b' + t)\} \in U$, then $\{\theta \log_2(2b + t) - \theta\}$ is in a small neighborhood of U, so $\{\theta \log_2(2b + t)\} \notin U$. Therefore, $2b + t$ and $b + b' + t$ cannot both belong to A simultaneously.

Consider now $U_0 = [0, 1/3 + \varepsilon)$, $U_1 = [1/3, 2/3 + \varepsilon)$ and $U_2 = [2/3, 1 + \varepsilon)$ where $0 < \varepsilon < 1/6$, so that $\{U_1, U_2, U_3\}$ is a cover of $\mathbb{T}$ with small overlapping. Notice that for $i \in \{0, 1, 2\}$, the set

$$A_i = \{m \in \mathbb{N} \colon \{1/2 \log_2(m)\} \in U_i\}$$

does not contain any $B + B + t$ for any infinite set $B \subset \mathbb{N}$ and shift $t \geq 0$. Now, we define the set A as

$$A = (A_0 \cap 3\mathbb{N}) \cup (A_1 \cap (3\mathbb{N} + 1)) \cup (A_2 \cap (3\mathbb{N} + 2)).$$

Then A is clearly syndetic, because $A_0 \cup A_1 \cup A_2 = \mathbb{N}$, each A_i is a disjoint union of discrete intervals and each interval in A_i is followed by an interval of $A_{i+1 \pmod 3}$, $i = 1, 2, 3$. Moreover, as the sets A_i overlap it is clear that the gaps of A have size at most 2, that is, $A \cup (A - 1) \cup (A - 2) = \mathbb{N}$ (this is why we need to include a small parameter $\varepsilon > 0$ in the definition of the sets $U_i \subset \mathbb{T}$).

Finally A does not contain any $B + B + t$ for any infinite set $B \subset \mathbb{N}$ and shift $t \geq 0$. Otherwise, without loss of generality, we could assume that the elements of B are equal mod 3, that is, there exists $j \in \{0, 1, 2\}$ such that for every $b \in B$, $b \equiv j \mod 3$. Then $B + B + t \subset 3\mathbb{N} + i$, where $i \equiv 2j + t \mod 3$. But this would mean that $B + B + t \subset A \cap (3\mathbb{N} + i) \subset A_i$, which is a contradiction. $\square$

Remark. We note that the sets A_i defined above can also be described as

$$A_i = \mathbb{N} \cap \bigcup_{n\in\mathbb{N}} [4^{i/3} \cdot 4^n, 4^{(i+1)/(3+\varepsilon)} \cdot 4^n),$$

and this alternative description illustrates the relationship between this example and those in Propositions 3.5.1 and 3.5.2.

As an immediate corollary of Proposition 3.6.1 we recover Proposition 3.2.2. Indeed, using the set A constructed in Proposition 3.6.1 we can build a three-colouring of $\mathbb{N}$ with syndetic colours and without any monochromatic infinite sumset $B + B$, because $\mathbb{N} = A \cup (A - 1) \cup (A - 2)$.

Remark. One can also prove Proposition 3.2.2 directly by considering the sets A_i, $i = 1, 2, 3$ as in the proof of Proposition 3.6.1, but for $U_0 = [0, 1/3)$, $U_1 = [1/3, 2/3)$, and $U_2 = [2/3, 1)$ and picking colours C_1, C_2, C_3 given by

$$\begin{aligned} C_1 &= (A_0 \cap 3\mathbb{N}) \cup (A_1 \cap (3\mathbb{N} + 1)) \cup (A_2 \cap (3\mathbb{N} + 2)) \\ C_2 &= (A_0 \cap (3\mathbb{N} + 2)) \cup (A_1 \cap 3\mathbb{N}) \cup (A_2 \cap (3\mathbb{N} + 1)) \\ C_3 &= (A_0 \cap (3\mathbb{N} + 1)) \cup (A_1 \cap (3\mathbb{N} + 2)) \cup (A_2 \cap 3\mathbb{N}) \end{aligned}$$

Our example in Proposition 3.6.1 is a set $A \subset \mathbb{N}$ with $\mathrm{d}_*(A) = 1/3$ that does not contain any infinite sumset $B + B + t$ for any $t \in \mathbb{N}$. By the trick of gluing to A the odd integers which belong to its complement, as in the second part of Proposition 3.5.1, we obtain a set $A' = A \cup (2\mathbb{N} + 1)$ with $\mathrm{d}_*(A') = 2/3$ and such that there is no infinite $B \subset \mathbb{N}$ so that $B + B \subset A'$. However, we do not yet have a characterization in terms of information provided by the lower Banach density of a set of integers and so we ask the following question.

Question 3.6.2. Let $A \subset \mathbb{N}$.

(1) If $\mathrm{d}_*(A) > 1/3$, does there exist an infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$ so that $B + B + t \subset A$?

(2) If $\mathrm{d}_*(A) > 2/3$, does there exist an infinite set $B \subset \mathbb{N}$ so that $B + B \subset A$?

3.7 Digression on monochromatic B+B in 2-colourings

As we mentioned in the introduction, the following question of C.J. Owings is an old problem that was very recently resolved in [34]. The results of this section thus become obsolete in some sense, but are nevertheless interesting observations on the conclusions that can be drawn for this partition regularity problem from our main theorems.

Question 3.7.1. [36, Problem E2494] Is it true that, no matter how $\mathbb{N}$ is partitioned into two sets, one of the sets must contain $B + B$ for some infinite $B \subset \mathbb{N}$?

Remark 3.7.2. By the parity argument repeated in the previous section, this question is equivalent to finding a partition in the even numbers such that one of the sets contains a $B+B$ for some infinite $B \subset \mathbb{N}$. But, by translation, this is equivalent to the analogous statement for the odd numbers and shifted patterns $B + B + 1$. Joining both statements we notice that the question is equivalent to whether any colouring of the natural numbers $\mathbb{N}$ in two colours, must contain a monochromatic $B + B + t$ for some infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$.

Using Proposition 3.3.8, we are able to prove the following result which reduces the problem to a narrower case by adding density restrictions on both sets in the partition.

Proposition 3.7.3. *Let $\mathbb{N} = C_1 \cup C_2$ be a partition of $\mathbb{N}$. If there is no infinite $B \subset \mathbb{N}$, no number $t \in \mathbb{N}$ and $i \in \{1, 2\}$ such that $B + B + t \subset C_i$, then it must be the case that*

$$\overline{\mathrm{d}}(C_i) + \underline{\mathrm{d}}(C_i) = 1, \tag{3.7.1}$$

for both $i = 1, 2$ and so, in particular,

$$\overline{\mathrm{d}}(C_1) = \overline{\mathrm{d}}(C_2) \text{ and } \underline{\mathrm{d}}(C_1) = \underline{\mathrm{d}}(C_2). \tag{3.7.2}$$

Proof. By Theorem 3.3.8 we know that if $\overline{\mathrm{d}}(C_i) + \underline{\mathrm{d}}(C_i) > 1$, then there is $B \subset \mathbb{N}$ infinite and $t \in \mathbb{N}$ such that $B + B + t \subset C_i$ and so, under our assumption, it must be that

$$\overline{\mathrm{d}}(C_i) + \underline{\mathrm{d}}(C_i) \leq 1 \tag{3.7.3}$$

for both $i = 1, 2$. As $\{C_1, C_2\}$ forms a partition of $\mathbb{N}$ we have that $\mathbb{N} \setminus C_1 = C_2$ and $\mathbb{N} \setminus C_2 = C_1$, so that

$$\overline{\mathrm{d}}(C_1) + \underline{\mathrm{d}}(C_2) = \overline{\mathrm{d}}(C_2) + \underline{\mathrm{d}}(C_1) = 1, \tag{3.7.4}$$

which follows trivially from the definitions of upper and lower density. Summing the two leftmost sides in (3.7.4) we get

$$\overline{\mathrm{d}}(C_1) + \underline{\mathrm{d}}(C_1) + \overline{\mathrm{d}}(C_2) + \underline{\mathrm{d}}(C_2) = 2 \tag{3.7.5}$$

and this forces (3.7.3) to take the form

$$\overline{\mathrm{d}}(C_i) + \underline{\mathrm{d}}(C_i) = 1,$$

for both $i = 1, 2$. This is (3.7.1) and along with the equalities in (3.7.4) we also obtain (3.7.2). □

Remark. Using Theorem 1.3.2 one can also place further constrictions on the values of the densities of the sets involved in the two-colouring. Indeed, combining Proposition 3.7.3 with Theorem 1.3.2, if the answer to Question 3.7.1 was negative, then $\overline{\mathrm{d}}(C_1) = \overline{\mathrm{d}}(C_2) \in [1/2, 2/3]$ and $\underline{\mathrm{d}}(C_1) = \underline{\mathrm{d}}(C_2) \in [1/3, 1/2]$.

Going a step further, if there was a colouring $\mathbb{N} = C_1 \cup C_2$ which gave a negative answer to Question 3.7.1, then it would have to be the case that $\overline{\mathrm{d}}(C_1 \cap (2^n\mathbb{N}+2\ell)) = \overline{\mathrm{d}}(C_2 \cap (2^n\mathbb{N}+2\ell)) \in [1/2^{n+1}, 2/(3\cdot 2^n)]$ and $\underline{\mathrm{d}}(C_1 \cap (2^n\mathbb{N}+2\ell)) = \underline{\mathrm{d}}(C_2 \cap (2^n\mathbb{N}+2\ell)) \in [1/(3\cdot 2^n), 1/2^{n+1}]$, for every natural numbers $n, \ell \in \mathbb{N}$. We merely sketch how one deduces this fact.

The first step is to use Theorem 3.3.5 in order to show that if $A \subset \mathbb{N}$ satisfies $\overline{\mathrm{d}}(A \cap 2\mathbb{N}) + \underline{\mathrm{d}}(A \cap 2\mathbb{N}) > 1/2$, then there exists an infinite $B \subset \mathbb{N}$ such that $B+B \subset A$. This is done exactly like part (2) of Corollary 3.2.1 is deduced from Theorem 1.3.1. Using this new fact as our base case we can prove by induction that if $A \subset \mathbb{N}$ satisfies $\overline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell)) + \underline{\mathrm{d}}(A \cap (2^n\mathbb{N} + 2\ell)) > 1/2^n$, for any $n, \ell \in \mathbb{N}$, then there exists an infinite $B \subset \mathbb{N}$ such that $B + B \subset A$. This is established just like Proposition 3.5.4. It's now easy to extend Proposition 3.7.3. Namely, in the context of its statement, we actually have that $\overline{\mathrm{d}}(C_1 \cap (2^n\mathbb{N} + 2\ell)) = \overline{\mathrm{d}}(C_2 \cap (2^n\mathbb{N} + 2\ell))$ and $\underline{\mathrm{d}}(C_1 \cap (2^n\mathbb{N} + 2\ell)) = \underline{\mathrm{d}}(C_2 \cap (2^n\mathbb{N} + 2\ell))$, for any $n, \ell \in \mathbb{N}$. Finally, the claim follows from this latter fact and the bounds established in Proposition 3.5.4.

Chapter 4

Unrestricted sumsets in countable abelian groups

4.1 Overview

This chapter is based on the results of [10] and generalises the results of Chapter 3 in two ways. First, we expand the framework from the integers to countable abelian groups. Secondly, we consider more general density schemes, even in the integers, as opposed to Chapter 3 where densities were only considered over the initial Følner sequence $N \mapsto [1, N]$.

The second point is in a sense forced by the first one, since there is no obvious analogue of the initial Følner sequence in $\mathbb{Z}$ for a general group. This leads us to understand more deeply what structural properties of $N \mapsto [1, N]$ were indirectly used in the proofs of Chapter 3 and formulate an abstract definition that captures analogous structural properties for general Følner sequences, in order to run analogous arguments.

The transition from $\mathbb{Z}$ to a general abelian group $(G, +)$ introduces another issue. Namely, the parity obstructions in $\mathbb{Z}$ become structural obstructions governed by the behaviour of the doubling map

$$D : G \to G, \qquad D(g) = 2g$$

In $\mathbb{Z}$, the even integers form the image of the doubling map. In a general abelian group, the image is the subgroup $D(G) = 2G := \{g + g : g \in G\}$, and the relevant quantities for our purposes are

$$\ell = [G : 2G], \text{ and } r = |\ker(D)|.$$

As we will see, it is necessary for our results that both ℓ and r are finite and also that the Følner sequences we consider interact well with doubling.

In analogy to the results of Chapter 3, the precursor to the results presented here, and once again, an ingredient in the proofs, is the generalisation of Theorem C to countable abelian groups by Charamaras and Mountakis [11].

Let $(G,+)$ be a countable abelian group such that the subgroup $2G$ has finite index and the doubling map $g \mapsto g+g$ has finite kernel. We establish lower bounds on the upper density of a set $A \subset G$ with respect to an appropriate Følner sequence, so that A contains a sumset of the form $\{b_1+b_2+t\colon b_1,b_2 \in B\}$ or $\{b_1+b_2\colon b_1,b_2 \in B\}$, for some infinite $B \subset G$ and some $t \in G$. Both assumptions on G are necessary for our results to be true. We also characterize the Følner sequences for which this is possible. Finally, we show that our lower bounds are optimal in a strong sense.

4.2 Introduction

For convenience, we recall once again the resolution of Erdős' conjecture by Kra, Moreira, Richter and Robertson [41].

Theorem C. *For any $A \subset \mathbb{N}$ with positive upper Banach density there exist an infinite set $B \subset A$ and some $t \in \mathbb{N}$ such that*

$$B \oplus B + t = \{b_1 + b_2 + t\colon b_1,b_2 \in B,\ b_1 \neq b_2\} \subset A.$$

It is natural to ask whether Theorem C could be extended to other countable amenable (semi)groups. This is explored in [11], where Charamaras and Mountakis establish an extension of Theorem C for a wide class of such groups. This class includes all finitely generated nilpotent groups, and all abelian groups $(G,+)$ with the property that $2G := \{2g\colon g \in G\}$, where $2g := g+g$, has finite index.

Throughout, unless explicitly stated otherwise, all the groups we consider are countable. Then [11, Corollary 1.13] asserts that if $(G,+)$ is a countable abelian group with index $[G:2G] < \infty$ and $A \subset G$ has positive upper Banach density, there exist an infinite set $B \subset A$ and some $t \in G$ such that $B \oplus B + t \subset A$.

Another question arising from Theorem C is whether the restriction $b_1 \neq b_2$ in the sumset can be removed and it seems natural to explore what – if any at all – density assumptions on the set $A \subset \mathbb{N}$ would allow one to drop the restriction $b_1 \neq b_2$. This was initiated in Chapter 3, and hopefully the results presented there convinced the reader that this is an interesting line of enquiry. In that chapter it is shown, for example, that for any $A \subset \mathbb{N}$ such that $\overline{\mathrm{d}}_{([1,N])}(A) > 2/3$, there are an

infinite set $B \subset \mathbb{N}$ and some $t \in \{0, 1\}$ such that $B + B + t \subset A$. We remark that taking the density with respect to the initial Følner sequence $N \mapsto [1, N]$ in $\mathbb{N}$ is essential, as it is clear from [44, Example 2.3] that not all Følner sequences can be used to guarantee such density threshold values.

In view of these observations, we repeat the main question addressed in this chapter, that was already mentioned in Section 1.3.2. Let G be an abelian group, such that $[G : 2G] < \infty$.

(a) Can we find a Følner sequence Φ and a constant $c = c(G, \Phi) > 0$ such that any set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) > c$ contains an unrestricted sumset of the form $B + B + t$ for some infinite set $B \subset G$ and some $t \in G$?

(b) Can we answer (a) in an optimal way?

We refer to this problem as *the (unrestricted) $B + B$ problem.* Throughout, G denotes a countable abelian group and D denotes the doubling map $D : G \to G, D(g) = 2g$. In addition, given a set $A \subset G$, we denote the set $D(A)$ by $2A$ and the set $D^{-1}(A)$ by $A/2$.

As we alluded to earlier, not all Følner sequences have density thresholds for the unrestricted $B + B$ problem. However, we are able to pinpoint structural properties of a Følner sequence that allow for this to happen and to describe those we introduce the next definition.

Definition 4.2.1. Let G be a countable abelian group.

(i) We define *the doubling ratio* of a Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ in G as

$$\alpha_\Phi = \liminf_{N \to \infty} \frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|}. \tag{4.2.1}$$

Whenever $\alpha_\Phi > 0$, we say that the Følner sequence Φ is *quasi-invariant with respect to doubling with ratio* α_Φ, and we abbreviate this as *q.i.d. with ratio* α_Φ.

(ii) We define the *group doubling ratio* as $\alpha_G = \sup\{\alpha_\Phi \colon \Phi \in \mathcal{F}_G\}$, where $\mathcal{F}_G$ denotes the set of all Følner sequences in G.

As we show in Section 4.7, any abelian group G with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$ admits Følner sequences that are quasi-invariant with respect to doubling. More precisely, we prove that $\alpha_G = \min\{1, \frac{r}{\ell}\}$ and that the value α_G is attained, i.e., there is a Følner sequence Φ in G so that $\alpha_\Phi = \alpha_G$.

Next, we state our main theorem, which asserts that the Følner sequences defined in Definition 4.2.1 possess the necessary structural properties to provide an affirmative answer to question (a). This allows us to resolve this question for all abelian groups G with $[G : 2G] < \infty$ and $|\ker(D)| < \infty$.

Theorem 4.2.2. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $A \subset G$ and Φ be any Følner sequence in G that is quasi-invariant with respect to doubling with ratio α_Φ. Then the following hold:*

(1) If $\overline{\mathrm{d}}_\Phi(A) > 1 - \dfrac{\ell\alpha_\Phi}{\ell + r}$, then there exists an infinite set $B \subset G$ and some $t \in G$ such that $B + B + t \subset A$.

(2) If $\overline{\mathrm{d}}_\Phi(A) > 1 - \dfrac{\alpha_\Phi}{\ell + r}$, then there exists an infinite set $B \subset G$ such that $B + B \subset A$.

(3) If $\overline{\mathrm{d}}_\Phi(A \cap 2G) > \dfrac{1}{\ell} - \dfrac{\alpha_\Phi}{\ell + r}$, then there exists an infinite set $B \subset G$ such that $B + B \subset A$.

The statements (1), (2) and (3) of Theorem 4.2.2 are logically equivalent; this is proved in Section 4.3.1. Hence, it suffices to establish only one of the statements. In Section 5.4.1, we reformulate statement (2) into a dynamical statement, which we then prove in Section 4.4, concluding the proof of Theorem 4.2.2.

The second main objective of this paper is to show sharpness of the lower bounds in Theorem 4.2.2. This is not evident from the proof: the bounds are directly derived from a variant of Furstenberg's correspondence principle given in Lemma 4.3.4. The next theorem addresses this question, showing optimality with respect to the parameters ℓ, r, and $\alpha_G = \min\{1, \frac{r}{\ell}\}$. This indicates that the constant α_Φ is not an artificial device of the proof, but rather captures an existing phenomenon. We highlight that in all abelian groups G, both $[G : 2G]$ and $|\ker(D)|$, whenever finite, are powers of 2 (see Lemma 4.5.1).

Theorem 4.2.3. *Let $\ell, r \in \mathbb{N}$ be powers of 2. Then, there exist a countable abelian group G with $[G : 2G] = \ell$ and $|\ker(D)| = r$, a Følner sequence Φ in G which is quasi-invariant with respect to doubling with ratio $\alpha_\Phi = \alpha_G$, and a set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) = 1 - \frac{\alpha_G}{\ell + r}$, for which $B + B \not\subset A$ for any infinite $B \subset G$.*

This result was already known in $\mathbb{Z}$ (see Chapter 3). The proof of Theorem 4.2.3 is carried out in Section 4.5 where for every choice of ℓ, r, we construct examples using the groups $\mathbb{Z}$, $\mathbb{F}_p^\omega$, $\mathbb{Z}(1/2)/\mathbb{Z} = \{\frac{k}{2^n} \mod 1 \mid n, k \in \mathbb{Z}\}$ and their products. The most technical and intricate task in the proof is the case $\ell = 2^{d_1}$ and $r = 2^{d_2}$,

with $d_1, d_2 \geq 1$. The following table summarizes the groups where we build the corresponding examples with their respective value of α_G.

	$\ell = 1$	$\ell = 2^{d_1}, d_1 \geq 1$
$r = 1$	$\mathbb{F}_3^\omega, \quad \alpha_G = 1$	$\mathbb{Z}^{d_1}, \ \alpha_G = 2^{-d_1}$
$r = 2^{d_2}, d_2 \geq 1$	$(\mathbb{Z}(\frac{1}{2})/\mathbb{Z})^{d_2}, \quad \alpha_G = 1$	$\mathbb{Z}^{d_1} \times (\mathbb{Z}(\frac{1}{2})/\mathbb{Z})^{d_2}, \ \alpha_G = \min\{1, 2^{d_2 - d_1}\}$

In a similar spirit, we show that the assumptions concerning the group G and the Følner sequence in Theorem 4.2.2 are necessary (see Section 4.6). In particular, we construct an abelian group with infinite $\ker(D)$, and a set of full density along a Følner sequence that is quasi-invariant with respect to doubling, which contains no infinite sumsets $B + B$ (see Section 4.6.1). In addition, in Section 4.6.2 we show that along any Følner sequence that is not quasi-invariant with respect to doubling, there are sets of full upper density that contain no infinite sumsets $B + B$. Finally, recall that the assumption of the finiteness of $[G : 2G]$ is imposed by [1]. However, the results there do not strictly rule out the possibility that there is an abelian group G with $[G : 2G] = \infty$ such that any set of full upper Banach density contains an infinite sumset $B + B + t$. In Section 4.6.3, we show that this is the case only if $2G$ is finite.

We end this section with open questions arising naturally from our work. The first is related to recent results [3] for abelian groups G with finite $[G\colon 6G]$ and sets that contain a pattern of the form $B \oplus B \oplus B$.

Question 4.2.4. Is it possible to characterize the abelian groups and the Følner sequences for which there is a constant $c > 0$ such that if $A \subset G$ and $\overline{\mathrm{d}}_\Phi(A) > c$, then there exists $B \subset G$ infinite such that $B + B + B \subset A$?

This problem is significantly different from the one we consider. Indeed, while $B + B = 2B \cup (B \oplus B)$, $B + B + B$ is not equal to $3B \cup (B \oplus B \oplus B)$ as one must additionally account for the sets $\{b_1 + 2b_2\colon b_1, b_2 \in B, b_1 < b_2\}$, $\{2b_1 + b_2\colon b_1, b_2 \in B, b_1 < b_2\}$. Those two extra components change the nature of the combinatorial problem; loosely speaking, it is like handling four linear equations instead of two.

The next two questions deal with stronger optimality results for the bounds in Theorem 4.2.2.

Question 4.2.5. Let G be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$, and let Φ be a Følner sequence in G that is quasi-invariant with respect to doubling. Does there exist a set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) = 1 - \frac{\alpha_\Phi}{\ell + r}$, such that

$B + B \not\subset A$ for any infinite $B \subset G$?

Question 4.2.5 asks for the strongest possible notion of optimality for the bounds in our main result. The following, weaker, natural question has better chances of having a positive answer.

Question 4.2.6. Let G be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Can one always find a Følner sequence Φ in G that is quasi-invariant with respect to doubling and a set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) = 1 - \frac{\alpha_\Phi}{\ell+r}$, such that $B + B \not\subset A$ for any infinite $B \subset G$?

We stress that Theorem 4.2.3 does not provide an answer to Question 4.2.6. Indeed, given the parameters ℓ and r, Theorem 4.2.3 asserts the existence of some group G with those values, a set $A \subset G$ and a Følner sequence Φ in G such that $\overline{\mathrm{d}}_\Phi(A) = 1 - \frac{\alpha_G}{\ell+r}$ and $B + B \not\subset A$ for any infinite $B \subset G$.

Finally, although these questions remain open, they partially motivate the results of Chapter 5 which generalise those presented in this chapter.

4.3 Translation of Theorem 4.2.2 to a dynamical statement

The proof of Theorem 4.2.2 is accomplished via a dynamical systems reformulation. We start the present section by showing that any one of the statements (1), (2) or (3) of Theorem 4.2.2 implies the others, and hence it suffices to prove (2) in order to establish the theorem. The reader may find the background material needed in order to realize the proof in Section 2.5. Finally, in Section 4.3.2 we state our main dynamical result, Theorem 4.3.5, and prove that it implies Theorem 4.2.2.

4.3.1 Equivalence of the statements in the main theorem

G always denotes a countable abelian group with $\ell = [G : 2G] < \infty$. We also fix $g_1, \ldots, g_\ell \in G$ so that $G = \bigsqcup_{i=1}^{\ell} 2G + g_i$. We omit these assumptions from the statements of this section.

Lemma 4.3.1. *Fix a subset $A \subset G$. The following are equivalent:*

(1) A contains $B + B$ for some infinite set $B \subset G$.

(2) $A \cap 2G$ contains $B + B$ for some infinite set $B \subset G$.

(3) $(A \cap 2G) \cup (G \backslash 2G)$ contains $B + B$ for some infinite set $B \subset G$.

Proof. For any infinite $B \subset G$, by the pigeonhole principle, there is an infinite subset of B which is contained in some coset $2G + g_i$. Equivalently, there exists an infinite $B' \subset 2G$ and some $i \in \{1, \dots, \ell\}$, such that $B' + g_i \subset B$. If $B + B \subset A$, then for $B' + g_i$ chosen as before we have $(B' + g_i) + (B' + g_i) \subset A \cap 2G$, therefore proving (1) $\implies$ (2). The implication (2) $\implies$ (3) is obvious. Finally, that (3) implies (2) is a special case of the implication (1) $\implies$ (2), because $((A \cap 2G) \cup (G \setminus 2G)) \cap 2G = A \cap 2G$. □

Lemma 4.3.2. *Fix a subset $A \subset G$. The following are equivalent:*

(1) A contains $B + B + t$ for some $t \in G$ and infinite $B \subset G$.

(2) A contains $B + B + g_i$ for some $i \in \{1, \dots, \ell\}$ and infinite $B \subset G$.

(3) $(A - g_i) \cap 2G$ contains $B + B$ for some $i \in \{1, \dots, \ell\}$ and infinite $B \subset G$.

Proof. The implication (1) $\implies$ (2) uses the fact that any $t \in G$ can be written as $2s + g_i$ for some $s \in G$ and $i \in \{1, \dots, \ell\}$. Thus if we define $B' = B + s$ we get that $B' + B' + g_i = B + B + 2s + g_i \subset A$. Assuming (2), we have that $B + B \subset A - g_i$ and then (3) follows directly from the equivalence of (1) and (2) in Lemma 4.3.1. Finally, (3) $\implies$ (1) is obvious. □

Using the previous lemmas we deduce the following proposition.

Proposition 4.3.3. *Let Φ be a Følner sequence in G, $A \subset G$ and $\beta > 0$. Then the following statements are equivalent:*

(1) If $\overline{\mathrm{d}}_\Phi(A) > \ell\beta$, then A contains $B + B + t$ for some $t \in G$ and some infinite set $B \subset G$.

(2) If $\overline{\mathrm{d}}_\Phi(A) > \beta + \frac{\ell - 1}{\ell}$ then A contains $B + B$ for some infinite set $B \subset G$.

(3) If $\overline{\mathrm{d}}_\Phi(A \cap 2G) > \beta$ then A contains $B + B$ for some infinite set $B \subset G$.

Proof. (1) $\implies$ (3): If $\overline{\mathrm{d}}_\Phi(A \cap 2G) > \beta$, then we define $\tilde{A} = \bigsqcup_{i=1}^{\ell} (A \cap 2G) + g_i$, and we have $\overline{\mathrm{d}}_\Phi(\tilde{A}) > \ell\beta$. By (1), $\tilde{A}$ contains a $B + B + t$ for some infinite $B \subset G$ and $t \in G$. By Lemma 4.3.2 we can reduce to the case where $t = g_i$ for some $i \in \{1, \dots, \ell\}$. Using Lemma 4.3.1, we can also assume that $B + B \subset 2G$. Therefore, we have that $B + B + g_i \subset (A \cap 2G) + g_i$ which concludes the proof.

(3) $\implies$ (1): If $\overline{\mathrm{d}}_\Phi(A) > \ell\beta$, then, by sub-additivity of the density, $\overline{\mathrm{d}}_\Phi(A \cap (2G + g_i)) > \beta$ for some $i \in \{1, \dots, \ell\}$. By translation invariance of the density, we see that $\overline{\mathrm{d}}_\Phi((A - g_i) \cap 2G) > \beta$ and therefore by (3), A contains $B + B + g_i$ for some $B \subset G$ infinite.

(2) $\Longrightarrow$ (3): If $\overline{\mathrm{d}}_\Phi(A\cap 2G) > \beta$, then $\overline{\mathrm{d}}_\Phi((A\cap 2G)\cup(G\backslash 2G)) > \beta + \frac{\ell-1}{\ell}$. Therefore, by (2), $(A\cap 2G)\cup(G\backslash 2G)$ contains $B+B$ and we conclude using Lemma 4.3.1.

(3) $\Longrightarrow$ (2): If $\overline{\mathrm{d}}_\Phi(A) > \beta + \frac{\ell-1}{\ell}$ then, using sub-additivity of the density and the fact that each coset has density $\frac{1}{\ell}$, we get $\overline{\mathrm{d}}_\Phi(A\cap 2G) > \beta$, so we conclude using (3). □

From Proposition 4.3.3, it is now immediate that (1), (2) and (3) of Theorem 4.2.2 are equivalent. Therefore, it suffices to prove (2) in order to establish the theorem.

4.3.2 Dynamical reformulation of Theorem 4.2.2

To prove Theorem 4.2.2, we follow an ergodic-theoretic approach. In particular, we translate the problem of finding infinite sumset configurations in subsets of a group G to a statement in ergodic theory, and particularly to the existence of Erdős progressions in products of G-systems. These ideas and basic tools necessary for their implementation were developed in [43] and [41] in the context of finding infinite configurations in $\mathbb{N}$, and were subsequently exploited for finding other patterns in $\mathbb{N}$ (see Chapter 3) and generalized in the context of countable amenable groups (see [11]).

Let $(G,+)$ be an abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. For the rest of this section Σ denotes the space $\{0,1\}^G$, and it is endowed with the product topology, so that it becomes a compact metrizable space. We also consider the shift action $S\colon \Sigma \to \Sigma$ given by $S_g(x(h)) = x(h+g)$, for any $h, g \in G$, $x = (x(g))_{g\in G} \in \Sigma$, and note that S is an action of G on X by homeomorphisms.

We use the following variant of Furstenberg's correspondence principle to reduce Theorem 4.2.2 to a dynamical statement. This variant crucially exploits the special structure of q.i.d. Følner sequences. A similar version appears in Lemma 3.3.3.

Lemma 4.3.4. *Let $A \subset G$ and Φ be any Følner sequence in G that is q.i.d. with ratio α_Φ. Then there exist a $\beta \geq \alpha_\Phi$, an ergodic G-system $(\Sigma\times\Sigma, \mu, S^2\times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ', such that $(a,a) \in \mathtt{gen}(\mu, \Phi')$, $A = \{g \in G\colon S_g a \in E\}$ and*

$$\ell\mu(\Sigma\times E) + \mu(E\times\Sigma) \geq \frac{\ell+r}{\beta}\left(\overline{\mathrm{d}}_\Phi(A) - 1\right) + \ell + 1. \tag{4.3.1}$$

Proof. By definition, and passing to a subsequence of (Φ_N) if necessary, we may

assume that there exists some $\beta \geq \alpha_\Phi$ so that

$$\lim_{N\to\infty} \frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|} = \beta \quad \text{and} \quad \overline{\mathrm{d}}_\Phi(A) = \lim_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|}.$$

Associate to the set A a point $a \in \Sigma = \{0,1\}^G$ via

$$a(g) = \begin{cases} 1, & \text{if } g \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Define the clopen set $E = \{x \in \Sigma \colon x(e_G) = 1\}$ and observe that, by construction, $A = \{g \in G \colon S_g a \in E\}$. Since Φ is quasi-invariant with respect to doubling, by Lemma A.0.5 we have that $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ given by $\Psi_N = \Phi_N/2 \cap \Phi_N$ is a Følner sequence, and then we can define the sequence of Borel probability measures $(\mu_N)_{N\in\mathbb{N}}$ on $\Sigma \times \Sigma$ given by

$$\mu_N = \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \delta_{(S_{2g}\times S_g)(a,a)}.$$

We let μ' be a weak* accumulation point of $(\mu_N)_{N\in\mathbb{N}}$, and then it is easy to see that μ' is an $(S^2 \times S)$-invariant measure. Recall that $\lim_{N\to\infty} \frac{|\Psi_N|}{|\Phi_N|} = \beta$.

For each N, $\Psi_N \subset \Phi_N$, so we have

$$\mu_N(\Sigma \times E) = \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \delta_{S_g a}(E) \geq \frac{|\Phi_N|}{|\Psi_N|} \frac{1}{|\Phi_N|}(|A \cap \Phi_N| - |\Phi_N| + |\Psi_N|),$$

and then sending $N \to \infty$ yields

$$\mu'(\Sigma \times E) \geq \frac{1}{\beta}\left(\overline{\mathrm{d}}_\Phi(A) - 1 + \beta\right) = \frac{\overline{\mathrm{d}}_\Phi(A)}{\beta} - \frac{1}{\beta} + 1. \tag{4.3.2}$$

From Lemma A.0.3 we know that $N \mapsto F_N = \bigcup_{g\in\Psi_N} g + \ker(D) \supset \Psi_N$ is a Følner in G, and $\frac{|F_N|}{|\Psi_N|} \to 1$ as $N \to \infty$. From the definition of F_N we get

$$\sum_{g\in F_N} \delta_{S_{2g}a}(E) = r \sum_{g\in 2\Psi_N} \delta_{S_g a}(E),$$

and therefore

$$\left| \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \delta_{S_{2g}a}(E) - \frac{r}{|\Psi_N|} \sum_{g\in 2\Psi_N} \delta_{S_g a}(E) \right| \leq \frac{|F_N \setminus \Psi_N|}{|\Psi_N|}$$

which goes to 0 as $N \to \infty$. In addition, $2\Psi_N = 2(\Phi_N/2 \cap \Phi_N) \subset \Phi_N \cap 2\Phi_N \subset \Phi_N$, for each $N \in \mathbb{N}$, and so combining with the previous we get

$$\begin{aligned}\mu_N(E \times \Sigma) &= \frac{1}{|\Psi_N|} \sum_{g \in \Psi_N} \delta_{S_{2g}a}(E) = \frac{r}{|\Psi_N|} \sum_{g \in 2\Psi_N} \delta_{S_g a}(E) + \mathrm{o}_{N\to\infty}(1) \\ &\geq r \frac{|\Phi_N|}{|\Psi_N|} \frac{1}{|\Phi_N|} (|A \cap \Phi_N| - |\Phi_N| + |2\Psi_N|) + \mathrm{o}_{N\to\infty}(1),\end{aligned}$$

and then sending $N \to \infty$ yields

$$\mu'(E \times \Sigma) \geq \frac{r}{\beta} \left(\overline{\mathrm{d}}_\Phi(A) - 1 + \frac{\beta}{r} \right) = \frac{r \overline{\mathrm{d}}_\Phi(A)}{\beta} - \frac{r}{\beta} + 1, \tag{4.3.3}$$

where we used that by Lemma A.0.3, $\frac{|2\Psi_N|}{|\Psi_N|} \to \frac{1}{r}$, so $\frac{|2\Psi_N|}{|\Phi_N|} \to \frac{\beta}{r}$ as $k \to \infty$. Combining (4.3.2) and (4.3.3) we obtain (4.3.1) for μ'. The conclusion of the lemma follows similarly to the proof of Lemma 3.3.3 (for the ergodic decomposition theorem for group actions see [55, Theorem 6.6] and the analogue of [19, Proposition 3.9] is straightforward). □

We now state the main dynamical result, which, along with Lemma 4.3.4 and Lemma 4.3.6 below, allows us to prove Theorem 4.2.2. For that, we need the notion of Erdős progressions, which were introduced in $\mathbb{N}$ by the authors in [41]. Given a topological G-system (X, T), a triple $(x_0, x_1, x_2) \in X^3$ is called a (3-term) Erdős progression if there exists an infinite sequence $(g_n)_{n\in\mathbb{N}}$ in G (that is, the set $\{g_n : n \in \mathbb{N}\}$ is infinite) such that $(T_{g_n} \times T_{g_n})(x_0, x_1) \to (x_1, x_2)$ as $n \to \infty$.

Theorem 4.3.5. *Let (X, μ, T) be an ergodic G-system, $a \in \mathtt{gen}(\mu, \Phi)$ for some Følner sequence Φ in G, and $E_1, E_2 \subset X$ be open sets satisfying*

$$\ell \mu(E_2) + \mu(E_1) > \ell. \tag{4.3.4}$$

Then, there exists an Erdős progression (a, x_1, x_2) such that $(x_1, x_2) \in E_1 \times E_2$.

We note here that Theorem 4.3.5 was already suggested by Tao (see the discussion after Theorem 7 in [59]). We postpone the proof of Theorem 4.3.5 to Section 4.4.2.

Lemma 4.3.6 ([11, Lemma 3.4])**.** *Let (X, T) be a topological G-system and let $E, F \subset X$ be open. Assume there exists an Erdős progression $(x_0, x_1, x_2) \in X^3$ with $x_1 \in E$ and $x_2 \in F$. Then, there exists an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset \{g \in G : T_g(x_0) \in E\}$ such that $B \oplus B = \{b_n + b_m : n, m \in \mathbb{N}, n \neq m\} \subset \{g \in G : T_g(x_0) \in F\}$.*

As Lemma 4.3.6 suggests, the existence of the Erdős progressions in the context of Theorem 4.3.5 is what allows us to recover the combinatorial statements of Theorem 4.2.2. Indeed, the proof that Theorem 4.3.5 implies part (2) of Theorem 4.2.2 – and thus the whole theorem in view of Proposition 4.3.3 – is completely analogous to the proof that Theorem 3.3.1 implies Theorem 1.3.1, and as such is omitted.

4.4 Measures on Erdős progressions and the proof of Theorem 4.3.5

In this section we prove Theorem 4.3.5. In Section 4.4.1 we collect some useful tools, and then in Section 4.4.2 we present the proof of the theorem.

4.4.1 Erdős progressions, measures and their properties

Again, let us fix a countable abelian group $(G,+)$ with $\ell = [G : 2G] < \infty$, and $g_1 = e_G, g_2, \dots, g_\ell \in G$ such that $G = \bigsqcup_{i=1}^{\ell}(g_i + 2G)$. Moreover, we fix an ergodic G-system (X, μ, T) admitting a continuous factor map $\pi : X \to Z$ to its Kronecker factor (Z, m, R), a Følner sequence Φ in G and a point $a \in \mathbf{gen}(\mu, \Phi)$.

In Theorem 4.3.5 we care about Erdős progressions with first coordinate equal to a. To this end, we utilize a natural measure σ_a on $X \times X$ with the property that σ_a-almost every pair (x_1, x_2) is such that (a, x_1, x_2) projects under the factor map π to an Erdős progression $(\pi(a), \pi(x_1), \pi(x_2))$ in the Kronecker factor Z. Although the definition of this measure is not necessary here, we include it for completeness.

Let $z \mapsto \eta_z$ denote the disintegration of μ over the factor map π (for details see [55, Section 6]) and for every $(x_1, x_2) \in X \times X$, consider the measure

$$\lambda_{(x_1,x_2)} = \int_Z \eta_{z+\pi(x_1)} \times \eta_{z+\pi(x_2)} \, dm(z) \tag{4.4.1}$$

on $X \times X$. Then $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a *continuous ergodic decomposition* of $\mu \times \mu$, that is, a disintegration of $\mu \times \mu$ where the measures $\lambda_{(x_1,x_2)}$ are ergodic for $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$, and the map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is continuous in the weak* topology. We define the measure σ_a on $X \times X$ via

$$\sigma_a = \int_Z \eta_z \times \eta_{2z-\pi(a)} \, dm(z) = \int_Z \eta_{\pi(a)+z} \times \eta_{\pi(a)+2z} \, dm(z). \tag{4.4.2}$$

The previous can be found in [41] for the case of $\mathbb{N}$ and in [11] for general amenable groups G.

Let us denote by $\pi_1, \pi_2 \colon X \times X \to X$ the projections $(x_1, x_2) \mapsto x_1$, $(x_1, x_2) \mapsto x_2$ respectively and by $\pi_i \sigma_a$ the pushforward of σ_a under π_i, $i = 1, 2$.

Proposition 4.4.1. *Let (x_1, x_2) be a point in $X \times X$ and $\lambda_{(x_1,x_2)}$, σ_a be the measures on $X \times X$ defined respectively in* (4.4.1) *and* (4.4.2)*. Then*

(1) $\pi_1 \sigma_a = \mu$ and $\frac{1}{\ell} \sum_{i=1}^{\ell} T_{g_i} \pi_2 \sigma_a = \mu$.

(2) For σ_a-almost every $(x_1, x_2) \in X \times X$ we have that $(x_1, x_2) \in \boldsymbol{supp}(\lambda_{(a,x_1)})$.

(3) There exists a Følner sequence Ψ, such that for μ-almost every $x_1 \in X$ the point (a, x_1) belongs to $\boldsymbol{gen}(\lambda_{(a,x_1)}, \Psi)$.

The fact that $\pi_1 \sigma_a = \mu$ is immediate from the definition. The proofs of (2) and (3) can be found respectively in [41, Lemma 3.7 and Proposition 3.11] and [41, Lemma 3.12] for the case of $\mathbb{N}$, and in [11, Theorem 4.9 and Lemma 4.14] and [11, Theorem 4.10 and Lemma 4.14] for the case of more general groups G. Hence we only need to prove that $\frac{1}{\ell} \sum_{i=1}^{\ell} T_{g_i} \pi_2 \sigma_a = \mu$. To do this, we need the following lemma, which asserts that we can find a subcollection $(g_{i_j})_j$ of $\{g_1, \dots, g_\ell\}$ so that Z can be written as a disjoint union of the cosets $\theta(g_{i_j}) + 2Z$, where $\theta \colon G \to Z$ is as in equation (2.5.1).

Lemma 4.4.2. *There exist an integer k with $k \mid \ell$ and integers $i_1, \dots, i_k \in \{1, \dots, \ell\}$ such that*

$$Z = \bigsqcup_{j=1}^{k} (\theta(g_{i_j}) + 2Z). \tag{4.4.3}$$

Moreover, for each $1 \leq j \leq k$, we have $|\{1 \leq i \leq \ell \colon \theta(g_i) + 2Z = \theta(g_{i_j}) + 2Z\}| = \ell/k$.

Proof. From the assumptions on G and θ we have that

$$Z = \overline{\theta(G)} = \bigcup_{i=1}^{\ell} \left(\theta(g_i) + 2\overline{\theta(G)}\right) = \bigcup_{i=1}^{\ell} (\theta(g_i) + 2Z).$$

The sets $\theta(g_i) + 2Z$ are cosets of $2Z$ in Z, hence any two such sets either coincide or they are disjoint. Consider a $i_1, \dots, i_k \in \{1, \dots, \ell\}$ for some $1 \leq k \leq \ell$ such that the cosets $\theta(g_{i_j}) + 2Z$ are pairwise distinct, and then (4.4.3) follows. It remains to show that $k \mid \ell$ and that for each $1 \leq j \leq k$, we have $|\{1 \leq i \leq \ell \colon \theta(g_i) + 2Z = \theta(g_{i_j}) + 2Z\}| = \ell/k$. The homomorphism $\theta : G \to Z$ induces a homomorphism $\widetilde{\theta} : G/2G \to Z/2Z$, mapping $g_i + 2G$ to $\theta(g_i) + 2Z$ for each $1 \leq i \leq \ell$. It follows by (4.4.3) that $\widetilde{\theta}$ is a surjective homomorphism of finite groups. Then, by the first

isomorphism theorem, we have that

$$Z/2Z \simeq \left(G/2G\right)/\ker(\widetilde{\theta}),$$

hence $k = \ell/|\ker(\widetilde{\theta})|$. This implies that $k \mid \ell$. Moreover, for fixed $1 \leq j \leq k$, the above isomorphism implies that there exist exactly $|\ker(\widetilde{\theta})| = \ell/k$ integers $1 \leq i \leq \ell$ such that $\theta(g_i) + 2Z = \theta(g_{i_j}) + 2Z$. This concludes the proof. □

We are now ready to prove (1) of Proposition 4.4.1.

Proof of (1) *of Proposition 4.4.1.* Let k and $i_1, i_2, \ldots, i_k$ as in Lemma 4.4.2. It is immediate from Lemma 4.4.2 that $m(2Z) = 1/k$. Now for each $u \in \{1, \ldots, \ell\}$ we define

$$m_{2,u} = k \cdot m|_{2Z+\theta(g_u)}.$$

Let m_2 denote the unique probability Haar measure on $2Z$. The pushforward Dm of the Haar measure m under the doubling map is translation invariant in $2Z$, and it is a probability measure on $2Z$, so $m_2 = Dm$. Also, it is not difficult to see that the probability measure $k \cdot m|_{2Z}$ is also translation invariant in $2Z$, so after all, $m_2 = Dm = k \cdot m|_{2Z}$. Now for each u, let D_u be the map sending z to $2z + \theta(g_u)$. Using the previous we then have that $m_{2,u} = D_u m$.

We first prove that for each $u \in \{1, \ldots, \ell\}$ we have

$$\frac{1}{k} \sum_{j=1}^{k} R_{g_{i_j}} m_{2,u} = m. \tag{4.4.4}$$

Fix $u \in \{1, \ldots, \ell\}$. There exists a rearrangement $(i'_j)_{1 \leq j \leq k}$ of $(i_j)_{1 \leq j \leq k}$ such that for each j we have $2Z + \theta(g_{i'_j}) = 2Z + \theta(g_{i_j}) + \theta(g_u)$. Using Lemma 4.4.2 and the definition of $m_{2,u}$, we have that for any measurable $C \subset Z$,

$$\begin{aligned} m(C) &= m\left(C \cap \left(\bigsqcup_{j=1}^{k} (\theta(g_{i'_j}) + 2Z)\right)\right) = m\left(\bigsqcup_{j=1}^{k} (C \cap (\theta(g_{i_j}) + \theta(g_u) + 2Z))\right) \\ &= \sum_{j=1}^{k} m(C \cap (\theta(g_{i_j}) + \theta(g_u) + 2Z)) = \sum_{j=1}^{k} m((C - \theta(g_{i_j})) \cap (2Z + \theta(g_u))) \\ &= \frac{1}{k} \sum_{j=1}^{k} m_{2,u}(C - \theta(g_{i_j})) = \frac{1}{k} \sum_{j=1}^{k} R_{g_{i_j}} m_{2,u}(C). \end{aligned}$$

Now we prove that for each $u \in \{1, \dots, \ell\}$ we have

$$\frac{1}{\ell} \sum_{i=1}^{\ell} R_{g_i} m_{2,u} = m. \tag{4.4.5}$$

We fix $u \in \{1, \dots, \ell\}$. By the last assertion of Lemma 4.4.2, for each $1 \leq j \leq k$, we can consider $1 \leq i_1^{(j)}, \dots, i_{\ell/k}^{(j)} \leq \ell$ such that for each $1 \leq v \leq \ell/k$, we have $\theta(g_{i_j}) + 2Z = \theta(g_{i_v^{(j)}}) + 2Z$, which implies that $R_{g_{i_j}} m_{2,u} = R_{g_{i_v^{(j)}}} m_{2,u}$. Moreover, (4.4.3) implies that $\{\{i_v^{(j)} \colon 1 \leq v \leq \ell/k\} \colon 1 \leq j \leq k\}$ is a partition of $\{1, \dots \ell\}$. Combining all the above and using (4.4.4), we have that

$$m = \frac{1}{k} \sum_{j=1}^{k} R_{g_{i_j}} m_{2,u} = \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{\ell/k} \sum_{v=1}^{\ell/k} R_{g_{i_v^{(j)}}} m_2 \right) = \frac{1}{\ell} \sum_{i=1}^{\ell} R_{g_i} m_{2,u},$$

proving (4.4.5).

Now we can conclude the proof. In view of (4.4.3), there is $u \in \{1, \dots, k\}$ and $z_0 \in Z$ such that $\pi(a) = \theta(g_u) + 2z_0$. First, we compute $\pi_2 \sigma_a$ and using the translation invariance of m we have that

$$\pi_2 \sigma_a = \int_Z \eta_{2z+\pi(a)} \, \mathrm{d}m(z) = \int_Z \eta_{2z+\theta(g_u)} \, \mathrm{d}m(z) = \int_Z \eta_z \, \mathrm{d}m_{2,u}(z), \tag{4.4.6}$$

where the last equality follows from the fact that $m_{2,u} = D_u m$. Combining (4.4.5) and (4.4.6), we obtain that

$$\begin{aligned} \frac{1}{\ell} \sum_{i=1}^{\ell} T_{g_i} \pi_2 \sigma_a &= \frac{1}{\ell} \sum_{i=1}^{\ell} \int_Z T_{g_i} \eta_z \, \mathrm{d}m_{2,u}(z) = \frac{1}{\ell} \sum_{i=1}^{\ell} \int_Z \eta_{R_{g_i}(z)} \, \mathrm{d}m_{2,u}(z) \\ &= \int_Z \eta_z \, \mathrm{d}\left(\frac{1}{\ell} \sum_{i=1}^{\ell} R_{g_i} m_{2,u} \right)(z) = \int_Z \eta_z \, \mathrm{d}m(z) = \mu. \end{aligned}$$

□

Using Proposition 4.4.1 we can guarantee the existence of many Erdős progressions starting at the point a.

Proposition 4.4.3. *Let (X, μ, T) be an ergodic G-system and assume there is a continuous factor map $\pi \colon X \to Z$ to its Kronecker factor. Let $a \in \boldsymbol{gen}(\mu, \Phi)$, for some Følner sequence Φ. Then for σ_a-almost every $(x_1, x_2) \in X \times X$, the point (a, x_1, x_2) is an Erdős progression.*

The proof of Proposition 4.4.3 is analogous to the proof of Proposition 3.4.3.

4.4.2 The proof of Theorem 4.3.5

We are now ready to prove Theorem 4.3.5. We will only prove the following special case, but the reader should easily be able to deduce the general case following the arguments of Section 3.4.2, with the role of Proposition 3.4.1 being replaced by [11, Proposition 3.7].

Theorem 4.4.4. *Let (X, μ, T) be an ergodic G-system admitting a continuous factor map to its Kronecker factor, $a \in \mathtt{gen}(\mu, \Phi)$ for some Følner sequence Φ in G, and $E_1, E_2 \subset X$ be open sets satisfying*

$$\ell\mu(E_2) + \mu(E_1) > \ell. \tag{4.4.7}$$

Then, there exist an Erdős progression (a, x_1, x_2) such that $(x_1, x_2) \in E_1 \times E_2$.

Proof. From Proposition 4.4.3, for σ_a-almost every $(x_1, x_2) \in X \times X$, the point (a, x_1, x_2) is an Erdős progression. Hence it suffices to verify that $\sigma_a(E_1 \times E_2) > 0$. As

$$E_1 \times E_2 = (E_1 \times X) \cap (X \times E_2),$$

this reduces to showing that

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) > 1.$$

Now,

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) = \pi_1\sigma_a(E_1) + \pi_2\sigma_a(E_2)$$

and by (1) of Proposition 4.4.1, the right hand side equals

$$\mu(E_1) + \ell\mu(E_2) - \pi_2\sigma_a(T_{g_2}^{-1}E_2) - \cdots - \pi_2\sigma_a(T_{g_\ell}^{-1}E_2).$$

Therefore,

$$\sigma_a(E_1 \times X) + \sigma_a(X \times E_2) \geq \mu(E_1) + \ell\mu(E_2) - (\ell - 1) > 1,$$

where the last inequality follows by (4.4.7). □

4.5 The proof of Theorem 4.2.3: optimality of the lower bounds

The goal of this section is to prove Theorem 4.2.3, so that the bounds that we get from Theorem 4.2.2 can be tested to be optimal for any possible value of ℓ and r. We begin by proving that ℓ and r can only be powers of 2.

Lemma 4.5.1. *If G is an abelian group with $\ell = [G\colon 2G]$ and $r = |\ker(D)|$ then*

- *if $\ell < \infty$, there exists $d_1 \in \mathbb{N}_0$ such that $\ell = 2^{d_1}$ and*
- *if $r < \infty$, there exists $d_2 \in \mathbb{N}_0$ such that $r = 2^{d_2}$.*

Proof. Since G is an abelian group, $G/2G$ and $\ker(D)$ are also abelian groups and therefore they have a $\mathbb{Z}$-module structure. Furthermore, $G/2G$ and $\ker(D)$ have an $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$-module structure. Indeed, if $g \in \ker(D)$ then $2g = g + g = 0$ by definition. Likewise, if $h \in G/2G$ then $h = \gamma + 2G$ for some $\gamma \in G$ and $2h = 2\gamma + 2G = 0 + 2G$ which is the identity element. Since $\mathbb{F}_2$ is a field, then $G/2G$ and $\ker(D)$ are vector spaces over $\mathbb{F}_2$, in particular they are isomorphic to $\bigoplus_{i\in I_1} \mathbb{F}_2$ and $\bigoplus_{i\in I_2} \mathbb{F}_2$ for some index set I_1 and I_2 respectively.

We conclude by observing that $\ell < \infty$ (respectively $r < \infty$) if and only if $|I_1| < \infty$ (respectively $|I_2| < \infty$) and in that case $\ell = 2^{|I_1|}$ (respectively $r = 2^{|I_2|}$). □

In what follows, for each possible value of ℓ and r we provide a group G, a Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ and a subset A of that group the density of which achieves the bound established in Theorem 4.2.3, while not containing an infinite subset of the form $B + B$. The main idea is to reverse-engineer the proof of Lemma 4.3.4 and construct a subset $A \subset G$ whose density achieves the bounds derived from that result. Even though this was done for the group $\mathbb{Z}$ (see Section 3.5) and is directly generalized to $\mathbb{Z}^d$ in Section 4.5.2, the construction becomes less clear for other groups. For example, in the case of $\mathbb{Z}^{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$ we must consider a Følner sequence different from the product Følner sequence to properly account for the distinct behavior that the doubling map induces in this group, where, informally speaking, it expands along the $\mathbb{Z}^{d_1}$ coordinates and contracts along the $(\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$ coordinates; see Sections 4.5.4 and 4.5.5 for more details.

We highlight that as a consequence of Proposition 4.3.3, if we show that one of the bounds in Theorem 4.2.2 is sharp, then the other two are also sharp. Thus, for each value of ℓ and r we shall provide an example that achieves optimal density for (2) of Theorem 4.2.2.

4.5.1 Case $\ell = 1$ and $r = 1$

We recall that for a prime number p, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and for any group G, G^ω is the group of sequences $(g_i)_{i\in\mathbb{N}}$ of elements in G such that $g_i \neq e_G$ for finitely many $i \in \mathbb{N}$. We construct this example in $\mathbb{F}_p^\omega$ for some odd prime p, such that there is a set $E_p \subset \mathbb{F}_p$ with $E_p \cap 2E_p = \emptyset$ and $|E_p| = (p-1)/2$. This is the case for $p = 3$ with $E_3 = \{1\}$, and $p = 11$ with $E_{11} = \{1, 3, 4, 5, 9\}$. However, there is no such set E_p when $p = 7$.[1] Let p be an odd prime with this property.

We denote $\Phi_N = \{x \in \mathbb{F}_p^\omega : x_i = 0 \text{ for all } i > N\}$, $\Phi_0 = \{0\}$, and by e_N we denote the canonical vector, that is $(e_N)_i = 1$ if $i = N$ and $(e_N)_i = 0$ otherwise. Notice that $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence in $\mathbb{F}_p^\omega$ with $\alpha_\Phi = \alpha_{\mathbb{F}_p^\omega} = 1$. For each N, let $A_N = \bigsqcup_{i\in E_p} \Phi_{N-1} + i \cdot e_N$, and take

$$A = \bigsqcup_{N\geq 1} A_N = \bigsqcup_{N\geq 1} \left(\bigsqcup_{i\in E_p} \Phi_{N-1} + i \cdot e_N \right) \tag{4.5.1}$$

Notice that

$$\frac{|A \cap \Phi_N|}{|\Phi_N|} = \frac{p-1}{2} \cdot \frac{1 + p + \cdots + p^{N-1}}{p^N} = \frac{p-1}{2} \left(\frac{1 - 1/p^N}{p-1} \right).$$

Therefore, taking limit as $N \to \infty$ we get $\mathrm{d}_\Phi(A) = \frac{1}{2} = 1 - \frac{\alpha_{\mathbb{F}_p^\omega}}{\ell_{\mathbb{F}_p^\omega} + r_{\mathbb{F}_p^\omega}}$.

Lemma 4.5.2. *If $B + B \subset A$ for A as in (4.5.1) and $B \subset \mathbb{F}_p^\omega$, then B is finite.*

Proof. Suppose there is an infinite $B \subset \mathbb{F}_p^\omega$ such that $B + B \subset A$. For $x \in \mathbb{F}_p^\omega$, we denote by $\iota(x)$ the greatest index i such that $x_i \neq 0$. We can take a subset $\{b(j)\}_{j\in\mathbb{N}} = B' \subset B$ such that $\iota(b(j)) < \iota(b(j+1))$ for all $j \in \mathbb{N}$.

Let $b, b' \in B'$ distinct, without loss of generality $\iota(b') < \iota(b)$. Since $b + b' \in A$, there exists $N \geq 1$ such that $b + b' \in A_N$. In particular, $\iota(b + b') = N$ and $(b + b')_N \in E_p$. But since $\iota(b') < \iota(b)$, we have that $\iota(b) = N$, $b'_N = 0$ and $b_N \in E_p$. Finally, $\iota(2b) = N$, $2b_N \in 2E_p$ which implies $2b \in \bigcup_{i\in 2E_p} (\Phi_{N-1} + i \cdot e_N)$ but this set is disjoint from A, contradicting the fact that $2b \in A$. □

Lemma 4.5.2 concludes the proof of the bound's sharpness in $\mathbb{F}_p^\omega$.

Remark. Another natural example of a group with $r = \ell = 1$ is the rational numbers $\mathbb{Q}$. We have constructed an example of a Følner sequence Φ and a set

[1] If $\pi_p : \mathbb{F}_p^X \to \mathbb{F}_p^X$ is the permutation induced by multiplication by 2 in the group of units of $\mathbb{F}_p$, then such a set E_p exists if and only if the disjoint decomposition of π_p consists exclusively of odd cycles. For instance, $\pi_3 = (1\ 2)$, $\pi_{11} = (1\ 2\ 4\ 8\ 5\ 10\ 9\ 7\ 3\ 6)$, but $\pi_7 = (1\ 2\ 4)(3\ 6\ 5)$.

A in $\mathbb{Q}$ such that $\alpha_\Phi = 1$, $\underline{d}_\Phi(A) \geq \frac{1}{2}$ and A contains no $B + B$ for an infinite set $B \subset \mathbb{Q}$, therefore proving optimality of the lower bounds in Theorem 4.2.2 for $(\mathbb{Q}, +)$. However, since the construction is somewhat lengthy, we decided to not include it in the paper.

4.5.2 Case $r = 1$ and $\ell = 2^d$, $d \geq 1$

The following examples are defined in $\mathbb{Z}^d$ for $d \geq 1$ and they are a direct generalization of the one given in Section 3.5 that was constructed in $\mathbb{N}$. For a vector $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we denote $\|x\|_\infty = \max_{i=1,\ldots,d} |x_i|$.

Lemma 4.5.3. *Let $A' \subset \mathbb{N}$ be such that if $B' + B' + t' \subset A'$ for some $B' \subset \mathbb{N}$ and $t' \in \mathbb{N}$, then B' is finite. Let $A = \{x \in \mathbb{Z}^d : \|x\|_\infty = a, a \in A'\}$. Then, if there exist $B \subset \mathbb{Z}^d$ and $t \in \mathbb{Z}^d$ such that $B + B + t \subset A$, it necessarily holds that B is finite.*

Proof. First, reasoning by contradiction, suppose that there exists an element $t \in \mathbb{Z}^d$ and an infinite set $B \subset Z^d$ such that $B + B + t \subset A$. By pigeonhole principle, we know that there exists an infinite subset of B (that we keep calling B) all the elements of which have the same sign in each coordinate, that is, for every $x, y \in B$, $x_i \cdot y_i \geq 0$ for all $i \in \{1, \ldots, d\}$. Without loss of generality we can assume that the coordinates of t have the same signs as the respective coordinates of the elements in B.

Using again the pigeonhole principle, we can suppose that there exists a fixed index $i_0 \in \{1, \ldots, d\}$ such that $\|x\|_\infty = x_{i_0}$ for all $x \in B$. Now, choose two arbitrary elements $x, y \in B$. Since $x + y + t \in A$, there exists $a' \in A'$ such that $a' = |x_{i_0} + y_{i_0} + t_{i_0}|$, in particular, by symmetry of A we can suppose that the i_0-th coordinate of every element in B is nonnegative.

After all the previous reductions, define $B_{i_0} = \{x_{i_0} : x \in B\} \subset \mathbb{N}$ and therefore, by construction, $B_{i_0} + B_{i_0} + t_{i_0} \subset A'$. Thus, B_{i_0} is finite. Finally, if $m = \max\{n \in B_{i_0}\}$ we notice that $B \subset \{x : \|x\|_\infty \leq m\}$ which implies that B is finite (contradiction). □

In this and the following sections we use the interval notation for discrete intervals, for example if $a, b \in \mathbb{R}$, we write $[a, b]$ for $[a, b] \cap \mathbb{Z}$. Consider the set

$$A' = \bigcup_{n \in \mathbb{N}} [4^n, (2 - 1/n) \cdot 4^n),$$

which, as was proven in Chapter 3, does not contain an infinite sumset of the form $B' + B' + t'$ with $B' \subset \mathbb{N}$ and $t' \in \mathbb{N}$. Let $A_d = \{x \in \mathbb{Z}^d : \|x\|_\infty = a, a \in A'\}$. Another

way of writing the set A_d (more similar to the one used in the next sections) is

$$A_d = \bigsqcup_{n\in\mathbb{N}} (-(2-1/n)\cdot 4^n, (2-1/n)\cdot 4^n)^d \backslash (-4^n, 4^n)^d.$$

Notice that $\mathbb{Z}^d$ has a natural Følner sequence given by $\Phi_N^d = [-N, N]^d$ for $N \in \mathbb{N}$ and that $\alpha_{\Phi^d} = \alpha_{\mathbb{Z}^d} = \frac{1}{2^d}$. Computing the density of A_d with respect to that Følner sequence we find

$$\begin{aligned}\overline{\mathrm{d}}(A_d) &= \lim_{N\to\infty} \frac{|[-(2-1/N)4^N, (2-1/N)4^N]^d \cap A_d|}{|[-(2-1/N)4^N, (2-1/N)4^N]^d|} \\ &= \lim_{N\to\infty} \frac{1}{(2-1/N)^d 4^{Nd}} \left((2^d-1)\frac{4^{d(N+1)}-1}{4^d-1} \right) = 1 - \frac{1}{2^d+1} = 1 - \frac{\ell_{\mathbb{Z}^d}\alpha_{\mathbb{Z}^d}}{r_{\mathbb{Z}^d}+\ell_{\mathbb{Z}^d}}.\end{aligned}$$

Using Lemma 4.5.3, the set $A_d \subset \mathbb{Z}^d$ gives us the sharpness of the corresponding bound.

4.5.3 Case $\ell = 1$ and $r = 2^d$, $d \geq 1$

In this subsection and the following ones we use the group of dyadic points in $\mathbb{R}/\mathbb{Z}$, that is, $G = \mathbb{Z}(1/2)/\mathbb{Z} = \{k/2^N \mod 1 : k, N \in \mathbb{N}\}$ and the disjoint family of subsets $(C_n)_{n\geq 0}$ given by

$$C_0 = \{0\}, C_1 = \{1/2\} \text{ and in general } C_n = \left\{ \frac{k}{2^n} : 0 \leq k < 2^n, k \text{ odd} \right\} \text{ for } n \geq 0. \tag{4.5.2}$$

Notice that $G = \bigcup_{n\geq 0} C_n = \{(2k+1)/2^n \in \mathbb{Q}/\mathbb{Z} : k, n \geq 0\}$. Also notice that

$$C_0/2 = \{0, 1/2\} = C_0 \cup C_1, \text{ and } C_n/2 = C_{n+1} \text{ for } n \geq 1. \tag{4.5.3}$$

The equation (4.5.3) is a key feature for this and the following examples. With this family of sets, one can also describe a natural Følner sequence $F = (F_N)_{N\geq 0}$ in G given by $F_N = \{K/2^N : 0 \leq k < 2^N\} = \bigsqcup_{n=0}^N C_n$. Using (4.5.3) we deduce that $F_N/2 = F_{N+1} = F_N \cup C_{N+1}$ and therefore $\alpha_F = 1$.

Consider the group G^d and notice that $\ell_{G^d} = 1$, $r_{G^d} = 2^d$. We construct something similar to the example given in the previous subsection. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be the Følner sequence in G^d defined by $\Phi_N = \underbrace{F_N \times \cdots \times F_N}_{d \text{ times}}$. As before, for each N, $\Phi_N/2 \supset \Phi_N$, which implies that $\alpha_\Phi = 1$.

Before enouncing the example we define some functions that are also useful in the following examples. Consider $\theta\colon G \to \mathbb{N}_0$ be the function such that

$$\theta\left(\frac{2k+1}{2^n}\right) = n \quad \text{for all } g = \frac{2k+1}{2^n} \in G. \tag{4.5.4}$$

That is, for every $g \in G$, $\theta(g) = n$ if and only if $g \in C_n$ (see (4.5.2)). Notice that, for $g, g' \in G$, if $\theta(g) \neq \theta(g')$, then $\theta(g+g') = \max(\theta(g), \theta(g'))$, while if $\theta(g) = \theta(g')$, then $\theta(g+g') < \theta(g)$. Using the function $\theta\colon G \to \mathbb{N}_0$ we define two functions $w\colon G^d \to N_0$ and $\eta\colon G^d \to \{1, \dots, d\}$ as follows: for $y = (y_1, \dots, y_d) \in G^d$

$$w(y) = \max\{\theta(y_j)\colon j = 1, \dots, d\} \tag{4.5.5}$$

$$\eta(y) = \min\{i \in \{1, \dots, d\}\colon \theta(y_i) = w(y)\} \tag{4.5.6}$$

where $\eta(y)$ should be interpreted as the index $i \in \{1, \dots, d\}$ where the maximum defined in $w(y)$ is achieved, but since that index is not necessarily unique we pick the minimum for convenience. For instance in G^3, $\eta\left(0, \frac{1}{8}, \frac{3}{8}\right) = 2$.

To understand the following example, it might be useful to think that we try to replicate the example in $\mathbb{Z}^d$, where now the function $\theta\colon G \to \mathbb{N}_0$ plays an analogous role to the one of the absolute value and $w\colon G^d \to \mathbb{N}_0$ to the uniform norm.

Lemma 4.5.4. *Let $A \subset (\mathbb{Z}(\frac{1}{2})/\mathbb{Z})^d$ be the set given by*

$$A = \bigcup_{n\geq 0} (F_{2n+1} \times \cdots \times F_{2n+1}) \backslash (F_{2n} \times \cdots \times F_{2n}) = \bigcup_{n \geq 0} (\Phi_{2n+1} \backslash \Phi_{2n}). \tag{4.5.7}$$

If $B + B \subset A$ for some $B \subset G^d$ then B is finite.

Proof. By contradiction, suppose that there exists an infinite set $B \subset G^d$ such that $B + B \subset A$. Notice that, using (4.5.5) and (4.5.6), we can rewrite A as

$$A = \{y \in G^d\colon w(y) \text{ is odd}\}.$$

Since B is infinite, without loss of generality, one can suppose that there exists $i \in \{1, \dots, d\}$ such that for all $b \in B$, $\eta(b) = i$. Moreover, since B is infinite and each C_n is finite, one can suppose that for every distinct $b, b' \in B$, $w(b) \neq w(b')$.

Take $b^{(1)}, b^{(2)} \in B$, and assume that $w(b^{(1)}) < w(b^{(2)})$. Then $2b^{(1)}, 2b^{(2)} \in A$, so for $w(2b^{(1)}) = w(b^{(1)}) - 1$ is odd, so $w(b^{(1)})$ is even. Using the same reasoning, $w(b^{(2)})$ is also even.

But then, since $w(b^{(1)}) < w(b^{(2)})$ and $\eta(b^{(1)}) = \eta(b^{(2)})$, we have that $w(b^{(1)} + b^{(2)}) = \theta(b_i^{(1)} + b_i^{(2)}) = \max\{\theta(b_i^{(1)}), \theta(b_i^{(2)})\} = \theta(b_i^{(2)}) = w(b^{(2)})$, which is even,

and therefore $b^{(1)} + b^{(2)}$ is not in A, which is a contradiction and we conclude the lemma. □

We conclude the sharpness of the bound given by Theorem 4.2.2 by computing the density of A,

$$\begin{aligned}\overline{\mathrm{d}}_\Phi(A) &= \lim_{N\to\infty} \frac{\sum_{n=0}^{N} |(F_{2n+1}\times\cdots\times F_{2n+1})\backslash(F_{2n}\times\cdots\times F_{2n})|}{|\Phi_{2N+1}|} \\ &= \lim_{N\to\infty} \frac{\sum_{n=0}^{N}(2^{2n+1})^d - (2^{2n})^d}{(2^{2N+1})^d} = \frac{2^d}{2^d+1} = 1 - \frac{\alpha_{G^d}}{\ell_{G^d}+r_{G^d}}.\end{aligned}$$

4.5.4 Case $\ell = 2^{d_1}$ and $r = 2^{d_2}$ with $d_1 \geq d_2 \geq 1$

In this subsection, we construct the desired examples for general $\ell, r > 1$ with $\ell \geq r$. From Lemma 4.5.1 we know that for any countable abelian group G, the quantities $r = |\ker(D)|, \ell = [G : 2G]$ are integer powers of 2. Let $d_1, d_2 \in \mathbb{N}$ with $d_1 \geq d_2$, and consider the group $G = \mathbb{Z}^{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$. Then $r = 2^{d_2}$ and $\ell = 2^{d_1}$ and $\ell \geq r$. For an element $z \in G$, we write $z = (z^{(1)}, z^{(2)})$, where $z^{(1)} \in \mathbb{Z}^{d_1}$ and $z^{(2)} \in (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$.

Let $c(N), v(N)$ be two strictly increasing sequences of natural numbers so that $c(N)$ is always even and $v(N+1) > v(N) + c(N) + 1$ for all $N \in \mathbb{N}$ (for example, take $v(N) = 3^N, c(N) = 2N$). For $k \in \mathbb{N}$, we recall that $C_k \subset \mathbb{Z}(\frac{1}{2})/\mathbb{Z}$ is the set defined in (4.5.2) and we also write $I_k = [-2^k, 2^k] \setminus \{0\}$. For each N, let

$$\Phi_N = \bigsqcup_{m=0}^{c(N)-1} I_{c(N)-m}^{d_1} \times C_{v(N)+m+1}^{d_2}. \tag{4.5.8}$$

We highlight that, in the definition of I_k we remove 0 from the discrete interval $[-2^k, 2^k]$ only to simplify the computations it what follows, but the same result is true if we do not remove it.

As $v(N) + c(N) < v(N+1)$, the Φ_N's are pairwise disjoint. We prove that $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence in G, and that $\alpha_\Phi = \frac{r}{\ell} = \min\{1, \frac{r}{\ell}\}$.

For each N,

$$|\Phi_N| = \sum_{m=0}^{c(N)-1} \left(2\cdot 2^{c(N)-m}\right)^{d_1} \left(2^{v(N)+m}\right)^{d_2} = 2^{d_1 c(N)+d_1+d_2 v(N)} \sum_{m=0}^{c(N)-1} 2^{(d_2-d_1)m}. \tag{4.5.9}$$

Calculating in (4.5.9), one sees that

$$|\Phi_N| = \begin{cases} c(N)2^{d_1c(N)+d_1v(N)+d_1}, \text{ if } d_1 = d_2 \\ 2^{d_1c(N)+d_1+d_2v(N)} \cdot \frac{1-(2^{d_2-d_1})^{c(N)}}{1-2^{d_2-d_1}}, \text{ if } d_1 \neq d_2. \end{cases} \tag{4.5.10}$$

Let $(x, y) \in G$, where $x = (x_1, \ldots, x_{d_1})$ and $y = (y_1, \ldots, y_{d_2})$. Recall the definition of $w(y)$ given in (4.5.5). Then for all integers $k_1, \ldots, k_{d_2} > w(y)$, $y + C_{k_1} \times \cdots \times C_{k_{d_2}} = C_{k_1} \times \cdots \times C_{k_{d_2}}$. In particular if $v(N) > w(y)$, then $C^{d_1}_{v(N)+m+1} + y = C^{d_1}_{v(N)+m+1}$ for every $m \in \mathbb{N}$.

For each N, by construction $v(N) \geq N$, so for $N > w(y)$, $v(N) > w(y)$, and therefore

$$\big((x, y) + \Phi_N\big) \triangle \Phi_N = \bigsqcup_{m=0}^{c(N)-1} \Big(\big(x + I^{d_1}_{c(N)-m}\big) \triangle I^{d_1}_{c(N)-m}\Big) \times C^{d_2}_{v(N)+m+1}.$$

Computing the cardinality we get

$$\big|\big(x + I^{d_1}_{c(N)-m}\big) \triangle I^{d_1}_{c(N)-m}\big| \leq 2^{d_1} \big(2^{c(N)-m}\big)^{d_1-1} \sum_{i=1}^{d_1} 2|x_i|,$$

and therefore

$$\begin{aligned} \big|\big((x, y) + \Phi_N\big) \triangle \Phi_N\big| &\leq \sum_{m=0}^{c(N)-1} 2^{d_1} \big(2^{c(N)-m}\big)^{d_1-1} \Big(\sum_{i=1}^{d_1} 2|x_i|\Big) \big(2^{v(N)+m}\big)^{d_2} \\ &= \Big(\sum_{i=1}^{d_1} 2|x_i|\Big) 2^{d_1+d_1c(N)+d_2v(N)-c(N)} \sum_{m=0}^{c(N)-1} \big(2^{d_2+1-d_1}\big)^m. \end{aligned}$$

Finally,

$$\begin{aligned} &\big|\big((x, y) + \Phi_N\big) \triangle \Phi_N\big| \leq \\ &\begin{cases} c(N)\big(\sum_{i=1}^{d_1} 2|x_i|\big) 2^{d_1+d_1c(N)+d_2v(N)-c(N)}, & \text{if } d_1 = d_2 + 1 \\ \big(\sum_{i=1}^{d_1} 2|x_i|\big) 2^{d_1+d_1c(N)+d_2v(N)-c(N)} \cdot \frac{1-(2^{d_2+1-d_1})^{c(N)}}{1-2^{d_2+1-d_1}}, & \text{else.} \end{cases} \end{aligned} \tag{4.5.11}$$

Using (4.5.10) and (4.5.11), we can prove the following lemma:

Lemma 4.5.5. *For all $d_1 \geq d_2$, $(\Phi_N)_{N\in\mathbb{N}}$ defined in* (4.5.8) *is a Følner sequence*

in $\mathbb{Z}^{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$.

Proof. If $d_1 = d_2$, then

$$\frac{|((x,y)+\Phi_N)\triangle\Phi_N|}{|\Phi_N|} \leq \frac{\sum_{i=1}^{d_1} 2|x_i|}{c(N)}(1-2^{-c(N)}) \xrightarrow{N\to\infty} 0$$

Similarly, if $d_1 = d_2 + 1$, then

$$\frac{|((x,y)+\Phi_N)\triangle\Phi_N|}{|\Phi_N|} \leq \frac{c(N)\big(\sum_{i=1}^{d_1} 2|x_i|\big)2^{d_2+1+d_2c(N)+d_2v(N)}}{(1-2^{-c(N)})2^{d_2c(N)+c(N)+d_2v(N)+d_2+2}} = \frac{\sum_{i=1}^{d_1} 2|x_i|}{1-2^{-c(N)}}\cdot\frac{c(N)}{2^{c(N)+1}}. \tag{4.5.12}$$

Since $\lim_{h\to\infty} \frac{h}{2^{h+1}} = 0$ and $c(N) \to \infty$, we have that $\frac{c(N)}{2^{c(N)+1}} \to 0$ as $N \to \infty$. Also, $1 - 2^{-c(N)} \to 1$ as $N \to \infty$, so from (4.5.12) we see that $\frac{|((x,y)+\Phi_N)\triangle\Phi_N|}{|\Phi_N|} \to 0$ as $N \to \infty$.

Finally, if $d_1 \neq d_2$ and $d_1 \neq d_2 + 1$, then $d_1 \geq d_2 + 2$ and

$$\frac{|((x,y)+\Phi_N)\triangle\Phi_N|}{|\Phi_N|} \leq \frac{\big(1-2^{d_2-d_1}\big)\big(\sum_{i=1}^{d_1} 2|x_i|\big)}{1-2^{d_2+1-d_1}} \cdot \frac{1-2^{(d_2+1-d_1)c(N)}}{1-2^{(d_2-d_1)c(N)}} \cdot 2^{-c(N)}. \tag{4.5.13}$$

Since $d_1 \geq d_2 + 2$, we have that $2^{(d_2+1-d_1)c(N)}, 2^{(d_2-d_1)c(N)} \to 0$ as $N \to \infty$, so from (4.5.13) we see that again $\frac{|((x,y)+\Phi_N)\triangle\Phi_N|}{|\Phi_N|} \to 0$ as $N \to \infty$. □

Lemma 4.5.6. *For all* $d_1 \geq d_2$*, the Følner sequence* $(\Phi_N)_{N\in\mathbb{N}}$ *defined in* (4.5.8) *has ratio* $\alpha_\Phi = \alpha_G = 2^{d_2-d_1}$.

Proof. For each N,

$$\Phi_N/2 = \bigsqcup_{m=0}^{c(N)-1} I_{c(N)-m-1}^{d_1} \times C_{v(N)+m+2}^{d_2} = \bigsqcup_{m=1}^{c(N)} I_{c(N)-m}^{d_1} \times C_{v(N)+m+1}^{d_2},$$

so $\Phi_N \setminus (\Phi_N/2) = I_{c(N)}^{d_1} \times C_{v(N)+1}^{d_2}$.

If $d_1 = d_2$, then

$$\frac{|\Phi_N \setminus (\Phi_N/2)|}{|\Phi_N|} = \frac{2^{d_1(c(N)+1)+d_1v(N)}}{c(N)2^{d_1c(N)+d_1v(N)+d_1}} = \frac{1}{c(N)} \xrightarrow{N\to\infty} 0$$

Therefore, $\frac{|\Phi_N\cap(\Phi_N/2)|}{|\Phi_N|} = 1 - \frac{|\Phi_N\setminus(\Phi_N/2)|}{|\Phi_N|} \to 1 = \frac{r}{\ell}$ as $N \to \infty$.

Similarly, if $d_1 > d_2$, then

$$\frac{|\Phi_N \setminus (\Phi_N/2)|}{|\Phi_N|} = \frac{2^{d_1(c(N)+1)+d_2v(N)}}{2^{d_1c(N)+d_1+d_2v(N)} \cdot \frac{1-(2^{d_2-d_1})^{c(N)}}{1-2^{d_2-d_1}}} = \frac{1-2^{d_2-d_1}}{1-(2^{d_2-d_1})^{c(N)}} \xrightarrow{N\to\infty} 1-2^{d_2-d_1}$$

where in the final limit we use that $\lim_{N\to\infty} 2^{(d_2-d_1)c(N)} = 0$. Thus, $\frac{|\Phi_N \cap (\Phi_N/2)|}{|\Phi_N|} \to 2^{d_2-d_1} = \frac{r}{\ell}$ as $N \to \infty$. □

To conclude this section, we build a set $A \subset G$ with $\mathrm{d}_\Phi(A) = 1 - \frac{r}{\ell(\ell+r)} = 1 - \frac{\alpha_\Phi}{\ell+r}$ so that A contains no $B+B$ for some infinite B. We highlight that for this example the density of A exists, that is the upper and lower density coincide.

Recall that for each $N \in \mathbb{N}$, by construction, $c(N)$ is even. Let

$$A_{2,N} = \bigsqcup_{m=0}^{\frac{c(N)}{2}-1} \left(I_{c(N)-2m} \cap 2\mathbb{Z}\right)^{d_1} \times C^{d_2}_{v(N)+2m+1},$$

that is $A_{2,N}$ consists of the elements of Φ_N for which every $\mathbb{Z}$-coordinates are even and their $(\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$ part belongs to $C^{d_2}_{v(N)+j}$ for an odd index $j \in \{1, \dots, c(N)\}$.

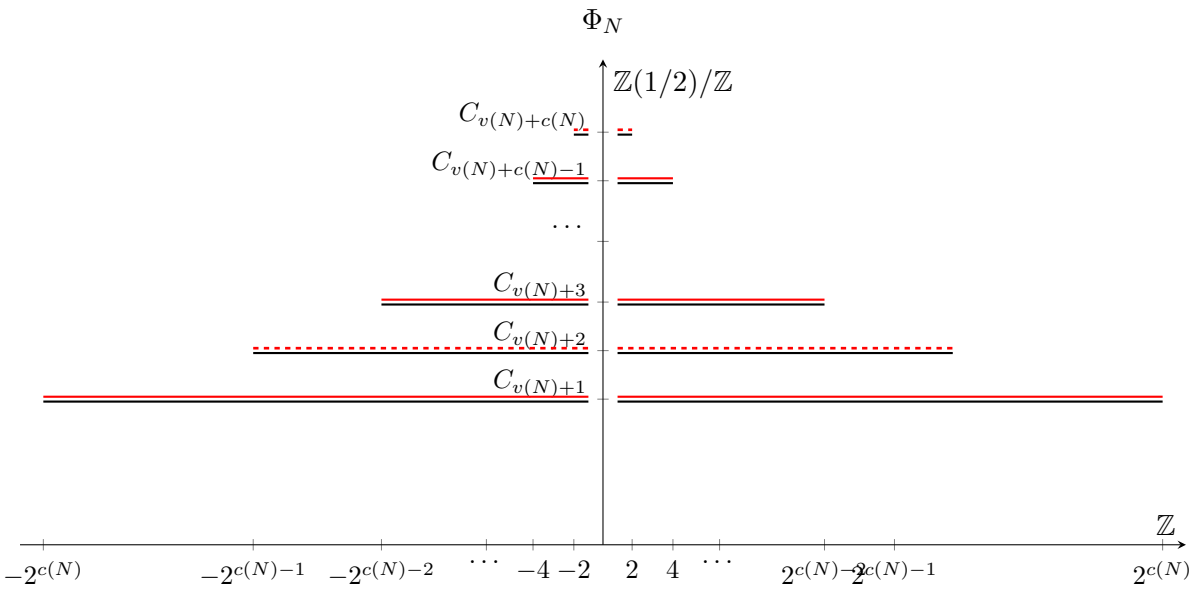
Observe that the $A_{2,N}$'s are pairwise disjoint. Let $A_2 = \bigsqcup_{N\in\mathbb{N}} A_{2,N}$. We also define a useful set that is used in this and in the next example,

$$O_{d_1} = (\mathbb{Z}^{d_1} \setminus (2\mathbb{Z})^{d_1}) = \{(x_1, \dots, x_{d_1}) \in \mathbb{Z}^{d_1} : \text{ at least one } x_i \text{ is odd}\} \tag{4.5.14}$$

and then $A_1 = O_{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$. Finally, take $A = A_1 \sqcup A_2$. Notice that

$$A \cap \Phi_N = \left[\bigsqcup_{m=0}^{c(N)-1} I^{d_1}_{c(N)-m} \cap O_{d_1} \times C^{d_2}_{v(N)+m+1}\right] \sqcup A_{2,N}. \tag{4.5.15}$$

We provide a figure below to illustrate Φ_N (in black) and $A \cap \Phi_N$ (in red) in the case $d_1 = d_2 = 1$, that is $G = \mathbb{Z} \times \mathbb{Z}(1/2)/\mathbb{Z}$. The dotted lines indicate that we only take elements with odd $\mathbb{Z}$-coordinate.



With this formula we can compute the density of A. First notice that for each $m \in \{0, 1, \ldots, c(N)-1\}$,

$$\frac{\left|I^{d_1}_{c(N)-m} \cap O_{d_1}\right|}{\left|I^{d_1}_{c(N)-m}\right|} = 1 - \frac{\left|I^{d_1}_{c(N)-m} \cap 2\mathbb{Z}^{d_1}\right|}{\left|I^{d_1}_{c(N)-m}\right|} = \frac{2^{d_1}-1}{2^{d_1}}. \tag{4.5.16}$$

Therefore,

$$\frac{\left|\bigsqcup_{m=0}^{c(N)-1}\left(I^{d_1}_{c(N)-m} \cap O_{d_1}\right) \times C^{d_2}_{v(N)+m+1}\right|}{|\Phi_N|} = \frac{2^{d_1}-1}{2^{d_1}}.$$

On the other hand, $|A_{2,N}| = \begin{cases} \frac{c(N)}{2} 2^{d_1 c(N)+d_1 v(N)} & \text{if } d_1 = d_2 \\ 2^{d_1 c(N)+d_2 v(N)} \cdot \frac{1-2^{(d_2-d_1)c(N)}}{1-2^{2(d_2-d_1)}} & \text{if } d_1 \neq d_2 \end{cases}$

Therefore, if $d_1 = d_2$, then using (4.5.10), (4.5.15), (4.5.17) and the previous calculation for $|A_{2,N}|$ one sees that

$$\frac{|A \cap \Phi_N|}{|\Phi_N|} = \frac{2^{d_1}-1}{2^{d_1}} + \frac{1}{2^{d_1+1}} = \frac{2^{d_1+1}-1}{2^{d_1+1}} = 1 - \frac{\alpha_\Phi}{\ell + r}.$$

Therefore, the density $d_\Phi(A)$ exists and is equal to $1 - \frac{\alpha_\Phi}{\ell+r}$.

Now, if $d_1 \neq d_2$, then again using (4.5.10), (4.5.15), (4.5.17) and the previous calculation for $|A_{2,N}|$ one sees that

$$\frac{|A \cap \Phi_N|}{|\Phi_N|} = \frac{2^{d_1}-1}{2^{d_1}} + \frac{1}{2^{d_1}+2^{d_2}} = \frac{2^{2d_1}+2^{d_1+d_2}-2^{d_2}}{2^{d_1}(2^{d_1}+2^{d_2})} = 1 - \frac{\alpha_\Phi}{\ell + r}.$$

Therefore, also in the case $d_1 \neq d_2$, the density $d_\Phi(A)$ exists and is equal to $1 - \frac{\alpha_\Phi}{\ell+r}$.

Now, it remains to prove that there is no infinite B so that $B+B \subset A$. Assume that there is some infinite B so that $B+B \subset A$. Since B is infinite, we may assume that all the elements of B have the same parity in their $\mathbb{Z}$-coordinates, that is if $b=(x,y)$ and $b'=(x',y')$ two elements in B, then $x_i = x_i' \mod 2$ for all $i=1,\dots,d_1$.

Fix $b=(x,y) \in B$. Then by assumption $2b \in A$ and also $2b \in (2\mathbb{Z})^{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$. Therefore $2b \in A_2$, in particular there is unique $N_1 \in \mathbb{N}$ so that $2b \in A_{2,N_1}$. Since B is infinite and $\bigcup_{j \leq N_1} A_{2,j}$ is finite, there exists $b'=(x',y') \in B$ so that $2b' \in A_{2,N_2}$ for some $N_2 > N_1$.

Then, from the definition of the $A_{2,N}$'s we see that there are $m_1 \in \{0,\dots,\frac{c(N_1)}{2}-1\}$ and $m_2 \in \{0,\dots,\frac{c(N_2)}{2}-1\}$ so that $2b \in \left(I_{c(N_1)-2m_1} \cap 2\mathbb{Z}\right)^{d_1} \times C^{d_2}_{v(N_1)+2m_1+1}$ and $2b' \in \left(I_{c(N_2)-2m_2} \cap 2\mathbb{Z}\right)^{d_1} \times C^{d_2}_{v(N_2)+2m_2+1}$. From the previous we infer that

$$x \in I^{d_1}_{c(N_1)-2m_1-1},\ y \in C^{d_2}_{v(N_1)+2m_1+2},\ x' \in I^{d_1}_{c(N_2)-2m_2-1} \ \text{ and } \ y' \in C^{d_2}_{v(N_2)+2m_2+2}.$$

By the parity assumption in B, $x+x' \in (2\mathbb{Z})^{d_1}$, so from the definition of A and since $b+b' \in A$, we obtain that $b+b' \in A_2$. Now since $v(N_2)+2m_2+2 > v(N_1)+2m_1+2$ and using the properties of the function $\theta\colon \mathbb{Z}(1/2)/\mathbb{Z} \to \mathbb{N}_0$ defined in (4.5.4), we have that for all $j=1,\dots,d_2$, $\theta(y_j+y_j') = \max\{\theta(y_j),\theta(y_j')\} = \max\{v(N_1)+2m_1+2, v(N_2)+2m_2+2\}$ and therefore $y_j+y_j' \in C^{d_2}_{v(N_2)+2m_2+2}$.

Thus $b+b' = (x+x', y+y') \in (2\mathbb{Z})^{d_1} \times C^{d_2}_{v(N_2)+2m_2+2}$, which is disjoint from A_2. This is a contradiction and therefore, there is no infinite B so that $B+B \subset A$, concluding the construction of the example in the case $\ell \geq r$.

4.5.5 Case $\ell = 2^{d_1}$ and $r = 2^{d_2}$ with $1 \leq d_1 < d_2$

In this subsection, for $d_1, d_2 \in \mathbb{N}$ with $d_1 < d_2$, we consider again the group $G = \mathbb{Z}^{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$, so $\ell = 2^{d_1}$ and $r = 2^{d_2}$, but this time we have $\ell < r$. As before, for every $z \in G$, we write $z = (z^{(1)}, z^{(2)})$, where $z^{(1)} \in \mathbb{Z}^{d_1}$ and $z^{(2)} \in (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$. For constructing the correspondent Følner sequence and the set A, again we use the sets $C_k \subset \mathbb{Z}(1/2)/\mathbb{Z}$ defined in (4.5.2), $I_k = [-2^k, 2^k] \setminus \{0\} \subset \mathbb{Z}$ and $O_{d_1} \subset \mathbb{Z}^{d_1}$ as in (4.5.14).

For this example, again we use $c(N), v(N)$ be two strictly increasing sequences of natural numbers with similar properties. In particular, $c(N)$ is even and $v(N+1) - c(N+1) > v(N) + 1$ for all $N \in \mathbb{N}$. In this case we still can take can take $v(N) = 3^N$ and $c(N) = 2N$. In this subsection, many computations are omitted, as they are very similar to the ones carried out in Section 4.5.4.

The first major change is the definition of the Følner sequence. For each N, let Φ_N be defined via

$$\Phi_N = \bigsqcup_{m=0}^{c(N)-1} I_{c(N)+m}^{d_1} \times C_{v(N)-m}^{d_2}. \tag{4.5.17}$$

As $v(N+1)-c(N+1) > v(N)+1$, the Φ_N's are pairwise disjoint. The proof that $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence in G is analogous to the one carried out in Lemma 4.5.5, so it is omitted. Similarly to Section 4.5.4 we have:

Lemma 4.5.7. *For all $d_1 < d_2$, the Følner sequence $(\Phi_N)_{N\in\mathbb{N}}$ defined in* (4.5.17) *has ratio $\alpha_\Phi = 1$.*

Using the same structure as in Section 4.5.4, we end this section by building a set $A \subset G$ such that $\mathrm{d}_\Phi(A) = 1 - \frac{1}{\ell+r} = 1 - \frac{\alpha_\Phi}{\ell+r}$ and if $B + B \subset A$ then B is finite. We highlight that, as last time, the density of the set A exists. The construction of A is also similar. First, for all $N \in \mathbb{N}$ set

$$A_{2,N} = \bigsqcup_{m=0}^{\frac{c(N)}{2}-1} \left(I_{c(N)+2m} \cap 2\mathbb{Z}\right)^{d_1} \times C_{v(N)-2m}^{d_2}.$$

Observe that the $A_{2,N}$'s are pairwise disjoint, so conveniently we define $A_2 = \bigsqcup_{N\in\mathbb{N}} A_{2,N}$. Like last time, $A_1 = O_{d_1} \times (\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$ and $A = A_1 \sqcup A_2$. The proof that A does not contain an infinite sumset $B + B$ is completely analogous to the one in the previous section. The only slightly different part in this argument is computing the density of A along Φ but this is an easy exercise.

We note here that in both Sections 4.5.4 and 4.5.5 the the Følner sequences have the same triangular shape: they are of the form $\Phi_N = \bigsqcup_m I_{c_1(m)}^{d_1} \times C_{c_2(m)}^{d_2}$, where as $c_2(m)$ grows larger, so we take more elements from $(\mathbb{Z}(1/2)/\mathbb{Z})^{d_2}$, $c_1(m)$ becomes smaller, so we have less elements from $\mathbb{Z}^{d_1}$. However, the Følner sequence Φ defined in Section 4.5.4 is not a Følner sequence in the case $\ell > r$ (because then $2^{d_2-d_1} > 1$). This is the reason why we have to consider different Φ's in the two cases.

4.6 Necessity of the assumptions in the main theorem

Herein, we show that the assumptions on the group and the Følner sequence in Theorem 4.2.2 are necessary.

4.6.1 The kernel of the doubling map has to be finite

We first prove that the assumption $r = |\ker(D)| < \infty$ is necessary.

Proposition 4.6.1. *There is a group G with $r = |\ker(D)| = \infty$, a Følner sequence Φ in G with $\alpha_\Phi = 1$, and a set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) = 1$ so that for any infinite $B \subset G$ we have $B + B \not\subset A$.*

Proof. Let $G = \mathbb{Z}(\frac{1}{2})/\mathbb{Z}$ and consider the group G^ω defined by

$$G^\omega = \bigoplus_{i\in\mathbb{N}} G = \{\boldsymbol{g} = (g_i)_{i\in\mathbb{N}} \in G^{\mathbb{N}} \mid g_i \neq 0 \text{ for finitely many } i\text{'s}\}.$$

Then $\ell_{G^\omega} = 1, r_{G^\omega} = \infty$. Recall the definition of C_n given in (4.5.2). We have $|C_0| = 1$, $|C_n| = 2^{n-1}$ for $n \in \mathbb{N}$, the sets $(C_n)_{n\in\mathbb{N}_0}$ are pairwise disjoint and $G = \bigcup_{n\in\mathbb{N}_0} C_n$. Also, the sequence $F = (F_N)_{N\in\mathbb{N}}$ defined by $F_N = \bigcup_{0\leq n\leq N} C_n$ is a Følner sequence in G and $F_N/2 = F_{N+1} = F_N \cup C_{N+1} \supset F_N$, so $\frac{|F_N/2 \cap F_N|}{|F_N|} = 1$ and hence $\alpha_F = 1$. Also, for all $N \in \mathbb{N}$, $|F_N| = 2^N$.

Now, consider the sequence $(\Phi_N)_{N\in\mathbb{N}}$ of subsets of G^ω defined by

$$\Phi_N = \underbrace{F_N \times \cdots \times F_N}_{N \text{ times}} \times \{0\}^\omega = \{\boldsymbol{g} = (g_i)_{i\in\mathbb{N}} \colon g_i \in F_N \text{ for } 1 \leq i \leq N, g_i = 0 \text{ for } i > N\}.$$

Then $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence in G^ω. This Følner sequence shares some properties with the one in G^d studied in Section 4.5.3. In particular,

$$\Phi_N/2 = \underbrace{F_N/2 \times \cdots \times F_N/2}_{N \text{ times}} \times \{0, 1/2\}^\omega = \underbrace{F_{N+1} \times \cdots \times F_{N+1}}_{N \text{ times}} \times \{0, 1/2\}^\omega,$$

and therefore, $\Phi_N \subset \Phi_N/2$, which implies that $\alpha_\Phi = 1$.

Consider now the set

$$A = \bigsqcup_{m\in\mathbb{N}} \left[\left(\underbrace{F_{2m+1} \times \cdots \times F_{2m+1}}_{2m+1 \text{ times}} \times \{0\}^\omega \right) \setminus \left(\underbrace{F_{2m} \times \cdots \times F_{2m}}_{2m+1 \text{ times}} \times \{0\}^\omega \right) \right].$$

Then for each $N \in \mathbb{N}$ we have that

$$A \cap \Phi_{2N+1} \supset \left(\underbrace{F_{2N+1} \times \cdots \times F_{2N+1}}_{2N+1 \text{ times}} \times \{0\}^\omega \right) \setminus \left(\underbrace{F_{2N} \times \cdots \times F_{2N}}_{2N+1 \text{ times}} \times \{0\}^\omega \right),$$

so $|A \cap \Phi_{2N+1}| \geq |F_{2N+1}|^{2N+1} - |F_{2N}|^{2N+1} = 2^{(2N+1)^2} - 2^{2N(2N+1)}$, which in turn implies that

$$\lim_{N\to\infty} \frac{|A \cap \Phi_{2N+1}|}{|\Phi_{2N+1}|} \geq \lim_{N\to\infty} \frac{2^{(2N+1)^2} - 2^{2N(2N+1)}}{2^{(2N+1)^2}} = 1 - \lim_{N\to\infty} \frac{1}{2^{2N+1}} = 1$$

Thus, we have that $\overline{\mathrm{d}}_\Phi(A) = 1$. We are left with proving there is no infinite set $B \subset G^\omega$ so that $B + B \subset A$.

We denote denote by $\mathbf{0}$ the identity element $(0, 0, \dots, 0, \dots)$ of G^ω. Consider the map $\theta\colon G \to \mathbb{N}_0$ defined in (4.5.4). Similarly to (4.5.5), let $w\colon G^\omega \to \mathbb{N}_0$ be the function given by $w(\boldsymbol{g}) = \max\{\theta(g_i)\colon i \in \mathbb{N}\}$. We also define $\tau\colon G^\omega \to \mathbb{N}_0$ by $\tau(\mathbf{0}) = 0$ and $\tau(\boldsymbol{g}) = \max\{i \in \mathbb{N} : g_i \neq 0\}$ for $\boldsymbol{g} \neq \mathbf{0}$. As in the finite dimensional case, for $\boldsymbol{g}, \boldsymbol{g}' \in G^\omega, w(\boldsymbol{g} + \boldsymbol{g}') \leq \max\{w(\boldsymbol{g}), w(\boldsymbol{g}')\}$ and if $w(\boldsymbol{g}) \neq w(\boldsymbol{g}')$, then $w(\boldsymbol{g} + \boldsymbol{g}') = \max\{w(\boldsymbol{g}), w(\boldsymbol{g}')\}$. Observe that

$$A = \bigsqcup_{m \in \mathbb{N}} \{\boldsymbol{g} \in G^\omega \mid w(\boldsymbol{g}) = 2m + 1, \tau(\boldsymbol{g}) \leq 2m + 1\}. \tag{4.6.1}$$

Assume that there is an infinite set $B \subset G^\omega$ so that $B + B \subset A$, and without loss of generality assume that $\mathbf{0} \notin B$. We want to reach a contradiction, and for that we separate cases.

First, assume that the set $\{w(\boldsymbol{b}) : \boldsymbol{b} \in B\}$ is finite, and take $M \in \mathbb{N}$ so that $w(\boldsymbol{b}) \leq 2M + 1$ for all $\boldsymbol{b} \in B$. Then for all $\boldsymbol{b}, \boldsymbol{b}' \in B$ we have that $w(\boldsymbol{b} + \boldsymbol{b}') \leq \max\{w(\boldsymbol{b}), w(\boldsymbol{b}')\} \leq 2M + 1$. Therefore, using (4.6.1) we get

$$B + B \subset \bigsqcup_{1 \leq m \leq M} \{\boldsymbol{g} \in G^\omega\colon w(\boldsymbol{g}) = 2m + 1 \text{ and } \tau(\boldsymbol{g}) \leq 2m + 1\} \subset \Phi_{2M+1}, \tag{4.6.2}$$

which implies that $B + B$ is finite and in particular $2B$ is finite. Since B is infinite, by pigeonhole principle, there exists $\mathbf{a} \in \Phi_{2M+1}$ and an infinite subset $B' \subset B$ such that $2\boldsymbol{b} = \mathbf{a}$ for all $\boldsymbol{b} \in B'$. Also by pigeonhole principle, we can suppose, without loss of generality, that the first $2M + 1$ coordinates of the elements in B' are equal. Let $\boldsymbol{b}, \boldsymbol{b'}$ be two distinct elements in B'. Since $b_i = b'_i$ for all $i \leq 2M + 1$, there exists a coordinate $j > 2M + 1$ such that $b_j \neq b'_j$. Fixing that index $j > 2M + 1$, since $a \in \Phi_{2M+1}$, $a_j = 0$ and hence $2b_j = 2b'_j = 0$. This implies that $b_j, b'_j \in \{0, \frac{1}{2}\}$ and therefore, using that $b_j \neq b'_j$, we must have $b_j + b'_j = \frac{1}{2}$, contradicting (4.6.2). Thus, the set $\{w(\boldsymbol{b}) : \boldsymbol{b} \in B\}$ has to be infinite.

Now, similarly to the proof of Lemma 4.5.4, suppose $\{w(\boldsymbol{b}) : \boldsymbol{b} \in B\}$ is infinite and let $\boldsymbol{b}, \boldsymbol{b'} \in B$ with $w(\boldsymbol{b'}) > w(\boldsymbol{b}) > 0$. Notice that $w(2\boldsymbol{b}) = w(\boldsymbol{b}) - 1$ and therefore, since $2\boldsymbol{b} \in A$, from (4.6.1) we have that $w(2\boldsymbol{b})$ is odd, so $w(\boldsymbol{b})$ is even. The same is true for $w(\boldsymbol{b'})$. Then we have that $w(\boldsymbol{b} + \boldsymbol{b'}) = \max\{w(\boldsymbol{b}), w(\boldsymbol{b'})\} = w(\boldsymbol{b'})$ which is even, and from (4.6.1) this contradicts the fact that $\boldsymbol{b} + \boldsymbol{b'} \in A$.

To summarize, A has full upper density with respect to the q.i.d. Følner sequence Φ with ratio $\alpha_\Phi = \alpha_G = 1$, but there is no infinite set $B \subset G^\omega$ so that $B + B \subset A$. $\square$

4.6.2 The Følner sequence has to be quasi-invariant with respect to doubling

Here we show that, if G is a countable abelian group with $2G$ infinite (note that if $[G : 2G] < \infty$, then $2G$ is infinite), then the q.i.d. assumption is necessary for a density solution to the unrestricted $B + B$ problem. We include the following useful remark, the proof of which is straightforward, so it is omitted.

Remark 4.6.2. Let G be a countable abelian group and $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in G. Then

(i) If $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is a sequence of subsets of G so that $\Psi_N \subset \Phi_N$ for all N and $|\Psi_N|/|\Phi_N| \to 1$ as $N \to \infty$, then Ψ is also a Følner in G.

(ii) If H is a subgroup of G so that $[G : H] = \infty$, then for every $g \in G$ we have $\mathrm{d}_\Phi(g + H) = 0$.

(iii) Using the sub-additivity of the density, if $E_1, E_2 \subset G$ are such that the densities $\mathrm{d}_\Phi(E_1), \mathrm{d}_\Phi(E_2)$ exist and $\mathrm{d}_\Phi(E_1) = 1$, then the density $\mathrm{d}_\Phi(E_1 \cap E_2)$ exists and it is equal to $\mathrm{d}_\Phi(E_2)$.

To achieve our aim we need the following lemma.

Lemma 4.6.3. *Let G be a countable abelian group with $2G$ infinite, let Φ be a Følner sequence in G that is not quasi-invariant with respect to doubling, and let $G = \{x_1, x_2, x_3, \ldots\}$ be an enumeration of G. Then there is a subsequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ of Φ and a Følner sequence $F = (F_N)_{N\in\mathbb{N}}$ in G so that for all N, $F_N \subset \Psi_N$, $(F_j + x_i)/2 \cap F_N = \emptyset$ whenever $i, j < N$, and*

$$\lim_{N\to\infty} \frac{|F_N|}{|\Psi_N|} = 1 \text{ and } \lim_{N\to\infty} \frac{|F_N/2 \cap F_N|}{|F_N|} = 0.$$

Proof. Since Φ is not q.i.d., we may pass to a subsequence, which by abuse of notation we also denote by $\Phi = (\Phi_N)_{N\in\mathbb{N}}$, so that

$$\lim_{N\to\infty} \frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|} = 0. \tag{4.6.3}$$

From the first isomorphism theorem for groups we have that $G/\ker(D) \cong 2G$, and since $2G$ is infinite, we have that $[G : \ker(D)] = \infty$. Then from Remark 4.6.2 (ii), each coset of $\ker(D)$ in G has zero density with respect to Φ. We will inductively construct a strictly increasing sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$ and a sequence

of sets $F = (F_k)_{k\in\mathbb{N}}$ so that if $\Psi_k = \Phi_{N_k}$, then for all $k \in \mathbb{N}$, $F_k \subset \Psi_k$, $\frac{|F_k|}{|\Psi_k|} > 1 - \frac{1}{k}$, and for $1 \leq i, j < k$, $(F_j + x_i)/2 \cap F_k = \emptyset$.

Let $N_1 = 1$ and take $F_1 = \Phi_1$. Now, assume that for some $k \geq 1$ we have constructed $N_1 < N_2 < \ldots < N_k$ and $F_1, \ldots, F_k$ so that the previous hold.

Observe that $\bigcup_{i,j=1}^{k}(F_j + x_i)/2$ is a (possibly empty) finite union of cosets of $\ker(D)$, so

$$\mathrm{d}_\Phi\Big(\bigcup_{i,j=1}^{k}(F_j + x_i)/2\Big) = 0,$$

and therefore there is $N_{k+1} > N_k$ so that

$$\frac{|\Phi_{N_{k+1}} \setminus \bigcup_{i,j=1}^{k}(F_j + x_i)/2|}{|\Phi_{N_{k+1}}|} > 1 - \frac{1}{k+1}. \tag{4.6.4}$$

Taking $F_{k+1} = \Phi_{N_{k+1}} \setminus \bigcup_{i,j=1}^{k}(F_j + x_i)/2$ one sees that $F_1, \ldots, F_{k+1}$ have the desired properties.

Observe that $|F_k|/|\Psi_k| \to 1$ as $k \to \infty$, so from Remark 4.6.2 (i) we have that F is indeed a Følner in G. From the construction we get that $(F_j + x_i)/2 \cap F_k = \emptyset$ whenever $i, j < k$. Finally, using that $F_k \subset \Psi_k$, and that $\frac{|\Psi_k/2 \cap \Psi_k|}{|\Psi_k|} = \frac{|\Phi_{N_k}/2 \cap \Phi_{N_k}|}{|\Phi_{N_k}|} \to 0$ and $|\Psi_k|/|F_k| \to 1$ as $k \to \infty$, we get that $\frac{|F_k/2 \cap F_k|}{|F_k|} \to 0$ as $k \to \infty$. This concludes the proof of the lemma. □

We are now ready to prove the necessity of the q.i.d. assumption.

Proposition 4.6.4. *Let G be a countable abelian group so that $2G$ is infinite, and let Φ be a Følner sequence in G that is not quasi-invariant with respect to doubling. Then there exists a set $A \subset G$ with $\overline{\mathrm{d}}_\Phi(A) = 1$ such that $B + B + t \not\subset A$ for any infinite set $B \subset G$ and any element $t \in G$.*

Proof. Let us fix an enumeration of G and write $G = \{x_1, x_2, x_3, \ldots\}$. Since Φ is not quasi-invariant with respect to doubling, we may use Lemma 4.6.3 to find a subsequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ of Φ and a Følner sequence $F = (F_N)_{N\in\mathbb{N}}$ in G so that for all N, $F_N \subset \Psi_N$, $(F_j + x_i)/2 \cap F_N = \emptyset$ whenever $i, j < N$,

$$\lim_{N\to\infty} \frac{|F_N|}{|\Psi_N|} = 1 \text{ and } \lim_{N\to\infty} \frac{|F_N/2 \cap F_N|}{|F_N|} = 0. \tag{4.6.5}$$

We will now choose a subsequence $(F_{N_k})_{k\in\mathbb{N}}$ of $(F_N)_{N\in\mathbb{N}}$ as follows. For $k \in \mathbb{N}$, let $c(k)$ be a small positive constant to be determined later. Now, given $F_{N_1}, \ldots, F_{N_{k-1}}$, we choose $N_k > N_{k-1}$ large enough so that

(i) $|F_{N_k} \cap F_{N_k}/2| < c(k)|F_{N_k}|$

(ii) $\left|F_{N_k} \cap \bigcup_{i,j=1}^{k-1} 2F_{N_j} + x_i\right| < c(k)|F_{N_k}|$

(iii) $\left|\bigcup_{i=1}^{k-1} F_{N_k} \triangle (F_{N_k} + x_i)\right| < c(k)|F_{N_k}|$.

Let us comment on why such a choice of N_k is possible. For (i), it suffices use the second equation in (4.6.5) and (ii) is possible because $\bigcup_{i,j=1}^{k-1} 2F_{N_j} + x_i$ is a finite set, while $|F_N| \to \infty$ as $N \to \infty$. Finally, for (iii), one simply has to use the fact that F is a Følner sequence. With that in mind, we choose $c(k)$ so that, if

$$A_{N_k} := \left(F_{N_k} \cap \bigcap_{i=1}^{k-1} F_{N_k} + x_i\right) \setminus \left(F_{N_k}/2 \cup \bigcup_{i,j=1}^{k-1} 2F_{N_j} + x_i\right),$$

then $|A_{N_k}| > \left(1 - \frac{1}{k}\right)|F_{N_k}|$. Letting $A = \bigcup_{k\in\mathbb{N}} A_{N_k}$ it follows by the latter that $\lim_{k\to\infty} \frac{|A\cap F_{N_k}|}{|F_{N_k}|} = 1$. Since $F_{N_k} \subset \Psi_{N_k}$ and $\frac{|F_{N_k}|}{|\Psi_{N_k}|} \to 1$ as $k \to \infty$, the previous implies that $\lim_{k\to\infty} \frac{|A\cap\Psi_{N_k}|}{|\Psi_{N_k}|} = 1$, and since Ψ is a subsequence of Φ, this implies that $\overline{\mathrm{d}}_\Phi(A) = 1$.

It remains to prove that for any infinite $B \subset G$ and any $t \in G$, $B + B + t \not\subset A$. Assume that there is some $t \in G$ and some infinite $B \subset G$ so that $B+B+t \subset A$. Let $b_1 \in B$ so that $t + 2b_1 \in A_{N_{k_1}}$ for some $k_1 \in \mathbb{N}$. Let $i_1, i_2 \in \mathbb{N}$ so that $x_{i_1} = t + 2b_1$, $x_{i_2} = -(t+2b_1)$. Since the A_{N_k}'s are finite, there is $b_2 \in B$ so that $t+b_1+b_2 \in A_{N_{k_2}}$ for some $k_2 > \max\{i_1, i_2, k_1\}$. We have that $t + 2b_2 \in A_{N_{k_3}}$ for some $k_3 \in \mathbb{N}$.

If $k_3 > k_2$, then from the definition of $A_{N_{k_3}}$ we have that $t+2b_2 \notin 2F_{N_{k_2}}+x_{i_2}$. On the other hand, since $t+b_1+b_2 \in A_{N_{k_2}} \subset F_{N_{k_2}}$, we have that $2t+2b_1+2b_2 \in 2F_{N_{k_2}}$ which implies that $t + 2b_2 \in 2F_{N_{k_2}} + x_{i_2}$, so we reach a contradiction.

If $k_3 = k_2$, then we have that $t + b_1 + b_2 \notin F_{N_{k_2}}/2$, so $2t + 2b_1 + 2b_2 \notin F_{N_{k_2}}$. Since $k_2 > i_2$, $t + 2b_2 \in A_{N_{k_2}}$ implies that $t + 2b_2 \in F_{N_{k_2}} + x_{i_2}$, which in turn gives that $2t + 2b_1 + 2b_2 \in F_{N_{k_2}}$, so again we reach a contradiction.

Finally, if $k_3 < k_2$, then $t + 2b_2 \in A_{N_{k_3}}$ implies that $t + 2b_2 \in F_{N_{k_3}}$, so $2t + 2b_1 + 2b_2 \in F_{N_{k_3}} + x_{i_1}$, and therefore $t + b_1 + b_2 \in (F_{N_{k_3}} + x_{i_1})/2$. Since $k_3, i_1 < k_2$, we have that $F_{N_{k_2}} \cap (F_{N_{k_3}} + x_{i_1})/2 = \emptyset$, and therefore $t + b_1 + b_2 \notin F_{N_{k_2}}$, so also $t + b_1 + b_2 \notin A_{N_{k_2}}$, which again is a contradiction. Hence in every case we reach a contradiction, so after all, for any infinite $B \subset G$ and any $t \in G$, $B+B+t \not\subset A$. □

We also show that non-q.i.d Følner sequences always exist in countable abelian groups.

Lemma 4.6.5. *Let G be a countable abelian group. Then, there exist Følner sequences in G which are not quasi-invariant with respect to doubling.*

Proof. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in G. Then, as each Φ_N is a finite set we can find $g_N \in G$ such that $g_N \notin \Phi_N - 2\Phi_N$. Consider the sequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ defined via $\Psi_N = g_N + \Phi_N$ for every $N \in \mathbb{N}$. It follows by Lemma A.0.1 (iii) (see also Remark A.0.2) that Ψ is a Følner sequence in G. Moreover, for any $N \in \mathbb{N}$ we see by the choice of g_N that $\Psi_N/2 \cap \Psi_N = \emptyset$. Indeed, if this wasn't the case we would have $2(\Psi_N/2)\cap(2\Psi_N) \neq \emptyset$ for some $N \in \mathbb{N}$ and this in turn would imply that $(g_N + \Phi_N) \cap (2g_N + 2\Phi_N) \neq \emptyset$. But this contradicts the fact that $g_N \notin \Phi_N - 2\Phi_N$ and thus we conclude. □

As an immediate consequence of the construction in Proposition 4.6.4, combined with the fact that non-q.i.d. Følner sequences always exist, we deduce the following.

Corollary 4.6.6. *Let G be a countable abelian group so that $2G$ is infinite. Then, there exists a set $A \subset G$ with upper Banach density equal to 1 and such that $B + B + t \not\subset A$ for any infinite set $B \subset G$ and any element $t \in G$.*

4.6.3 The case of infinite index $[G : 2G] = \infty$

In [1] it is shown that if $[G : 2G] = \infty$, then for any $\varepsilon > 0$ there exists a set with upper Banach density at least $1-\varepsilon$ that contains no shift of an infinite sumset. However, it could be that sets of full density always contain $B+B+t$. An interesting dichotomy manifests itself in this case. Indeed, as we show below, if $2G$ is a finite set (e.g., when $G = \mathbb{F}_2^\omega$), then any set of full upper Banach density contains a shifted sumset $B + B + t$. On the other hand, once $2G$ is an infinite set (assuming $[G : 2G] = \infty$) we construct – along any given Følner sequence – a set of full density that fails to contain such sumsets.

We begin with a simple observation.

Lemma 4.6.7. *Let G be an abelian group with $[G : 2G] = \infty$ and let Φ be any Følner sequence in G. Then there is a set $A \subset G$ with $\mathrm{d}_\Phi(A) = 1$ so that for all infinite $B \subset G$, $B + B \not\subset A$.*

Proof. From Remark 4.6.2 (ii) we have that $\mathrm{d}_\Phi(2G) = 0$. Let $A = G \setminus 2G$. Then $\mathrm{d}_\Phi(A) = 1$ and $A \cap 2G = \emptyset$, so for each nonempty $B \subset G$, $2B \cap A = \emptyset$. This concludes the proof. □

Allowing for the possibility of shifted sumsets makes the situation more delicate.

Proposition 4.6.8. *Let G be a countable abelian group so that $2G$ is finite. If $A \subset G$ has upper Banach density 1, then for every $t \in A$ there is some infinite $B \subset A$ so that $B + B + t \subset A$.*

Proof. Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be a Følner sequence along which the density of A is equal to 1, i.e.

$$\lim_{N\to\infty} \frac{|A\cap\Psi_N|}{|\Psi_N|} = 1.$$

From the first isomorphism theorem for groups we have that $G/\ker(D) \cong 2G$, which implies that $s := [G : \ker(D)] < \infty$. Then, using [11, Lemma 5.4] we have that $\mathrm{d}_\Psi(\ker(D)) = \frac{1}{s} > 0$, where the previous density exists.

Let $t \in A$. Then $\mathrm{d}_\Psi(A) = 1$, so from Remark 4.6.2 (iii) we have that $\mathrm{d}_\Psi(A \cap \ker(D)) = \frac{1}{s}$. Pick $b_1 \in A \cap \ker(D)$. Since $b_1 \in \ker(D)$, we have $t + 2b_1 = t \in A$.

Since $\mathrm{d}_\Psi(A - t - b_1) = 1$, we have that $\mathrm{d}_\Psi(A \cap (A - t - b_1) \cap ker(D)) = \frac{1}{s}$, so we may pick $b_2 \in A \cap (A - t - b_1) \cap ker(D)$, with $b_2 \neq b_1$. Again, since $b_2 \in \ker(D)$, we have $t + 2b_2 = t \in A$, and since $b_2 \in A - t - b_1$ we have $t + b_1 + b_2 \in A$.

Next, since $\mathrm{d}_\Psi(A-t-b_2) = 1$, we have $\mathrm{d}_\Psi(A\cap(A-t-b_1)\cap(A-t-b_2)\cap ker(D)) = \frac{1}{s}$, so we may pick $b_3 \in A\cap(A-t-b_1)\cap(A-t-b_2)\cap ker(D)$, with $b_3 \neq b_1, b_2$. Again, since $b_3 \in \ker(D)$, we have $t + 2b_3 = t \in A$, and since $b_3 \in (A - t - b_1) \cap (A - t - b_2)$ we have $t + b_1 + b_2, t + b_1 + b_3 \in A$.

Continuing inductively, we end up finding a sequence of different elements $(b_j)_{j\in\mathbb{N}} \subset A$ so that for all $i, j \in \mathbb{N}$ with $i \neq j$ we have $t + b_i + b_j \in A$ and $t + 2b_i = t \in A$. Hence for the infinite set $B = \{b_j : j \in \mathbb{N}\} \subset A$ we have $B + B + t \subset A$. □

Remark 4.6.9. Recall that a set $A \subset G$ is thick if for any finite set $F \subset G$ there is some $t \in G$ such that $t + F \subset A$. It is not difficult to see that this happens if and only if for any finite set $F \subset G$ there is some $t \in A$ such that $t + F \subset A$. Now, it is an easy exercise to verify that sets of upper Banach density 1 are thick. Also, it is well-known (see for example [8, Lemma 4.5]) that thick sets are IP-sets. That is, if $A \subset G$ is thick, there exists an infinite set $B \subset G$ such that $FS(B) := \left\{ \sum_{b\in H} b : H \subset B,\ H \text{ is finite} \right\} \subset A$.

Therefore, in the setting of Proposition 4.6.8, the set A is thick and so there exists $t \in A$ so that $t + 2G \subset A$. Moreover, $A - t$ is also thick and thus we may find $B \subset A - t$ infinite with $FS(B) + t \subset A$. In particular, since $2B \subset 2G \subset A - t$, we actually have that $(B + B + t) \cup (t + FS(B)) \subset A$.

We conclude with the following strengthening of Proposition 4.6.4 in the case that $[G : 2G] = \infty$.

Proposition 4.6.10. *Let G be an abelian group so that $2G$ is infinite and $[G : 2G] = \infty$, and let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be any Følner sequence in G. Then there is a set $A \subset G$ so that $\overline{\mathrm{d}}_\Phi(A) = 1$ and for all infinite $B \subset G$ and $t \in G$, $B + B + t \not\subset A$.*

Proof. From the first isomorphism theorem for groups we have that $G/\ker(D) \cong 2G$, so in particular $[G : \ker(D)] = \infty$, and therefore from Remark 4.6.2 (ii) we infer that $\mathrm{d}_\Phi(g+\ker(D)) = 0$ for all $g \in G$. Also, since $[G : 2G] = \infty$, again from Remark 4.6.2 (ii) we have $\mathrm{d}_\Phi(g + 2G) = 0$ for all $g \in G$.

We inductively construct a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ of natural numbers and a sequence of pairwise disjoint finite sets $A_k \subset \Phi_{N_k}$, $k \in \mathbb{N}$, as follows. Take $N_1 = 1$ and $A_1 = \Phi_1$. Now, for $k \geq 2$, given $N_1, \ldots, N_{k-1}$ and $A_1, \ldots, A_{k-1}$, let $D_k = \bigcup_{i=1}^{k-1} A_i$, which is finite. Then $E_{1,k} = D_k + 2G$ is a finite union of cosets of $2G$, so $\mathrm{d}_\Phi(E_{1,k}) = 0$, and $E_{2,k} = D_k + \ker(D)$ is a finite union of cosets of $\ker(D)$, so also $\mathrm{d}_\Phi(E_{2,k}) = 0$. Hence, letting $E_k = E_{1,k} \cup E_{2,k}$ we have $\mathrm{d}_\Phi(E_k) = 0$, so there is $N_k > N_{k-1}$ with

$$\frac{|\Phi_{N_k} \setminus E_k|}{|\Phi_{N_k}|} > 1 - \frac{1}{k}. \tag{4.6.6}$$

Take $A_k = \Phi_{N_k} \setminus E_k$, and let $A = \bigcup_{k=1}^{\infty} A_k$. Then from (4.6.6) we have that $\lim_{k\to\infty} \frac{|A\cap\Phi_{N_k}|}{|\Phi_{N_k}|} = 1$, so in particular $\overline{\mathrm{d}}_\Phi(A) = 1$. Also, since each Φ_N is finite, from the construction one can see that for all $g \in G$, the sets $A \cap (g + 2G)$ and $A \cap (g + \ker(D))$ are finite.

Now, assume that there is some $t \in G$ and some infinite $B \subset G$ so that $B+B+t \subset A$. Suppose that some coset of $\ker(D)$ contains infinitely many elements of B, i.e. there is $g_0 \in G$ so that $B' = B \cap (g_0 + \ker(D))$ is infinite. It follows that $t + B' + B' \subset A \cap (t + 2g_0 + \ker(D))$, and therefore $A \cap (t + 2g_0 + \ker(D))$ is infinite, contradicting the fact that A has finite intersection with every coset of $\ker(D)$.

Thus, for every $g \in G$, $B \cap (g + \ker(D))$ is finite, and since B is infinite, there is an infinite $B' \subset B$ so that every two elements of B' belong to different cosets of $\ker(D)$. In particular for $b_1, b_2 \in B'$ with $b_1 \neq b_2$, we have $2b_1 \neq 2b_2$, so the set $\{t + 2b' : b' \in B'\}$ is infinite. We have $\{t + 2b' : b' \in B'\} \subset A$ and $\{t + 2b' : b' \in B'\} \subset t + 2G$, so $A \cap (t + 2G)$ is infinite, contradicting the fact that A has finite intersection with every coset of $2G$. Hence after all, if $t \in G$ and $B \subset G$ is infinite, then $B + B + t \not\subset A$, which concludes the proof. □

4.7 Existence of Følner sequences that are quasi-invariant with respect to doubling

The goal of this section is to prove that any abelian group G with $[G : 2G] < \infty$ and $|\ker(D)| < \infty$ admits a q.i.d. Følner sequence. In fact, we establish the following even stronger result. Recall that $\alpha_G = \sup\{\alpha_\Phi \colon \Phi \in \mathcal{F}_G\}$, where $\mathcal{F}_G$ denotes the collection of all Følner sequences in G.

Theorem 4.7.1. *If G is a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$, then the following hold:*

(a) $\alpha_G = \min\{1, \frac{r}{\ell}\}$. In particular, G admits Følner sequence that is quasi-invariant with respect to doubling.

(b) There exists a Følner sequence Φ in G such that $\alpha_\Phi = \alpha_G$.

We start by proving (b) of Theorem 4.7.1.

Proof of b in Theorem 4.7.1. If $\alpha_G = 0$, then for all $\Psi \in \mathcal{F}_G$, $\alpha_\Psi = 0$, and the result follows immediately.

On the other hand, if $\alpha_G > 0$, then there is a sequence $\alpha_k \to \alpha_G$ and a sequence $(\Psi^{(k)})_{k\in\mathbb{N}} = \big((\Psi_N^{(k)})_{N\in\mathbb{N}}\big)_{k\in\mathbb{N}}$ of Følner sequences in G so that for all k, $\alpha_k = \alpha_{\Psi^{(k)}}$. Let $G = \{x_1, x_2, x_3, \ldots\}$ be an enumeration of the elements of G. Then for each $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ so that for all $N \geq N_k$ we have

$$\frac{\big|\Psi_N^{(k)} \triangle \big(x_s + \Psi_N^{(k)}\big)\big|}{\big|\Psi_N^{(k)}\big|} < \frac{1}{k}, \text{ for all } s \in \{1, \ldots, k\}, \text{ and } \frac{\big|\Psi_N^{(k)} \cap (\Psi_N^{(k)}/2)\big|}{\big|\Psi_N^{(k)}\big|} > \alpha_k - \frac{1}{k}. \tag{4.7.1}$$

Take a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ so that for all $k \in \mathbb{N}$, (4.7.1) holds for N_k and for each k, let $\Phi_k = \Psi_{N_k}^{(k)}$. For $x = x_{n_0} \in G$ and $k \geq n_0$ we have $\frac{|\Phi_k \triangle (x+\Phi_k)|}{|\Phi_k|} < \frac{1}{k} \xrightarrow{k\to\infty} 0$, so $\Phi = (\Phi_k)_{k\in\mathbb{N}}$ is indeed a Følner in G. Also, $\frac{|\Phi_k \cap (\Phi_k/2)|}{|\Phi_k|} > \alpha_k - \frac{1}{k} \xrightarrow{k\to\infty} \alpha_G$, so $\alpha_\Phi \geq \alpha_G$. By definition, we also have that $\alpha_\Phi \leq \alpha_G$, so after all, $\alpha_\Phi = \alpha_G$. This concludes the proof. □

Now we move to the proof of (a) of Theorem 4.7.1. For that we need the following lemma.

Lemma 4.7.2. *Let G be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $g_1, \ldots, g_\ell \in G$ so that $g_1 = e_G$ and $G = \bigsqcup_{i=1}^{\ell} g_i + 2G$. If $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is any Følner sequence in G, and we let $F_N = \bigsqcup_{i=1}^{\ell}(g_i + 2\Psi_N)$, $N \in \mathbb{N}$, then $F = (F_N)_{N\in\mathbb{N}}$ is also a Følner sequence in G.*

Proof. Let $g \in G$ and $\varepsilon > 0$. Then there exist $1 \leq i_0 \leq \ell$ and $h \in G$ such that $g = g_{i_0} + 2h$. Then

$$(g + F_N) \cap F_N = \left(\bigsqcup_{i=1}^{\ell}(g_{i_0} + g_i + 2(h + \Psi_N))\right) \cap \left(\bigsqcup_{i=1}^{\ell}(g_j + 2\Psi_N)\right). \tag{4.7.2}$$

Since the cosets $g_j + 2G$ are disjoint, it follows that for each $1 \leq i \leq \ell$, there exists a unique $1 \leq j(i) \leq \ell$ such that $g_{i_0} + g_i \in g_{j(i)} + 2G$, so there is $y_i \in G$ such that

$g_{i_0} + g_i = g_{j(i)} + 2y_i$. In addition, if we assume that for $i_1 \neq i_2$ we have $j(i_1) = j(i_2)$, then we have that $g_{i_1} - g_{i_2} = 2y_{i_1} - 2y_{i_2} \in 2G$, so $g_{i_1} + 2G = g_{i_2} + 2G$, which is a contradiction. Therefore, the map $j : \{1, \dots, \ell\} \to \{1, \dots, \ell\}, i \mapsto j(i)$ is a bijection, and then (4.7.2) becomes

$$(g + F_N) \cap F_N = \bigsqcup_{i=1}^{\ell} \big(g_{j(i)} + (2(y_i + h + \Psi_N) \cap 2\Psi_N)\big).$$

We thus have that

$$|(g+F_N)\cap F_N| = \sum_{i=1}^{\ell} |2(y_i+h+\Psi_N)\cap 2\Psi_N)| \geq \sum_{i=1}^{\ell} |2((y_i+h+\Psi_N)\cap \Psi_N)|. \quad (4.7.3)$$

Now since Ψ is a Følner sequence in G, for N sufficiently large, we have that for every $0 \leq i \leq \ell$,

$$|(y_i + h + \Psi_N) \triangle \Psi_N| \leq \frac{\varepsilon}{r} |\Psi_N|,$$

and then we have that

$$\frac{|2((y_i + h + \Psi_N) \cap \Psi_N)|}{|2\Psi_N|} \geq 1 - \frac{r|\Psi_N \triangle (y_i + h + \Psi_N)|}{|\Psi_N|} \geq 1 - \varepsilon, \quad (4.7.4)$$

where for the second inequality above we used that $|2(\Psi_N \triangle (y_j + h + \Psi_N))| \leq |(\Psi_N \triangle (y_j + h + \Psi_N))|$ and that $|2\Psi_N| \geq \frac{|\Psi_N|}{r}$. Then, combining (4.7.3) and (4.7.4) we get that for N sufficiently large,

$$\frac{|(g + F_N) \cap F_N|}{|F_N|} \geq \frac{\ell(1-\varepsilon)|2\Psi_N|}{|F_N|} = 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{N\to\infty} \frac{|(g+F_N)\cap F_N|}{|F_N|} = 1$. Thus, $(F_N)_{N\in\mathbb{N}}$ is a Følner sequence in G. □

Proof of (a) in Theorem 4.7.1. Fix a countable abelian group G with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. We need to prove that $\alpha_G = \min\{1, r/\ell\}$.

Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in G. By definition, $\alpha_\Phi \leq 1$. On the other hand, from Lemma A.0.4 we know that $\frac{|\Phi_N/2|}{|\Phi_N|} \to \frac{r}{\ell}$ as $N \to \infty$. For each N, $\frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|} \leq \frac{|\Phi_N/2|}{|\Phi_N|}$, so taking $\liminf_{N\to\infty}$ we obtain that $\alpha_\Phi \leq \frac{r}{\ell}$. Therefore, for each $\Phi \in \mathcal{F}_G$, $\alpha_\Phi \leq \min\left\{1, \frac{r}{\ell}\right\}$, which implies that $\alpha_G \leq \min\{1, r/\ell\}$.

To conclude the proof, it suffices to prove that $\alpha_G \geq \min\{1, r/\ell\}$. We split the proof into three cases, according to whether $\ell > r$, $\ell < r$ or $\ell = r$.

The case $\ell > r$: In this case $\min\{1, r/\ell\} = r/\ell$. Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be any Følner

sequence in G. Let also $g_1, \dots, g_\ell \in G$ so that $g_1 = e_G$ and $G = \bigsqcup_{i=1}^{\ell} g_i + 2G$. For each $N \in \mathbb{N}$, let

$$H_N^{(1)} = \bigsqcup_{i=1}^{\ell} (g_i + 2\Psi_N) \text{ and } H_N^{(j)} = \bigsqcup_{i=1}^{\ell} \left(g_i + 2H_N^{(j-1)}\right), \text{ for } j > 1.$$

From Lemma 4.7.2, we inductively get that for all $j \in \mathbb{N}$, $H^{(j)} = \left(H_N^{(j)}\right)_{N\in\mathbb{N}}$ is a Følner sequence in G. For each $k, N \in \mathbb{N}$, let

$$F_N^{(k)} = \bigcup_{j=1}^{k} H_N^{(j)}.$$

From Lemma A.0.1 (i) we have that for all $k \in \mathbb{N}$, $F^{(k)} = \left(F_N^{(k)}\right)_{N\in\mathbb{N}}$ is also a Følner sequence in G. We have $H_N^{(1)}/2 = (2\Psi_N)/2 = \Psi_N + \ker(D)$ and for each $j > 1$, $H_N^{(j)}/2 = \left(2H_N^{(j-1)}\right)/2 = H_N^{(j-1)} + \ker(D)$. In addition, using Lemma A.0.3 we get that for each $j \in \mathbb{N}$, $|H_N^{(j)}| = \ell|2H_N^{(j-1)}| = \frac{\ell}{r}|H_N^{(j-1)}| + \mathrm{o}_{N\to\infty}(|H_N^{(j-1)}|)$, so inductively we get that for each j,

$$\left|H_N^{(j)}\right| = \left(\frac{\ell}{r}\right)^j |\Psi_N| + \mathrm{o}_{N\to\infty}(|\Psi_N|). \tag{4.7.5}$$

For each $k, N \in \mathbb{N}$ we have $F_N^{(k)}/2 = (\Psi_N + \ker(D)) \cup \bigcup_{j=1}^{k-1} \left(H_N^{(j)} + \ker(D)\right)$ and $(F_N^{(k)}/2) \setminus F_N^{(k)} \subset (\Psi_N + \ker(D)) \cup \bigcup_{j=1}^{k-1} \left(H_N^{(j)} + \ker(D)\right) \setminus H_N^{(j)}$. Also, from the proof of Lemma A.0.3 we have that $\frac{|\Psi_N + \ker(D)|}{|\Psi_N|} \to 1$ and $\frac{|H_N^{(j)} + \ker(D)|}{|H_N^{(j)}|} \to 1$ as $N \to \infty$. Combining those with (4.7.5) we infer that

$$\begin{aligned}
\frac{\left|(F_N^{(k)}/2) \setminus F_N^{(k)}\right|}{\left|F_N^{(k)}/2\right|} &\leq \frac{|\Psi_N + \ker(D)|}{\left|F_N^{(k)}/2\right|} + \sum_{j=1}^{k-1} \frac{\left|(H_N^{(j)} + \ker(D)) \setminus H_N^{(j)}\right|}{\left|F_N^{(j)}\right|} \\
&\leq \frac{|\Psi_N + \ker(D)|}{|\Psi_N|} \frac{|\Psi_N|}{\left|H_N^{(k-1)}\right|} + \sum_{j=1}^{k-1} \left(1 - \frac{\left|H_N^{(j)}\right|}{\left|H_N^{(j)} + \ker(D)\right|}\right) \xrightarrow{N\to\infty} \left(\frac{r}{\ell}\right)^{k-1},
\end{aligned}$$

which implies that

$$\limsup_{N\to\infty} \frac{\left|(F_N^{(k)}/2) \setminus F_N^{(k)}\right|}{\left|F_N^{(k)}/2\right|} \leq \left(\frac{r}{\ell}\right)^{k-1}.$$

From Lemma A.0.4 we know that $\frac{|F_N^{(k)}/2|}{|F_N^{(k)}|} \to \frac{r}{\ell}$ as $N \to \infty$ and therefore $\alpha_k =$

$\alpha_{F^{(k)}}$ equals

$$\liminf_{N\to\infty} \frac{|F_N^{(k)} \cap F_N^{(k)}/2|}{|F_N^{(k)}|} = \liminf_{N\to\infty} \frac{|F_N^{(k)}/2|}{|F_N^{(k)}|} \left(1 - \frac{|(F_N^{(k)}/2) \setminus F_N^{(k)}|}{|F_N^{(k)}/2|}\right) \geq \frac{r}{\ell}\left(1 - \left(\frac{r}{\ell}\right)^{k-1}\right).$$

Using that $(\frac{r}{\ell})^{k-1} \searrow 0$ as $k \to \infty$ we get $\alpha_G \geq \sup\{\alpha_k : k \in \mathbb{N}\} = \frac{r}{\ell}$, which concludes the proof in that case.

The case $\ell < r$: In this case we have $\min\{1, r/\ell\} = 1$. Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be any Følner sequence in G. By an application of Lemma A.0.1 (iv), we have that for each $j \in \mathbb{N}$, $(\Psi_N/2^j)_{N\in\mathbb{N}}$ is also a Følner sequence in G. By the same lemma, if for each $k, N \in \mathbb{N}$ we define

$$F_N^{(k)} = \bigcup_{j=0}^{k} \Psi_N/2^j,$$

then we have that $F^{(k)} = (F_N^{(k)})_{N\in\mathbb{N}}$ is also Følner sequence. In addition, for each $j \in \mathbb{N}$ we have that

$$|\Psi_N/2^j| = \left(\frac{r}{\ell}\right)^j |\Psi_N| + o_{N\to\infty}(|\Psi_N|),$$

by induction and Lemma A.0.4. Therefore, we see that

$$\frac{\left|F_N^{(k)} \setminus (F_N^{(k)}/2)\right|}{\left|F_N^{(k)}\right|} \leq \frac{|\Psi_N|}{|\Psi_N/2^k|} \xrightarrow{N\to\infty} \left(\frac{\ell}{r}\right)^k,$$

which implies that

$$\alpha_k := \alpha_{F^{(k)}} = \liminf_{N\to\infty} \frac{\left|F_N^{(k)} \cap (F_N^{(k)}/2)\right|}{|F_N^{(k)}|} = \liminf_{N\to\infty} \left(1 - \frac{\left|F_N^{(k)} \setminus (F_N^{(k)}/2)\right|}{|F_N^{(k)}|}\right) \geq 1 - \left(\frac{\ell}{r}\right)^k.$$

Using that $(\frac{\ell}{r})^k \searrow 0$ as $k \to \infty$ we get $\alpha_G \geq \sup\{\alpha_k : k \in \mathbb{N}\} = 1$, which concludes the proof in that case.

The case $\ell = r$: In case $\ell = r$, we have $\min\{1, r/\ell\} = 1$. Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be any Følner in G. As before, for each $k \in \mathbb{N}_0$ consider the Følner sequence $F^{(k)} = \left(F_N^{(k)}\right)_{N\in\mathbb{N}}$ defined by

$$F_N^{(k)} = \bigcup_{j=0}^{k} \Psi_N/2^j.$$

Also, for each $k \in \mathbb{N}_0$, let $\alpha_k = \alpha_{F^{(k)}} = \liminf_{N\to\infty} \frac{|F_N^{(k)} \cap (F_N^{(k)}/2)|}{|F_N^{(k)}|}$. By employing

a diagonal argument and passing to a subsequence, we may assume that the limits exist, so $\alpha_k = \lim_{N\to\infty} \frac{|F_N^{(k)} \cap (F_N^{(k)}/2)|}{|F_N^{(k)}|}$.

Assume that $\sup\{\alpha_k : k \in \mathbb{N}_0\} = \alpha < 1$. Then for each $j \in \mathbb{N}_0$ we have

$$\big(F_N^{(j)}/2\big) \setminus F_N^{(j)} = \big(\Psi_N/2^{j+1}\big) \setminus \big(\bigcup_{m=0}^{j} \Psi_N/2^m\big). \tag{4.7.6}$$

Also, since $r = \ell$ and $F^{(j)}$ is a Følner, from Lemma A.0.4 we have that $\lim_{N\to\infty} \frac{|F_N^{(j)}/2|}{|F_N^{(j)}|} = 1$. Using the previous we get that

$$\alpha_j = \lim_{N\to\infty} \frac{\big|(F_N^{(j)}/2) \cap F_N^{(j)}\big|}{\big|F_N^{(j)}\big|} = \lim_{N\to\infty} \frac{\big|(F_N^{(j)}/2) \cap F_N^{(j)}\big|}{\big|F_N^{(j)}/2\big|} = \lim_{N\to\infty} \Big[1 - \frac{\big|(F_N^{(j)}/2) \setminus F_N^{(j)}\big|}{\big|F_N^{(j)}/2\big|}\Big],$$

which implies that $\lim_{N\to\infty} \frac{|(F_N^{(j)}/2)\setminus F_N^{(j)}|}{|F_N^{(j)}/2|} = 1 - \alpha_j$, and therefore

$$\lim_{N\to\infty} \frac{\big|(F_N^{(j)}/2) \setminus F_N^{(j)}\big|}{\big|F_N^{(j)}\big|} = 1 - \alpha_j. \tag{4.7.7}$$

Observe that $\Psi_N \subset F_N^{(j)}$, and combining this with (4.7.6) and (4.7.7) we get that

$$\liminf_{N\to\infty} \frac{\big|\big(\Psi_N/2^{j+1}\big) \setminus \big(\bigcup_{m=0}^{j} \Psi_N/2^m\big)\big|}{\big|\Psi_N\big|} \geq 1 - \alpha_j \geq 1 - \alpha > 0. \tag{4.7.8}$$

For each $k \in \mathbb{N}$, utilizing (4.7.8) for $j \in \{0, 1, \ldots, k-1\}$ we can find an $N_k \in \mathbb{N}$ such that for all $N \geq N_k$,

$$\big|\big(\Psi_N/2^{j+1}\big) \setminus \big(\bigcup_{m=0}^{j} \Psi_N/2^m\big)\big| > (1-\alpha)(1-2^{-k})\big|\Psi_N\big|, \text{ for all } j \in \{0, 1, \ldots, k-1\}. \tag{4.7.9}$$

Observe that $F_N^{(k)} = \Psi_N \sqcup \bigsqcup_{j=1}^{k} \big(\Psi_N/2^j\big) \setminus \big(\bigcup_{m=0}^{j-1} \Psi_N/2^m\big) = \Psi_N \sqcup \bigsqcup_{j=0}^{k-1} \big(\Psi_N/2^{j+1}\big) \setminus \big(\bigcup_{m=0}^{j} \Psi_N/2^m\big)$, so for $N \geq N_k$ we have

$$|F_N^{(k)}| = |\Psi_N| + \sum_{j=0}^{k-1} \big|\big(\Psi_N/2^{j+1}\big) \setminus \big(\bigcup_{m=0}^{j} \Psi_N/2^m\big)\big| > k(1-\alpha)(1-2^{-k})|\Psi_N|. \tag{4.7.10}$$

Since $F_N^{(k)} \setminus (F_N^{(k)}/2) \subset \Psi_N$, combining with (4.7.10) we get that for $N \geq N_k$,

$$\frac{\left|F_N^{(k)} \setminus (F_N^{(k)}/2)\right|}{\left|F_N^{(k)}\right|} \leq \frac{1}{k(1-\alpha)(1-2^{-k})}, \tag{4.7.11}$$

which implies that

$$\alpha_k = \alpha_{F^{(k)}} = \liminf_{N\to\infty} \frac{\left|F_N^{(k)} \cap (F_N^{(k)}/2)\right|}{|F_N^{(k)}|} \geq 1 - \frac{1}{k(1-\alpha)(1-2^{-k})}.$$

Since $\frac{1}{k(1-\alpha)(1-2^{-k})} \to 0$ as $k \to \infty$ we get that $\alpha_k \to 1$ as $k \to \infty$, which contradicts our original assumption that $\sup\{\alpha_k : k \in \mathbb{N}_0\} < 1$. Therefore, $\sup\{\alpha_k : k \in \mathbb{N}_0\} = 1$, which implies that $\alpha_G \geq 1$ and concludes the proof in the case $\ell = r$, and with it the proof of (a) of Theorem 4.7.1. □

Chapter 5

Sharp density conditions for B+B sumsets in abelian groups

5.1 Overview

In this chapter, which is based on the results of [39], we continue our investigations around the unrestricted $B + B$ problem, started in Chapter 3 and generalised in Chapter 4. In particular, motivated by the results of Chapter 4 on infinite sumsets of the form $B+B = \{b_1 + b_2 : b_1, b_2 \in B\}$ in large subsets of abelian groups, and an old problem of Owings [36, Problem E2494] about the partition regularity of $B+B$ in 2 colours, we show the following theorem. Let $(G, +)$ be a countable abelian group such that the subgroup $\{g+g\colon g \in G\}$ has finite index and the doubling map $D : g \mapsto g + g$ has finite kernel. Let also $\Phi = (\Phi_N)_N$ be any Følner sequence in G and $\Phi/2 = (D^{-1}(\Phi_N))_N$. Then, if $A \subset G$ is such that $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$, there is an infinite set $B \subset G$ and some $t \in G$ for which $B + B + t \subset A$.

We prove that this result implies the main result of Chapter 4, namely Theorem 4.2.2, and construct an example to show the reverse implication does not hold. Moreover, we show that our main theorem is optimal in a strong sense. Namely, for any countable abelian group $(G, +)$ with the aforementioned assumptions – which are necessary – there exists a Følner sequence Φ and a set $A \subset G$ so that $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) = 1$, but there is no infinite set $B \subset G$ and $t \in G$ for which $B + B + t \subset A$.

5.2 Introduction

Motivated by recent breakthroughs [43, 41, 49] on the general problem of finding infinite sumsets in sets with positive density, and specifically by a conjecture of Kra,

Moreira, Richter and Robertson [44, Conjecture 3.7], Radić and the author [40] proved the first density regularity type of results for unrestricted infinite sumsets (presented in Chapter 3). In Theorems 1.3.2 and 1.3.1 it was shown, for example, that for $A \subset \mathbb{N}$ with $\underline{d}(A) > 1/2$, there is an infinite set $B \subset \mathbb{N}$ and some integer $t \geq 0$, such that $B + B + t \subset A$.

In consistency with Section 3.2, we refer to *the (unrestricted) $B + B$ problem* as the general problem of finding a density scheme – in a countable abelian group – for which any set with large enough density contains a sumset of the form $B + B + t$.

Our main result reads as follows.

Theorem 1.3.3. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $A \subset G$ and Φ be any Følner sequence in G. Pass to a subsequence of Φ if necessary so that all densities exist. Then the following hold:*

(1) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$, there exist an infinite set $B \subset G$ and some $t \in G$ such that $B + B + t \subset A$.

(2) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > \frac{2\ell-1}{\ell}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

(3) If $\mathrm{d}_\Phi(A \cap 2G) + \mathrm{d}_{\Phi/2}(A \cap 2G) > \frac{1}{\ell}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

We want to stress the need to pass to an appropriate subsequence in Theorem 1.3.3. As the proof in Section 5.4 reveals, we want to consider densities of the form

$$\mathrm{d}_{(\Phi_{N_k})}(A) + \mathrm{d}_{(\Phi_{N_k}/2)}(A),$$

along the same subsequence (N_k). As mentioned in the introduction, we may even have the following extreme behaviour if we don't consider densities along the same subsequence.

Proposition 1.3.4. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. There is a set $A \subset G$ and a Følner sequence Φ such that*

$$\overline{\mathrm{d}}_\Phi(A) + \overline{\mathrm{d}}_{\Phi/2}(A) = 2,$$

but there does not exist an infinite set $B \subset G$ and $t \in G$ with $B + B + t \subset A$.

Returning to our original discussion, the results in [40] were restricted to taking upper and lower densities with respect to the original Følner sequence $N \mapsto [1, N] =$

$\{1, \dots, N\}$ in $\mathbb{N}$. The threshold values established therein were proved to be optimal, and while it was known that not all Følner sequences can be used to obtain analogous results (see, for example, [44, Example 3.6]), it was not explored any further whether other density schemes, e.g. density of a set along other Følner sequences, could be used to address the $B + B$ problem in $\mathbb{N}$.

Around the same time, Charamaras and Mountakis [11] proved the analogue of the main result in [41] for the widest class of countable abelian groups possible. In particular, they showed that for countable abelian groups G with $[G : 2G] < \infty$, any set with positive upper Banach density contains an infinite sumset $B \oplus B + t = \{b_1 + b_2 + t : b_1, b_2 \in B, \text{ and } b_1 \neq b_2\}$. In an effort to extend the results of [11] to cover unrestricted sumsets, and to understand more deeply the necessary structural properties of Følner sequences that can be used as averaging schemes to a density solution for the $B + B$ problem, Charamaras, Mountakis, Radić and the author [10] introduced the notion of quasi-invariant to doubling Følner sequences. Recall that for a set $C \subset G$ of an abelian group $(G, +)$, we write $2C = D(C)$ and $C/2 = D^{-1}(C)$, where $D \colon G \to G$ is the *doubling map* $g \mapsto 2g$.

Definition 5.2.1 (cf. Definition 4.2.1)**.** Let G be a countable abelian group. The *the doubling ratio* of a Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ in G is defined as

$$\alpha_\Phi = \liminf_{N \to \infty} \frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|}. \tag{5.2.1}$$

If $\alpha_\Phi > 0$, we say that the Følner sequence Φ is *quasi-invariant with respect to doubling (q.i.d.) with ratio* α_Φ.

As explained in Chapter 4, the q.i.d. assumption can be used to provide density threshold values for the $B + B$ problem via the following theorem, which we restate for convenience.

Theorem 4.2.2. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $A \subset G$ and Φ be any Følner sequence in G that is quasi-invariant with respect to doubling with ratio α_Φ. Then the following hold:*

(1) If $\overline{\mathrm{d}}_\Phi(A) > 1 - \dfrac{\ell \alpha_\Phi}{\ell + r}$, there exist an infinite set $B \subset G$ and some $t \in G$ such that $B + B + t \subset A$.

(2) If $\overline{\mathrm{d}}_\Phi(A) > 1 - \dfrac{\alpha_\Phi}{\ell + r}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

(3) If $\overline{\mathrm{d}}_\Phi(A \cap 2G) > \dfrac{1}{\ell} - \dfrac{\alpha_\Phi}{\ell + r}$, there exists an infinite set $B \subset G$ such that $B + B \subset A$.

To complement this result, it was shown in Proposition 4.6.4 that if a Følner sequence is not q.i.d. there is a set of full density along this sequence that contains no sumset as in (1). In that sense, the q.i.d. assumption is necessary for Theorem 4.2.2. This proposition (and thus Definition 5.2.1) will be used repeatedly in the constructions of Section 5.6.

It is not obvious, but it turns out that Theorem 1.3.3 implies Theorem 4.2.2 and we prove this in Proposition 5.3.2. Perhaps even more interesting is the fact that the reverse implication is false. Namely, Theorem 4.2.2 cannot be used to deduce Theorem 1.3.3 as there exist sets that satisfy, say, condition (1) of Theorem 1.3.3 along some Følner sequence, but not condition (1) of Theorem 4.2.2 along any Følner sequence.

Proposition 5.2.2. *There is a set $A \subset \mathbb{Z}$ such that*

$$\overline{\mathrm{d}}_{\Phi}(A) \leq 1 - \frac{\ell \alpha_{\Phi}}{\ell + r},$$

for every Følner sequence Φ in $\mathbb{Z}$, but

$$\mathrm{d}_{F}(A) + \mathrm{d}_{F/2}(A) > 1,$$

for some Følner sequence F in $\mathbb{Z}$.

We also remark that the three statements in Theorem 1.3.3 are equivalent and we show this in Section 5.3.

The proof of Theorem 1.3.3 is dynamical, similarly to the proof of Theorem 4.2.2. In fact, the main dynamical result behind both theorems is the same; Theorem 4.3.5. The difference lies in the way we implement the correspondence principle, which is used to translate the combinatorial input of the statement into an assumption about a corresponding dynamical system.

The reason why the stronger Theorem 1.3.3 was not found in [10] has to do with the broader viewpoint of how the problem was formulated and thus approached there. In particular, from the perspective of the authors in [10] "the main question to address is whether for a given abelian group G one can find a Følner sequence Φ and a constant $c = c(G, \Phi) > 0$ such that any set $A \subset G$ with upper density

$$\overline{\mathrm{d}}_{\Phi}(A) = \limsup_{N \to \infty} \frac{|A \cap \Phi_N|}{|\Phi_N|} > c$$

contains an unrestricted sumset of the form $B + B$ for some infinite set $B \subset G$."

The reason why Theorem 1.3.3 was found now is twofold. Firstly, it appeared

naturally in considerations of questions about the optimality of Theorem 4.2.2 (optimality in the sense of sharpness of the threshold bounds). These questions were formulated in Questions 4.2.5 and 4.2.6. We present here their analogues for Theorem 1.3.3. The first question asks whether the result is optimal in any group where it holds.

Question 5.2.3 (cf. Question 4.2.6)**.** Let G be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Can one always find a Følner sequence Φ in G and a set $A \subset G$ with $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) = 1$, such that $B + B + t \not\subset A$ for any infinite $B \subset G$ and $t \in G$?

The answer to Question 4.2.6 is not known; however, the weaker assumptions in our main result allow us to address its analogue, Question 5.2.3, in full generality.

Theorem 1.3.5. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. There is a set $A \subset G$ such that*

$$\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) = 1,$$

for some Følner sequence Φ in G, but there does not exist an infinite set $B \subset G$ and $t \in G$ with $B + B + t \subset A$.

The second question asks for the strongest possible notion of optimality for the bounds in our main result.

Question 5.2.4 (cf. Question 4.2.5)**.** Let G be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$, and let Φ be a Følner sequence in G. Does there exist a set $A \subset G$ with $\mathrm{d}_\Psi(A) + \mathrm{d}_{\Psi/2}(A) = 1$, along some subsequence Ψ of Φ, and such that $B + B + t \not\subset A$ for any infinite $B \subset G$ and $t \in G$?

Answering Question 5.2.4 is probably very difficult in full generality (unless the answer is negative and we are missing an easy counterexample), but we can address it in the integers along sequences of intervals. Indeed, this relates to the second reason why Theorem 1.3.3 was found. An old Problem of C.J. Owings [36, Problem E2494] asks whether, for any colouring (partition) of $\mathbb{N}$ into two sets, one of the sets must contain $B + B$ for some infinite $B \subset \mathbb{N}$. This problem was studied by various authors (see, for example, [28, 29, 37]) and is rather interesting because Hindman [27] showed the analogue for three-colourings to have a negative answer. In the author's effort to address this problem with some type of a density argument (which was motivated by the joint work with Radić [40]), it was observed that if Theorem 1.3.3 was not optimal, Owings' question would have a positive answer. The details

are explained in Section 5.7. As an interesting historical remark, the same day that the paper [39] appeared on the arXiv, a solution to Owings' problem appeared as well [34]. This probably means that other than serving as a motivation for the main results of this chapter, the results of Section 5.7 are obsolete; nevertheless, we chose to keep it for completeness and because the arguments used there may turn out to be of some independent value.

Here, now, is the special case of Question 5.2.4 that we can address.

Proposition 5.2.5. *Let Φ be any Følner sequence of intervals in $\mathbb{Z}$. Then, there exists a set $A \subset \mathbb{Z}$ such that $\mathrm{d}_\Psi(A) + \mathrm{d}_{\Psi/2}(A) = 1$, where Ψ is a subsequence of Φ along which the densities exist, for which $B + B + t \not\subset A$, for any infinite set B and any $t \in \mathbb{Z}$.*

Finally, we want to briefly explain the assumptions that $[G : 2G] < \infty$ and $r = |\ker(D)| < \infty$ in all our results. In Section 4.6.1 it was shown that there exists an abelian group with infinite $\ker(D)$, and a set of full density along a Følner sequence that is q.i.d., which contains no infinite sumsets $B + B$. It was also proved in Section 4.6.3 that when $2G$ is infinite and $[G : 2G] = \infty$, Theorem 4.2.2 fails even with density 1. The second assumption was also imposed by the results in [1]. Since these assumptions were already known to be necessary for Theorem 4.2.2, they have to be assumed for Theorem 1.3.3 to hold as well, in view of the relation between the two results (Proposition 5.3.2).

5.3 Theorem 1.3.3 implies Theorem 4.2.2

As was mentioned before, G always denotes a countable abelian group with $\ell = [G : 2G] < \infty$. We also fix $g_1, \dots, g_\ell \in G$ so that $G = \bigsqcup_{i=1}^{\ell} 2G + g_i$. We omit these assumptions from the statements of this section.

The main purpose of this section is to prove that our main result implies the main result of [10], Theorem 4.2.2. Since the conclusion of both theorems is the same, this amounts to showing that the assumptions of Theorem 1.3.3 are weaker than those of Theorem 4.2.2. We begin by showing in Proposition 5.3.1 that the three statements of Theorem 1.3.3 are equivalent, in analogy to Section 4.3.1. This is also useful when showing that the bounds are optimal, because we only need to do this for one of the bounds.

Proposition 5.3.1. *Let $(G, +)$ be a countable abelian group with $\ell = [G : 2G] < \infty$. Let $A \subset G$ and Φ be any Følner sequence in G. Then the following statements are equivalent:*

(1) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$, *there exist an infinite set* $B \subset G$ *and some* $t \in G$ *such that* $B + B + t \subset A$.

(2) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > \frac{2\ell-1}{\ell}$, *there exists an infinite set* $B \subset G$ *such that* $B+B \subset A$.

(3) If $\mathrm{d}_\Phi(A \cap 2G) + \mathrm{d}_{\Phi/2}(A \cap 2G) > \frac{1}{\ell}$, *there exists an infinite set* $B \subset G$ *such that* $B + B \subset A$.

Proof. The proof is very similar to that of Proposition 4.3.3, and as an indicative example we prove the equivalence between (1) and (3).

(1) $\Longrightarrow$ (3): If $\mathrm{d}_\Phi(A\cap 2G)+\mathrm{d}_{\Phi/2}(A\cap 2G) > \frac{1}{\ell}$, we let $\tilde{A} = \bigsqcup_{i=1}^{\ell}(A\cap 2G)+g_i$, and the invariance of Φ and $\Phi/2$ gives that $\mathrm{d}_\Phi(\tilde{A}) + \mathrm{d}_{\Phi/2}(\tilde{A}) > 1$. By (1), $\tilde{A}$ contains a sumset of the form $B+B+t$ for some infinite $B \subset G$ and $t \in G$. By Lemma 4.3.2 we can reduce to the case where $t = g_i$ for some $i \in \{1, \dots, \ell\}$. Using Lemma 4.3.1, we can also assume that $B+B \subset 2G$. Therefore, we have that $B+B+g_i \subset (A\cap 2G)+g_i$ which concludes the proof.

(3) $\Longrightarrow$ (1): Suppose $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1$. By sub-additivity of the density,

$$\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) \leq \sum_{i=1}^{\ell} \mathrm{d}_\Phi(A \cap (2G + g_i)) + \mathrm{d}_{\Phi/2}(A \cap (2G + g_i)),$$

so there is $i \in \{1, \dots, \ell\}$ for which $\mathrm{d}_\Phi(A \cap (2G + g_i)) + \mathrm{d}_{\Phi/2}(A \cap (2G + g_i)) > \frac{1}{\ell}$. By translation invariance of the density and (3), it follows that A contains $B + B + g_i$ for some $B \subset G$ infinite. □

We now prove the main result of this section.

Proposition 5.3.2. *Theorem 1.3.3 implies Theorem 4.2.2.*

Proof. In view of Proposition 4.3.3 and Proposition 5.3.1, it suffices to show that (1) of Theorem 1.3.3 implies (1) of Theorem 4.2.2. As explained in the beginning of this section, we only have to prove that that the assumptions of the latter imply the assumptions of the former.

Let $\Phi = (\Phi_N)$ be a Følner sequence in G with doubling ratio

$$0 < \alpha_\Phi = \liminf_{N\to\infty} \frac{|\Phi_N/2 \cap \Phi_N|}{|\Phi_N|}$$

and, by passing to a subsequence of Φ if necessary, suppose that $A \subset G$ with $\mathrm{d}_\Phi(A) > 1-\frac{\ell\alpha_\Phi}{\ell+r}$ and that $\mathrm{d}_{\Phi/2}(A)$ exists as well. We will next verify that $\mathrm{d}_\Phi(A)+\mathrm{d}_{\Phi/2}(A) > 1$.

Observe that

$$|A \cap \Phi_N| = |A \cap \Phi_N \cap \Phi_N/2| + |A \cap (\Phi_N \setminus \Phi_N/2)|$$

and thus

$$|A \cap \Phi_N| \leq |A \cap \Phi_N \cap \Phi_N/2| + |\Phi_N| - |\Phi_N \cap \Phi_N/2|\,.$$

Using the density assumption and the doubling ratio property, it follows that

$$\limsup_{N\to\infty} \frac{|A \cap \Phi_N \cap \Phi_N/2|}{|\Phi_N|} > \alpha_\Phi \left(1 - \frac{\ell}{\ell + r}\right) = \left(\frac{r\alpha_\Phi}{\ell + r}\right). \tag{5.3.1}$$

Using Lemma A.0.4, we see that

$$\mathrm{d}_{\Phi/2}(A) = \lim_{N\to\infty} \frac{|A \cap \Phi_N/2|}{|\Phi_N/2|} = \lim_{N\to\infty} \frac{\ell\,|A \cap \Phi_N/2|}{r\,|\Phi_N|}.$$

Hence, in view of (5.3.1), we have that

$$\begin{aligned}
&\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > 1 - \frac{\ell\alpha_\Phi}{\ell + r} + \lim_{N\to\infty} \frac{\ell\,|A \cap \Phi_N/2|}{r\,|\Phi_N|} \\
&\geq 1 - \frac{\ell\alpha_\Phi}{\ell + r} + \limsup_{N\to\infty} \frac{\ell\,|A \cap \Phi_N \cap \Phi_N/2|}{r\,|\Phi_N|} > 1 - \frac{\ell\alpha_\Phi}{\ell + r} + \left(\frac{\ell\alpha_\Phi}{\ell + r}\right) = 1.
\end{aligned}$$

□

5.4 Proof of the main result

In this section we present the proof of Theorem 1.3.3.

5.4.1 A correspondence principle

As we explained in the introduction, the difference between the proofs of Theorems 1.3.3 and 4.2.2 lies at the level of the correspondence principle. More precisely, starting from a set $A \subset G$ and a Følner sequence Φ, the correspondence principle associates to (A, Φ) an ergodic system which, in some sense, encodes the statistical behavior of A along Φ. The improvement of this approach is that allowing for a more flexible averaging scheme, where we consider the density of a set along two distinct, but related, Følner sequences, allows us to obtain a better lower bound for the measures of some important sets in the ergodic system associated to A.

Again, let $(G, +)$ be an abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. By Σ we denote the space $\{0, 1\}^G$, endowed with the product topology, which

becomes a compact metrizable space. We also consider the shift action $S = (S_g)_{g\in G}$ defined by $S_g\colon \Sigma \to \Sigma$, $S_g(x(h)) = x(h+g)$, for any $h, g \in G$, $x = (x(g))_{g\in G} \in \Sigma$, and note that S is an action of G on X by homeomorphisms. We further denote the G-action $g \mapsto S_{2g}$ by S^2.

Via the following variant of Furstenberg's correspondence principle, we can reduce Theorem 1.3.3 to the dynamical statement of Theorem 4.3.5.

Lemma 5.4.1. *Let $A \subset G$ and Φ be any Følner sequence in G (and pass to a subsequence if necessary so that all densities exist). There exist an ergodic G-system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner sequence Φ', such that $(a, a) \in \mathtt{gen}(\mu, \Phi')$, $A = \{g \in G\colon S_g a \in E\}$ and*

$$\ell\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq \ell \cdot (\mathrm{d}_{\Phi/2}(A) + \mathrm{d}_{\Phi}(A)) - \ell + 1. \tag{5.4.1}$$

Proof. By definition,

$$\mathrm{d}_{\Phi/2}(A) = \lim_{N\to\infty} \frac{|A \cap \Phi_N/2|}{|\Phi_N/2|}.$$

We associate to the set A a point $a \in \Sigma = \{0,1\}^G$ via

$$a(g) = \begin{cases} 1, & \text{if } g \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Letting $E = \{x \in \Sigma\colon x(e_G) = 1\}$ – which, we note, is a clopen set – observe that, by construction, $A = \{g \in G\colon S_g a \in E\}$. We can define the sequence of Borel probability measures $(\mu_N)_{N\in\mathbb{N}}$ on $\Sigma \times \Sigma$ given by

$$\mu_N = \frac{1}{|\Phi_N/2|} \sum_{g\in\Phi_N/2} \delta_{(S_{2g}\times S_g)(a,a)}.$$

If μ' is a weak* accumulation point of $(\mu_N)_{N\in\mathbb{N}}$, then μ' is an $(S^2 \times S)$-invariant measure. Now, since for each $N \in \mathbb{N}$, we have

$$\mu_N(\Sigma \times E) = \frac{1}{|\Phi_N/2|} \sum_{g\in\Phi_N/2} \delta_{S_g a}(E) = \frac{|A \cap \Phi_N/2|}{|\Phi_N/2|},$$

sending $N \to \infty$ yields

$$\mu'(\Sigma \times E) = \mathrm{d}_{\Phi/2}(A). \tag{5.4.2}$$

Now, Lemma A.0.3 gives that

$$\lim_{N\to\infty} \frac{|\{g \in \Phi_N/2 : g + \ker(D) \subset \Phi_N/2\}|}{|\Phi_N/2|} = 1.$$

Moreover, if we consider the Følner sequence $N \mapsto F_N = \bigcup_{g\in\Phi_N/2} g + \ker(D) \supset \Phi_N/2$ (where (F_N) is Følner since $|\ker(D)| = r < \infty$), it is easy to see that $\sum_{g\in F_N} \delta_{S_{2g}a}(E) = r \sum_{g\in 2(\Phi_N/2)} \delta_{S_g a}(E)$. Therefore, it follows that

$$\left| \frac{1}{|\Phi_N/2|} \sum_{g\in\Phi_N/2} \delta_{S_{2g}a}(E) - \frac{r}{|\Phi_N/2|} \sum_{g\in 2(\Phi_N/2)} \delta_{S_g a}(E) \right| \leq \frac{|F_N \setminus \Phi_N/2|}{|\Phi_N/2|} \xrightarrow{N\to\infty} 0.$$

We thus see that

$$\begin{aligned} \mu_N(E\times\Sigma) = \frac{1}{|\Phi_N/2|} \sum_{g\in\Phi_N/2} \delta_{S_{2g}a}(E) &= \frac{r}{|\Phi_N/2|} \sum_{g\in 2(\Phi_N/2)} \delta_{S_g a}(E) + \mathrm{o}_{N\to\infty}(1) \\ &\geq \frac{\ell}{|\Phi_N|} |A \cap 2(\Phi_N/2)| + \mathrm{o}_{N\to\infty}(1), \quad (5.4.3) \end{aligned}$$

by Lemma A.0.4. Since $2(\Phi_N/2) = \Phi_N \cap 2G$, we have

$$|A \cap 2(\Phi_N/2)| = |A \cap \Phi_N| - |A \cap (\Phi_N \setminus 2(\Phi_N/2))| \geq |A \cap \Phi_N| - |\Phi_N| + |\Phi_N \cap 2G|.$$

Hence, using [11, Lemma 5.4], it follows by (5.4.3) as $N \to \infty$ that

$$\mu'(E\times\Sigma) \geq \ell \left(\mathrm{d}_\Phi(A) - 1 + \frac{1}{\ell} \right) = \ell\, \mathrm{d}_\Phi(A) - \ell + 1. \quad (5.4.4)$$

Combining (5.4.2) and (5.4.4) we obtain (5.5.1) for μ'. Although μ' is not necessarily ergodic, the argument that allows us to find an ergodic measure μ satisfying (5.5.1) follows just as in the proof of Lemma 3.3.3. □

5.4.2 The proof of Theorem 1.3.3

We can now use the correspondence principle of Lemma 5.4.1 and Lemma 4.3.6 in order to deduce (2) of Theorem 1.3.3, which, in view of Proposition 5.3.1 shows the full theorem.

Proof of Theorem 1.3.3. Let G, ℓ, r, Φ be as in the assumptions of Theorem 1.3.3. Let also $A \subset G$ satisfy $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) > \frac{2\ell-1}{\ell}$. Using Lemma 5.4.1 we can find an ergodic G-system $(\Sigma\times\Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, a point $a \in \Sigma$ and a Følner

sequence Φ' in G, such that $(a,a) \in \mathtt{gen}(\mu, \Phi')$, $A = \{g \in G \colon S_g a \in E\}$ and

$$\ell\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq \ell \cdot (\mathrm{d}_{\Phi/2}(A) + \mathrm{d}_{\Phi}(A)) - \ell + 1.$$

Because of the assumptions on A, we get

$$\ell\mu(\Sigma \times E) + \mu(E \times \Sigma) > \ell.$$

Applying Theorem 4.3.5 and Lemma 4.3.6 we can conclude similarly to the proof of Theorem 1.3.1 from Theorem 3.3.1. □

5.5 A slightly stronger result

With not much more effort, we can actually prove something slightly more general than Theorem 1.3.3.

Theorem 5.5.1. *Let $(G,+)$ be a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$. Let $A, A' \subset G$ and Φ be any Følner sequence in G. Then the following hold:*

1. *If $\mathrm{d}_{\Phi}(A') + \mathrm{d}_{\Phi/2}(A) > 1$, there exist an infinite set $B \subset G$ and some $t \in G$ such that $B \oplus B + t \subset A$ and $t + 2B \subset A'$.*

2. *If $\mathrm{d}_{\Phi}(A') + \mathrm{d}_{\Phi/2}(A) > \frac{2\ell-1}{\ell}$, there exists an infinite set $B \subset G$ such that $B \oplus B \subset A$ and $2B \subset A'$.*

3. *If $\mathrm{d}_{\Phi}(A' \cap 2G) + \mathrm{d}_{\Phi/2}(A \cap 2G) > \frac{1}{\ell}$, there exists an infinite set $B \subset G$ such that $B \oplus B \subset A$ and $2B \subset A'$.*

The proof of the equivalences between the statements in Theorem 5.5.1 is similar to the proof of Proposition 5.3.1 and so we omit it. In order to actually prove Theorem 5.5.1 we use the same dynamical result that we used for Theorem 1.3.3, that is Theorem 4.3.5, and a more general form of the correspondence principle to capture the statistics of both sets.

Lemma 5.5.2. *Let $A, A' \subset G$ and Φ be any Følner sequence in G (and pass to a subsequence if necessary so that all densities exist). There exist an ergodic G-system $(\Sigma \times \Sigma, \mu, S^2 \times S)$, an open set $E \subset \Sigma$, and points $a, a' \in \Sigma$ and a Følner sequence Φ', such that $(a,a') \in \mathtt{gen}(\mu, \Phi')$, $A = \{g \in G \colon S_g a \in E\}$ and $A' = \{g \in G \colon S_g a' \in E\}$ with*

$$\ell\mu(\Sigma \times E) + \mu(E \times \Sigma) \geq \ell \cdot (\mathrm{d}_{\Phi/2}(A) + \mathrm{d}_{\Phi}(A')) - \ell + 1. \tag{5.5.1}$$

Proof. The only difference with the proof of Lemma 5.4.1 is that we now introduce two points $a, a' \in \Sigma$ which correspond to the indicators of A and A', respectively, and then consider the measures $(\mu_N)_{N\in\mathbb{N}}$ on $\Sigma \times \Sigma$ given by

$$\mu_N = \frac{1}{|\Phi_N/2|} \sum_{g\in\Phi_N/2} \delta_{(S_{2g}\times S_g)(a',a)}.$$

The rest of the proof proceeds analogously. □

The question of whether Theorem 5.5.1 is optimal is easily seen to be a trivial one, which makes it less interesting than the more natural results, Theorem 4.2.2 and Theorem 1.3.3. Indeed, taking any Følner sequence Φ and any set $A \subset G$ with full density along Φ and $A' = \emptyset$, we see at once that the threshold values in (1) and (3) of the theorem are obtained, but the conclusion fails.

5.6 Optimality and relations between main theorems

We next show that Theorem 4.2.2 does not imply Theorem 1.3.3; that is, the density assumption of Theorem 4.2.2 cannot be used to deduce the density assumption of Theorem 1.3.3. This is already the case in the integers, which was rather unexpected. However, the construction seems to be adaptable in all groups satisfying the standing assumptions, although we do not pursue this in detail. This result was presented as Proposition 5.2.2 and we repeat the statement here for convenience.

Proposition 5.6.1. *There is a set $A \subset \mathbb{Z}$ such that*

$$\overline{\mathrm{d}}_\Phi(A) \leq 1 - \frac{\ell\alpha_\Phi}{\ell + r},$$

for every Følner sequence Φ in $\mathbb{Z}$, but

$$\mathrm{d}_F(A) + \mathrm{d}_{F/2}(A) > 1,$$

for some Følner sequence F in $\mathbb{Z}$.

Proof. We let $F = (F_N)_{N\in\mathbb{N}}$ be a Følner sequence that is not q.i.d. and such that $F_1, F_1/2, F_1/4, F_2, F_2/2, F_2/4, \ldots$ are pairwise disjoint. For example, let $F_N = [4^{2N}, 2 \cdot 4^{2N}) \cap \mathbb{N}$, for $N \in \mathbb{N}$. In particular, $F_N/2 = [2 \cdot 4^{2N-1}, 4^{2N}) \cap \mathbb{N}$ and $F_N/4 = [4^{2N-1}, 2 \cdot 4^{2N-1}) \cap \mathbb{N}$. We then consider a set $A \subset \mathbb{N}$ defined via

$$A = \left(\bigcup_{N=1}^{\infty} F_N\right) \bigcup \left(\bigcup_{N=1}^{\infty} F_N/2 \cap k\mathbb{N}\right),$$

with $k \in \mathbb{N}$, $k \geq 3$. Essentially, this could be generalized by replacing $k\mathbb{N}$ with a syndetic set of sufficiently small density.

First, we observe that

$$\mathrm{d}_F(A) + \mathrm{d}_{F/2}(A) = 1 + \frac{1}{k} > 1.$$

We remark that, in view of Theorem 1.3.3, this implies the existence of some infinite set $B \subset \mathbb{N}$ and $t \in \mathbb{N}_0$ such that $B + B + t \subset A$; this is, however, not a necessary part of this proof.

We proceed to prove that for every Følner sequence Φ in $\mathbb{Z}$, it holds that

$$\overline{\mathrm{d}}_\Phi(A) \leq 1 - \frac{\ell \alpha_\Phi}{\ell + r}. \tag{5.6.1}$$

By the definition of A, we only have to consider Følner sequences that only intersect non-trivially with F and $F/2$. For, if $\Phi_N = G_N \cup B_N$, where $G_N \subset \bigcup_{N=1}^\infty F_N \cup F_N/2$ and $B_N \cap (\bigcup_{N=1}^\infty F_N \cup F_N/2) = \emptyset$, it holds that

$$\overline{\mathrm{d}}_\Phi(A) = \limsup_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|} = \limsup_{N\to\infty} \frac{|A \cap G_N|}{|G_N| + |B_N|} \leq \limsup_{N\to\infty} \frac{|A \cap G_N|}{|G_N|}.$$

To easy the presentation, we first address two simple cases. If $\Phi = F$ or $\Phi = F/2$, we have that $\alpha_F = 0$, and so (5.6.1) holds by default. Next, we consider the case that $(\Phi_N)_{N\in\mathbb{N}} = (F_N \cup F_N/2)_{N\in\mathbb{N}}$. Then we have that

$$\alpha_\Phi = \lim_{N\to\infty} \frac{|F_N/2|}{|F_N| + |F_N/2|} = \frac{1}{3}.$$

In the integers we have that $\ell = 2$ and $r = 1$, so this means that the right hand side on (5.6.1) equals $7/9$. At the same time

$$\overline{\mathrm{d}}_\Phi(A) = \lim_{N\to\infty} \frac{|F_N| + |F_N/2|/k}{|F_N| + |F_N/2|} = \frac{2 + 1/k}{3} = \frac{2k+1}{3k} \leq \frac{7}{9},$$

for $k \geq 3$. Similar calculations show (5.6.1) in the case that $\Phi_N = (G_N \cup H_N)_{N\in\mathbb{N}}$, where $G_N \subset F_N$ and $H_N \subset F_N/2$.

Finally, we consider the general case where $\Phi_N = (G_N \cup H_N)_{N\in\mathbb{N}}$, with $G_N = G_{N_{k_1}} \cup \cdots \cup G_{N_{k_N}}$ for $G_{N_{k_i}} \subset F_{n(N,k_i)}$, and $H_N = H_{N_{m_1}} \cup \cdots \cup H_{N_{m_N}}$ for $H_{N_{m_i}} \subset F_{m(N,m_i)}/2$. Note that, $(G_N)_{N\in\mathbb{N}}$ and $(H_N)_{N\in\mathbb{N}}$ have to be Følner sequences also. In order to verify (5.6.1) it suffices to show that for every $N \in \mathbb{N}$,

$$|A \cap \Phi_N| + \frac{2}{3}|\Phi_N \cap \Phi_N/2| \leq |\Phi_N| + \mathrm{o}_{N\to\infty}(1),$$

which is implied by

$$3\,|A\cap\Phi_N| + 2\,|\Phi_N\cap\Phi_N/2| \leq 3\,|\Phi_N| + \mathrm{o}_{N\to\infty}(1). \tag{5.6.2}$$

Now, using the definition of Φ_N and A, we see that the left hand side in (5.6.2) equals

$$3|G_N| + \frac{3}{k}|H_N| + 2|G_N/2\cap H_N| + \mathrm{o}_{N\to\infty}(1),$$

which equals

$$3\sum_{i=1}^{k_N}\left|G_{N_{k_i}}\right| + \frac{3}{k}\sum_{j=1}^{m_N}\left|H_{N_{m_j}}\right| + 2\sum_{i,j}\left|G_{N_{k_i}}/2\cap H_{N_{m_j}}\right| + \mathrm{o}_{N\to\infty}(1).$$

Note that we have here used the fact that (H_N) is Følner and the assumption that $\{F_N, F_N/2, F_N/4 : N\in\mathbb{N}\}$ are all disjoint, because otherwise we could have other non-trivial intersections in

$$\Phi_N\cap\Phi_N/2 = (G_N\cup H_N)\cap(G_N/2\cup H_N/2).$$

But observe that $G_{N_{k_i}}/2\cap H_{N_{m_j}}\neq\emptyset$ only if $F_{n(N,k_i)}/2\cap F_{m(N,m_i)}/2\neq\emptyset$, i.e. only if $n(N,k_i)=m(N,m_i)$. Note also that in this case, we have $|G_{N_{k_i}}/2\cap H_{N_{m_j}}|\leq|H_{N_{m_j}}|$.

For $k\geq 3$ it thus follows that

$$\begin{aligned}&3\,|A\cap\Phi_N| + 2\,|\Phi_N\cap\Phi_N/2| \leq 3\sum_{i=1}^{k_N}\left|G_{N_{k_i}}\right| + 3\sum_{j=1}^{m_N}\left|H_{N_{k_j}}\right| + \mathrm{o}_{N\to\infty}(1)\\ =&3|G_N| + 3|H_N| + \mathrm{o}_{N\to\infty}(1) = 3|\Phi_N| + \mathrm{o}_{N\to\infty}(1),\end{aligned}$$

showing (5.6.2) and therefore the result. □

Now we prove that Theorem 1.3.3 is optimal in the sense of Question 5.2.3. That is, for any countable abelian group, we can find a Følner sequence along which some set achieves the threshold density of (1) Theorem 1.3.3, without containing the sumsets of the statement. This is Theorem 1.3.5 from the introduction.

Theorem 5.6.2. *Let $(G,+)$ be a countable abelian group with $\ell = [G:2G] < \infty$. There is a set $A\subset G$ such that*

$$\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) = 1,$$

for some Følner sequence Φ in G, but there does not exist an infinite set $B\subset G$ and

$t \in G$ *with* $B + B + t \subset A$.

Proof. In Corollary 4.6.6 it was shown that in every countable abelian group with $[G : 2G] < \infty$ there is a Følner sequence Φ that is not q.i.d. and a set $A \subset \bigcup_{N=1}^{\infty} \Phi_N$ with $\overline{\mathrm{d}}_\Phi(A) = 1$ and such that $B + B + t \not\subset A$ for any infinite set $B \subset G$ and any element $t \in G$.

It follows by the proof of this corollary, and in particular, the construction in Lemma 4.6.5 that Φ and A can be chosen so that $\overline{\mathrm{d}}_{\Phi/2}(A) = 0$. Hence, by passing to a subsequence of Φ, which we still write as Φ, we obtain

$$\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/2}(A) = 1,$$

and the proof is complete. □

A similar argument shows the need to consider the same subsequence in the densities over the two Følner sequences in Theorem 1.3.3, namely Proposition 1.3.4.

Proposition 5.6.3. *Let* $(G, +)$ *be a countable abelian group with* $\ell = [G : 2G] < \infty$ *and* $r = |\ker(D)| < \infty$. *There is a set* $A \subset G$ *and a Følner sequence* Φ *such that*

$$\overline{\mathrm{d}}_\Phi(A) + \overline{\mathrm{d}}_{\Phi/2}(A) = 2,$$

but there does not exist an infinite set $B \subset G$ *and* $t \in G$ *with* $B + B + t \subset A$.

Proof. We again rely on Corollary 4.6.6. Indeed, we can consider a non-q.i.d. Følner sequence $\Phi = (\Phi_N)$, so that $\Phi/2 = (\Phi_N/2)$ is also a non-q.i.d. Følner and even $\Phi_N \cap (\Phi_N/2) = \emptyset$ and $\Phi_N \cap (\Phi_N/4) = \emptyset$. Moreover, we can choose $\Phi = (\Phi_N)$ so that the sets Φ_N are pairwise disjoint and so are the sets $\Phi_N/2$ (see the proof of Lemma 4.6.5 for the construction of a non-q.i.d. Følner ; this is not necessary, we could alternatively refine the sequence so that $|\Phi_{N+1}| >> |\Phi_N|$).

Then, we consider the Følner sequence $F = (F_N)$ defined via $F_N = \Phi_{2N} \cup (\Phi_{2N+1}/2)$. Then, $F_N/2 = (\Phi_{2N}/2) \cup (\Phi_{2N+1}/4)$ and by the disjointness assumptions it follows that $F = (F_N)$ is non-q.i.d. Passing to a subsequence of (F_N) if needed, we then find a set $A = \bigcup_{N=1}^{\infty} A_N$ so that $A_N \subset F_N$ and $\mathrm{d}_F(A) = 1$ and there does not exist an infinite set $B \subset G$ and $t \in G$ with $B + B + t \subset A$. Finally, note that

$$\overline{\mathrm{d}}_\Phi(A) + \overline{\mathrm{d}}_{\Phi/2}(A) = \mathrm{d}_{(\Phi_{2N})}(A) + \mathrm{d}_{(\Phi_{2N+1}/2)}(A) = 2.$$

□

Addressing the optimality of Theorem 1.3.3 in the strong sense of Question 5.2.4 for all abelian groups under consideration and all Følner sequences seems to be

a very difficult task. For the moment, we confine ourselves to proving that our theorem is optimal in the integer setting for some special class of Følner sequences. This is already interesting, and useful – enriching our discussion in Section 5.7 – but the construction exploits the structure of the group $\mathbb{Z}$ crucially as the reader may observe.

We say that $\Phi = (\Phi_N)$ is a Følner sequence of intervals in $\mathbb{Z}$ if each Φ_N is an interval. In some sense, such special sequences are the building blocks of all Følner sequences in $\mathbb{Z}$. We do not make this fact precise, but the point is that the generality of the next result (which is Proposition 5.2.5 from the introduction) is non-trivial.

Proposition 5.6.4. *Let Φ be any Følner sequence of intervals in $\mathbb{Z}$. Then, there exists a set $A \subset \mathbb{Z}$ such that $\mathrm{d}_\Psi(A) + \mathrm{d}_{\Psi/2}(A) = 1$, along a subsequence Ψ of Φ for which the densities exist, for which $B + B + t \not\subset A$, for any infinite set B and any $t \in \mathbb{Z}$.*

Proof. We will choose a sequence $(N_m)_{m\in\mathbb{N}}$ and construct a Følner sequence $F = (F_m)_{m\in\mathbb{N}}$ that is not q.i.d. for which $F_m \subset \Phi_{N_m} \cup \Phi_{N_m}/2$, such that $A' = \bigcup_{m=1}^{\infty} F_m$ satisfies $\overline{\mathrm{d}}_\Phi(A') + \overline{\mathrm{d}}_{\Phi/2}(A') = 1$ and $(N_m)_{m\in\mathbb{N}}$ is chosen so that

$$\overline{\mathrm{d}}_\Phi(A') + \overline{\mathrm{d}}_{\Phi/2}(A') = \lim_{m\to\infty} \frac{|F_m \cap \Phi_{N_m}|}{|\Phi_{N_m}|} + \lim_{m\to\infty} \frac{|F_m \cap \Phi_{N_m}/2|}{|\Phi_{N_m}/2|}.$$

This can be arranged, for example, by choosing (N_m) so that $\frac{|\Phi_{N_m}|}{|\Phi_{N_{m+1}}|} \xrightarrow{m\to\infty} 0$.

Having done that, we can then use Proposition 4.6.4 in order to find a sequence $(m_k)_{k\in\mathbb{N}}$ and set $A \subset \mathbb{Z}$ such that $A = \bigcup_{k=1}^{\infty} A_{m_k}$, where $A_{m_k} \subset F_{m_k}$ and $\overline{\mathrm{d}}_F(A) = \mathrm{d}_{(F_{m_k})}(A) = 1$, for which $B + B + t \not\subset A$, for any infinite set B and any $t \in \mathbb{Z}$. These properties of A and the definition of F and A' in turn imply that $\mathrm{d}_\Psi(A) + \mathrm{d}_{\Psi/2}(A) = 1$, where $\Psi = (\Phi_{N_{m_k}})_{k\in\mathbb{N}}$, concluding the proof.

We are left with constructing $F = (F_m)_{m\in\mathbb{N}}$. To ease notation, without loss of generality we assume that $N_m = m$, so that (a posteriori) $A' = \bigcup_{N=1}^{\infty} F_N$ with $F_N \subset \Phi_N \cup \Phi_N/2$.

For each $N \in \mathbb{N}$, we let $S_{N,0} = \Phi_N \setminus \Phi_N/2$. Then, we define $S_{N,1} = S_{N,0}/2^2 = (\Phi_N/4) \setminus (\Phi_N/8)$, $S_{N,2} = S_{N,0}/2^4 = (\Phi_N/16) \setminus (\Phi_N/32)$ and more generally

$$S_{N,j} = S_{N,0}/2^{2j} = (\Phi_N/2^{2j}) \setminus (\Phi_N/2^{2j+1}),$$

for $j = 0, \dots, k_N$, where k_N is the largest integer j for which $\Phi_N/2^{2j} \cap (\Phi_N \cup \Phi_N/2) \neq \emptyset$.

We then define $F_N = \bigcup_{j=0}^{k_N} S_{N,j}$. Observe that by the definitions, the previous union is a disjoint union, and this will be important in the calculations to follow.

We remark that we have used our assumption that (Φ_N) is a sequence of intervals here.

Moreover, we observe that $(S_{N,0})$ is a sequence of intervals whose lengths increase to infinity and it readily follows that (F_N) is asymptotically invariant, i.e. a Følner sequence. Finally,

$$F_N/2 = \bigcup_{j=0}^{k_N} S_{N,j}/2 = \bigcup_{j=0}^{k_N} (\Phi_N/2^{2j+1}) \setminus (\Phi_N/2^{2(j+1)}),$$

so that $F_N \cap F_N/2 = \emptyset$. Thus (F_N) is a q.i.d. Følner sequence.

We then see that

$$\begin{aligned}\frac{|F_N \cap \Phi_N|}{|\Phi_N|} &= \sum_{j=0}^{k_N} \frac{|S_{N,j} \cap \Phi_N|}{|\Phi_N|} = \frac{|\Phi_N \setminus \Phi_N/2|}{|\Phi_N|} + \sum_{j=1}^{k_N} \frac{|(\Phi_N \cap \Phi_N/2^{2j}) \setminus \Phi_N/2^{2j+1}|}{|\Phi_N|} \\ =1 &- \frac{|\Phi_N \cap \Phi_N/2|}{|\Phi_N|} + \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j}|}{|\Phi_N|} - \sum_{j=1}^{k_N} \frac{|(\Phi_N \cap \Phi_N/2^{2j}) \cap \Phi_N/2^{2j+1}|}{|\Phi_N|}.\end{aligned} \tag{5.6.3}$$

Similarly, but noting that $S_{N,0} \cap \Phi_N/2 = \emptyset$, and since $|H_N/2| = |H_N|/2 + \mathrm{o}_{N\to\infty}(1)$, for any Følner sequence (H_N) in $\mathbb{Z}$, we get that

$$\frac{|F_N \cap \Phi_N/2|}{|\Phi_N/2|} = \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j-1}|}{|\Phi_N|} - \sum_{j=1}^{k_N} \frac{|(\Phi_N \cap \Phi_N/2^{2j-1}) \cap \Phi_N/2^{2j}|}{|\Phi_N|} + \mathrm{o}_{N\to\infty}(1). \tag{5.6.4}$$

Now, observe that $\Phi_N \cap \Phi_N/2^i \cap \Phi_N/2^{i+1} = \Phi_N \cap \Phi_N/2^{i+1}$, where we have crucially used that (Φ_N) is a sequence of intervals once again. Using this in (5.6.3) and (5.6.4) we obtain that

$$\frac{|F_N \cap \Phi_N|}{|\Phi_N|} = 1 - \frac{|\Phi_N \cap \Phi_N/2|}{|\Phi_N|} + \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j}|}{|\Phi_N|} - \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j+1}|}{|\Phi_N|} \tag{5.6.5}$$

and

$$\frac{|F_N \cap \Phi_N/2|}{|\Phi_N/2|} = \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j-1}|}{|\Phi_N|} - \sum_{j=1}^{k_N} \frac{|\Phi_N \cap \Phi_N/2^{2j}|}{|\Phi_N|} + \mathrm{o}_{N\to\infty}(1). \tag{5.6.6}$$

It follows that

$$\frac{|F_N \cap \Phi_N|}{|\Phi_N|} + \frac{|F_N \cap \Phi_N/2|}{|\Phi_N/2|} = 1 + \mathrm{o}_{N\to\infty}(1),$$

where we also used that $\Phi_N \cap \Phi_N/2^{2k_N+1} = \emptyset$. Indeed, if $\Phi_N \cap \Phi_N/2^{2k_N+1} \neq \emptyset$ then we would also have $\Phi_N/2 \cap \Phi_N/2^{2(k_N+1)} \neq \emptyset$, contradicting the definition of k_N. Therefore,

$$\mathrm{d}_{\Phi}(A') + \mathrm{d}_{\Phi/2}(A') = 1$$

and this concludes the proof. □

5.7 Owings' problem

We now present a connection between the main result and Owings' problem mentioned in the introduction. The observations presented herein merely serve the purpose of illustrating one of the reasons why our main result, Theorem 1.3.3, was discovered – as explained in Section 5.2. Indeed, Owings' problem was solved in the positive very recently in [34]. Perhaps one conclusion of the considerations to follow is, in our opinion, that there probably is no density-type of argument solution to the problem.

Question 5.7.1. Let $\mathbb{N} = C_1 \cup C_2$ be a 2-colouring of the natural numbers. Does there exist an infinite set $B = \{b_1 < b_2 < \cdots\} \subset \mathbb{N}$, such that $B + B = \{b_i + b_j : i, j \in \mathbb{N}\} \subset C_k$, for some $k \in \{1, 2\}$?

As was observed in Remark 3.7.2, in order to answer Question 5.7.1, one may allow a shift in the sumsets. This follows directly from the fact that $\mathbb{N} = C_1/2 \cup C_2/2$, where $C_k/2 = \{n \in \mathbb{N} : 2n \in C_k\}$, is also a 2-colouring of $\mathbb{N}$, and the observation that $B + B + t \subset C_k/2$ implies that $(2B + t) + (2B + t) \subset C_k$. Specifically, the following question is equivalent to Question 5.7.1.

Question 5.7.2. Let $\mathbb{N} = C_1 \cup C_2$ be a 2-colouring of the natural numbers. Does there exist an infinite set $B = \{b_1 < b_2 < \cdots\} \subset \mathbb{N}$, and some $t \in \mathbb{N}_0$, such that $B + B + t = \{t + b_i + b_j : i, j \in \mathbb{N}\} \subset C_k$, for some $k \in \{1, 2\}$?

We repeat that Questions 5.7.1 and 5.7.2 both have a positive answer, as shown in [34].

Theorem 5.7.3 ([34, Theorem 1.1])**.** *Let $\mathbb{N} = C_1 \cup C_2$. Then there exists $i \in \{1, 2\}$ and an infinite $B \subset \mathbb{N}$ such that $B + B \subset C_i$.*

As an immediate corollary of Theorem 1.3.3 for the integers we obtain the result in the natural numbers as well.

Corollary 5.7.4. *Let $A \subset \mathbb{N}$ be a set and $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ be a Følner sequence in $\mathbb{N}$. Then, if*

$$\mathrm{d}_{\Phi}(A) + \mathrm{d}_{\Phi/2}(A) > 1, \tag{5.7.1}$$

it follows that there exists an infinite set $B \subset \mathbb{N}$ *and some* $t \in \mathbb{N}_0$ *such that* $B+B \subset A-t$.

The important observation is that if Corollary 5.7.4 (or, Theorem 1.3.3 in the integers) was not optimal, then we would obtain an answer to Question 5.7.1 as a direct application of this. More precisely, assume there exists a Følner sequence Φ in $\mathbb{N}$ and some $\varepsilon > 0$ such that (5.7.1) in the context of Corollary 5.7.4 can be replaced by $\mathrm{d}_{\Phi}(A) + \mathrm{d}_{\Phi/2}(A) > 1-\varepsilon$, yielding the same conclusion. Then, given any colouring $\mathbb{N} = C_1 \cup C_2$, and passing to a subsequence of $\Phi = (\Phi_N)$ if necessary, it must be the case that

$$\mathrm{d}_{\Phi}(C_1) + \mathrm{d}_{\Phi/2}(C_1) + \mathrm{d}_{\Phi}(C_2) + \mathrm{d}_{\Phi/2}(C_2) = 2,$$

which implies that for some $k \in \{1,2\}$, $\mathrm{d}_{\Phi}(C_k) + \mathrm{d}_{\Phi/2}(C_k) > 1-\varepsilon$, and thus there exist infinite $B \subset \mathbb{N}$ and $t \in \mathbb{N}$ such that $B+B \subset C_k - t$.

We have already showed that Corollary 5.7.4 is optimal along Følner sequences of intervals and we believe the proof should extend to all Følner sequences. In particular, this would mean that an argument as the one laid out here cannot be used to answer Question 5.7.2, i.e. to given an alternative proof of Theorem 5.7.3.

Chapter 6

Asymmetric infinite sumsets

6.1 Overview

This chapter is based on the results of [38]. The first main result herein is a natural strengthening of Theorem C, and resolves a conjecture of Kra, Moreira, Richter and Robertson [44, Conjecture 3.10]. We show that for any set $A \subset \mathbb{N}$ with positive upper Banach density and any $\ell, m \in \mathbb{N}$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ so that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 < b_2\} + t \subset A$.

We also consider the patterns $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$, for infinite $B \subset \mathbb{N}$ and prove that any set $A \subset \mathbb{N}$ with lower density $\underline{d}(A) > 1/2$ contains such configurations up to a shift. We show that the value $1/2$ is optimal and obtain analogous results for values of upper density and when no shift to the sumsets is allowed. As in the preceding chapters, including the diagonal requires stronger density hypotheses, and the optimal thresholds depend on the precise density notion and on the parameters ℓ, m.

6.2 Introduction

In a survey paper, Kra, Moreira, Richter and Robertson, proposed a conjecture (see [44, Conjecture 3.10]) which would generalise Theorem C for weighted sumsets. Our first main result in this chapter verifies this conjecture and is the following.

Theorem 1.2.1. *For any $A \subset \mathbb{N}$ with positive upper Banach density and $\ell, m \in \mathbb{N}$, there exists some infinite set $B \subset \mathbb{N}$ and a number $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} + t \subset A.$$

Remark 6.2.1. If $\ell = m = 1$, Theorem 1.2.1 coincides with Theorem C and more generally, when $\ell = m$, Theorem 1.2.1 can easily be deduced from Theorem C. If $\ell \neq m$, shifts of the patterns $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 \neq b_2\}$ can not always be found in sets of positive density. In fact, these sumsets are not even partition regular as shown in [44, Example 3.9].

Our proof of Theorem 1.2.1 is ergodic-theoretic in nature. The main setup for it is laid out in Section 6.4 and it is completed in Section 6.5, along with a slightly stronger result (see Remark 6.5.3).

Analogously to the results of Chapter 3, we are also interested in an unrestricted version of Theorem 1.2.1. Including the diagonal in the sumsets is an important first step in this direction – perhaps even the best one possible.

Theorem 1.3.6. *Let $\ell, m \in \mathbb{N}$. For any $A \subset \mathbb{N}$ with $\underline{d}(A) > 1/2$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A.$$

Theorem 1.3.7. *Let $\ell, m \in \mathbb{N}$ and $k = m/\ell$. For any $A \subset \mathbb{N}$ with $\overline{d}(A) > 1 - 1/(k+2)$, there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A.$$

We will in fact show that both the previous bounds are optimal (in the sense of Propositions 3.5.1 and 3.5.2). We also stress that the bound established in Theorem 1.3.6 does not depend on the parameters m, ℓ, unlike the bound in Theorem 1.3.7, which does so implicitly, as it depends on the ratio m/ℓ. On another note, it is easy to see that the shift t in Theorems 1.2.1, 1.3.6 and 1.3.7 can be chosen from $\{0, 1, \ldots, \ell + m - 1\}$. Indeed, write $t = (\ell + m)j + i$, for some $j \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, \ell + m - 1\}$. Then, for example, the inclusion $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A$ can be rewritten as $\{mb_1 + \ell b_2 : b_1, b_2 \in (B + j) \text{ and } b_1 \leq b_2\} + i \subset A$.

We also prove similar results for the case of unshifted patterns.

Theorem 6.2.2. *Let $\ell, m \in \mathbb{N}$ and $k = m/\ell$. For any $A \subset \mathbb{N}$ with $\overline{d}(A) > 1 - 1/\left(\ell(k+1)(k+2)\right)$, there exists an infinite set $B \subset \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \subset A.$$

In Section 6.6 we prove that Theorems 1.3.7 and 6.2.2 are optimal. An analogue of Theorem 6.2.2 with lower density threshold of $\underline{d}(A) > 1 - 1/(2(\ell + m))$ is proven

in Section 6.7, where we also show that this and Theorem 1.3.6 are optimal. For a discussion about other potential unrestricted versions of Theorem 1.2.1 we refer the reader to Section 6.8.

Our proofs of the above results use ergodic theory. An overview of how the translation to a dynamical statement is achieved was described in Section 1.2.1.

For convenience, recall that we first relate the inclusion $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 < b_2\} + t \subset A$ to the existence of a specific system (X, μ, T), and a triple $(a, x_1, x_2) \in X^3$, with predetermined $a \in X$, such that $\left(T^\ell \times T^m\right)^{n_i} (a, x_1) \to (x_1, x_2)$, along a sequence $(n_i)_{i\in\mathbb{N}}$. In Section 6.3.1 we explain how a classical version of Furstenberg's correspondence principle allows us to translate Theorem 1.2.1 to a dynamical statement of the above form, that is, Theorem 6.3.2.

In this dynamical setting, in order to guarantee that $\{mb_1+\ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \subset A$, we moreover need to know that $(T^{j\ell})^{n_i} a \to x_2$, for some $j \in \mathbb{N}$ such that $j\ell = \ell + m$. To handle the general case when the ratio $k = m/\ell$ is not necessarily an integer, we consider a $(T^{(\lceil k\rceil+1)} \times T)$-invariant probability measure in $(\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, T^{(\lceil k\rceil+1)} \times T)$, where $\lceil k\rceil$ denotes the ceiling of k, arising from a generic pair of points (a', a), where a corresponds to the indicator of A, and a' corresponds to the indicator of an auxiliary set $A' \subset \mathbb{N}$.

In Section 6.3.2 we formulate this version of the correspondence principle as Lemma 6.3.8 and use it in order to deduce Theorem 1.3.7 from a dynamical statement, namely Theorem 6.3.6. The proof of Lemma 6.3.8, along with that of another correspondence principle which is used for the case of unshifted patterns $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$, is given in Section 6.4.5.

Finally, we should remark that since the publication of [38], Theorem 1.2.1 has been generalised by Hernández [22] with a far reaching result covering linear sumsets with any finite number of summands building on the groundbreaking work of Kra, Moreira, Richter and Robertson [45] proving a truncated density analogue of Hindman's finite sums theorem. This discussion will be particularly relevant in Chapter 7.

6.3 Translation to dynamics

6.3.1 Restricted sumsets

We next define the concept of dynamical progressions – which parallels that of Erdős progressions introduced in [41] – which as we shall see are connected to the combinatorial patterns that we are looking for in Theorem 1.2.1.

Definition 6.3.1. Given a topological system (X, T) and natural numbers ℓ, m, we say that a point $(x_0, x_1, x_3) \in X^3$ is an (ℓ, m)-*Erdős progression* if there exists a sequence $n_1 < n_2 < \cdots$ of integers such that $(T^\ell \times T^m)^{n_i}(x_0, x_1) \to (x_1, x_2)$ as $i \to \infty$.

Our first main dynamical result, the one behind Theorem 1.2.1, is the following theorem.

Theorem 6.3.2. *Let (X, μ, T) be an ergodic system, let $a \in \boldsymbol{gen}(\mu, T, \Phi)$ for some Følner sequence Φ and $E \subset X$ be an open set with $\mu(E) > 0$. Then, for each $\ell, m \in \mathbb{N}$, there exist $x_1, x_2 \in X$ and $t \in \mathbb{N}$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and $T^t x_2 \in E$.*

It turns out that Theorems 1.2.1 and 6.3.2 are actually equivalent, but we shall only deal with the required direction here. For this we need the following – parallel of [41, Theorem 2.2] – result.

Proposition 6.3.3. *Fix a topological system (X, T) and open sets $U, V \subset X$. If there exists an (ℓ, m)-Erdős progression $(x_0, x_1, x_2) \in X^3$ with $x_1 \in U$ and $x_2 \in V$, then there exists some infinite set $B \subset \{n \in \mathbb{N} \colon T^{\ell n} x_0 \in U\}$ such that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset \{n \in \mathbb{N} \colon T^n x_0 \in V\}$.*

Proof. By the definition of (ℓ, m)-Erdős progressions we can find a strictly increasing sequence $(c_n)_{n \in \mathbb{N}}$ such that $(T^\ell \times T^m)^{c_n}(x_0, x_1) \to (x_1, x_2)$ as $n \to \infty$ and each c_n is such that $T^{\ell c_n} x_0 \in U$. We will construct $B \subset \{c_n : n \in \mathbb{N}\}$ inductively and, the basis of the induction being simple, we only prove the inductive step. Suppose $b_1 < \cdots < b_n$ have been chosen so that

$$x_0 \in \bigcap_{1 \le i < j \le n} T^{-mb_i - \ell b_j} V \text{ and } x_1 \in \bigcap_{1 \le i \le n} T^{-mb_i} V.$$

Then, we can choose $b_{n+1} \in \{c_k : k \in \mathbb{N}\}$ with $b_{n+1} > b_n$ and such that

$$(T^\ell \times T^m)^{b_{n+1}}(x_0, x_1) \in \left(\bigcap_{1 \le i \le n} T^{-mb_i} V \right) \times V.$$

It follows that

$$x_0 \in \bigcap_{1 \le i < j \le n+1} T^{-mb_i - \ell b_j} V \text{ and } x_1 \in \bigcap_{1 \le i \le n+1} T^{-mb_i} V$$

and this concludes the induction. Letting $B = \{b_n : n \in \mathbb{N}\}$, the proof is complete. □

To prove that Theorem 6.3.2 implies Theorem 1.2.1 we shall use the following classical version of Furstenberg's correspondence principle.

Lemma 6.3.4 ([43, Theorem 2.10])**.** *For a set $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ there exists an ergodic system (X, μ, T), a Følner sequence Φ, a point $a \in \mathtt{gen}(\mu, T, \Phi)$ and a clopen set $E \subset X$ such that $\mu(E) > 0$ and $A = \{n \in \mathbb{N} : T^n a \in E\}$.*

Proof that Theorem 6.3.2 implies Theorem 1.2.1. Let $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ and (X, μ, T), $a \in X$, Φ and $E \subset X$ be those arising from Lemma 6.3.4. By Theorem 6.3.2, there exists a $t \in \mathbb{N}$ and an (ℓ, m)-Erdős progression $(a, x_1, x_2) \in \{a\} \times X \times T^{-t}E$. Invoking Proposition 6.3.3 we obtain an infinite set $B \subset \mathbb{N}$, such that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset \{n \in \mathbb{N} \colon T^n a \in T^{-t}E\}$. Since $A = \{n \in \mathbb{N} : T^n a \in E\}$, we see that $A - t = \{n \in \mathbb{N} \colon T^n a \in T^{-t}E\}$, so the theorem follows. □

Apropos of this discussion, we address the necessity of the shift in Theorem 1.2.1 and also the density threshold for the unshifted version. This is merely an observation, but for the reader's convenience we prove it in the next proposition.

Proposition 6.3.5. *Let $\ell, m \in \mathbb{N}$. If $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 1 - \frac{1}{(\ell+m)}$, then there is an infinite set $B \subset \mathbb{N}$ such that*

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset A.$$

Otherwise, the shift in Theorem 1.2.1 is in general necessary.

Proof. Observe that the set $A = \mathbb{N} \setminus (\ell + m)\mathbb{N}$ has natural density $1 - \frac{1}{(\ell+m)}$ and contains no infinite sumset of the form $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\}$. Indeed, the infinity of B allows us to choose an infinite subset of it, say $B' \subset B$, all the elements of which are equal modulo $(\ell + m)$. That is, there is some $j \in \{0, 1, \ldots, \ell + m - 1\}$ so that any $b \in B'$ is of the form $b = (\ell + m)n + j$, some $n \in \mathbb{N}$. It follows that $mb_1 + \ell b_2 \in (\ell + m)\mathbb{N}$, for any $b_1, b_2 \in B'$. This means that the shift above is necessary and the density threshold cannot be improved.

On the other hand, if $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 1 - \frac{1}{(\ell+m)}$, it is easy to see that $\mathrm{d}^*(A \cap (\ell + m)\mathbb{N}) > 0$ and then by Theorem 1.2.1 there is some infinite $B \subset \mathbb{N}$ such that

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset A \cap (\ell + m)\mathbb{N} \subset A.$$

Again, for the first inclusion we implicitly used the fact that infinity of B allows us to choose an infinite subset of it, all the elements of which are equal modulo $(\ell + m)$. □

6.3.2 Lifting restrictions

In order to prove the combinatorial results in Theorems 1.3.7 and 6.2.2 we will use the following dynamical results respectively. Recall that for a real number $r \in \mathbb{R}$, $\lceil r \rceil \in \mathbb{Z}$ denotes its ceiling.

Theorem 6.3.6. *Let (X, μ, T) be an ergodic system and $a \in \boldsymbol{gen}(\mu, \Phi)$ for some Følner sequence Φ. Moreover, let $\ell, m \in \mathbb{N}$, $q = \lceil m/\ell \rceil$ and consider open sets $E_1, \ldots, E_{\ell+m}, F_1, \ldots, F_{\ell+m} \subset X$ such that $F_j = T^{-(j-1)}F_1$, $j = 1, \ldots, \ell + m$, $E_{i+(q+1)} = T^{-1}E_i$, for $i = 1, \ldots, \ell + m - q - 1$, and also*

$$(\ell + m)\mu(F_1) + \ell\left(\mu(E_1) + \cdots + \mu(E_{q+1})\right) > \ell(q + 1). \tag{6.3.1}$$

Then, for some $j \in \{1, \ldots, \ell + m\}$, there exist $x_1, x_2 \in X$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and $(x_1, x_2) \in E_j \times F_j$.

Theorem 6.3.7. *Let (X, μ, T) be an ergodic system, let $a \in \boldsymbol{gen}(\mu, \Phi)$ for some Følner sequence Φ and $E, F \subset X$ be an open sets with*

$$(\ell + m)\mu(F) + \ell\mu(E) > 2\ell + m - 1, \tag{6.3.2}$$

for some $\ell, m \in \mathbb{N}$. Then, there exist $x_1, x_2 \in X$ such that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and also $(x_1, x_2) \in E \times F$.

To facilitate the transition from ergodic theory to combinatorics in this setting, we shall again utilise the notion of $(\ell, m)-$Erdős progressions as in Definition 6.3.1, as well as Proposition 6.3.3. However, the previously used, more classical version of Furstenberg's correspondence principle seems to no longer be useful and we need the adaptations presented in Lemmas 6.3.8 and 6.3.9 below, for Theorems 1.3.7 and 6.2.2, respectively.

Before stating the lemmas, we establish some notation; Σ denotes the space $\{0, 1\}^{\mathbb{Z}}$ and is endowed with the product topology so that it is compact metrizable. We also let $S \colon \Sigma \to \Sigma$ denote the shift transformation given by $S(x(n)) = x(n + 1)$, for any $n \in \mathbb{Z}$, $x = (x(n))_{n\in\mathbb{Z}} \in \Sigma$.

Lemma 6.3.8. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$ with $k = m/\ell$ and let $q = \lceil k \rceil$. Then, there exist an ergodic system $(\Sigma \times \Sigma, \mu, S^{(q+1)} \times S)$, an open set $E \subset \Sigma$, a pair of points $a, a' \in \Sigma$ and a Følner sequence Φ, such that $(a', a) \in \boldsymbol{gen}(\mu, \Phi)$ and*

$$(\ell+m)\mu(\Sigma\times E)+\ell\sum_{j=0}^{q}\mu(S^{-j}E\times\Sigma) \geq (\ell+m)\left((k+1)\cdot\overline{\mathrm{d}}(A)-k\right)+\ell(k+1)\cdot\overline{\mathrm{d}}(A)+\ell(q-k).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ *and* $(A-j)/(\ell+m) = \{n \in \mathbb{N} : S^{(q+1)\ell n + j} a' \in E\}$*, for each* $j = 0, 1, \ldots, \ell+m-1$*, where* $(A-j)/(\ell+m) = \{n \in \mathbb{N} : n(\ell+m)+j \in A\}$*.*

Lemma 6.3.9. *Let* $A \subset \mathbb{N}$ *and* $\ell, m \in \mathbb{N}$ *with* $k = m/\ell$*. Then, there exists an ergodic system* $(\Sigma \times \Sigma, \mu, S \times S)$*, an open set* $E \subset \Sigma$*, a pair of points* $a'', a \in \Sigma$ *and a Følner sequence* Φ*, such that* $(a'', a) \in \boldsymbol{gen}(\mu, \Phi)$ *and*

$$(\ell+m)\mu(\Sigma \times E) + \ell\mu(E \times \Sigma) \geq (2\ell+m)\left((k+1)\cdot \overline{\mathrm{d}}(A) - k\right).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ *and* $A/(\ell+m) = \{n \in \mathbb{N} : S^{\ell n} a'' \in E\}$*.*

We postpone the proofs of Lemmas 6.3.8 and 6.3.9 until the end of Section 6.4. Instead, we will finish this section by showing how to deduce Theorems 1.3.7 and 6.2.2 from their dynamical counterparts, using the tools we have acquired thus far. To this end, we reverse the order of presentation and start with the case of no shift because the proof is, at the very least notationally, lighter.

Proof that Theorem 6.3.7 implies Theorem 6.2.2. Let $\ell, m \in \mathbb{N}$ and let $k = m/\ell$. Given $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 1 - 1/\left(\ell(k+1)(k+2)\right)$, we find, by way of Lemma 6.3.9, an ergodic system $(\Sigma \times \Sigma, \mu, S \times S)$, an open set $E \subset \Sigma$, a pair of points $a'', a \in \Sigma$ and a Følner sequence Φ, such that $(a'', a) \in \mathbf{gen}(\mu, \Phi)$ and

$$\begin{aligned}(\ell+m)\mu(\Sigma \times E) + \ell\mu(E \times \Sigma) \geq (2\ell+m)\left((k+1)\cdot \overline{\mathrm{d}}(A) - k\right) > \\ (2\ell+m)(1 - \frac{1}{\ell(k+2)}) = 2\ell + m - 1,\end{aligned}$$

because $\ell + m = \ell(k+1)$, hence $\ell(k+2) = 2\ell + m$. It follows by Theorem 6.3.7 that there exist points $(x_{10}, x_{11}), (x_{20}, x_{21}) \in \Sigma \times \Sigma$ so that $((a'', a), (x_{10}, x_{11}), (x_{20}, x_{21})) \in (\Sigma \times \Sigma)^3$ is an (ℓ, m)-Erdős progression for $(\Sigma \times \Sigma, \mu, S \times S)$ and $((x_{10}, x_{11}), (x_{20}, x_{21})) \in (E \times \Sigma) \times (\Sigma \times E)$. Then, an application of Proposition 6.3.3 yields an infinite set $B \subset \mathbb{N}$ such that

$$B \subset \{n \in \mathbb{N} : (S \times S)^{\ell n}(a'', a) \in E \times \Sigma\} = \{n \in \mathbb{N} : S^{\ell n} a'' \in E\}$$

and similarly,

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 < b_2\} \subset \{n \in \mathbb{N} : S^n a \in E\}.$$

Since $S^{\ell n} a'' \in E \iff (\ell+m)n \in A$, the former inclusion rewrites as $(\ell+m)B \subset A$ and the latter as $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 < b_2\} \subset A$. Combining these two

we conclude that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \subset A$. □

Proof that Theorem 6.3.6 implies Theorem 1.3.7. Let $m, \ell \in \mathbb{N}$ and $k = m/\ell$, $q = \lceil k \rceil$. Given $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) > (k+1)/(k+2)$, we find, by way of Lemma 6.3.8, an ergodic system $(\Sigma \times \Sigma, \mu, S^{(q+1)} \times S)$, an open set $E \subset \Sigma$, a pair of points $a', a \in \Sigma$ and a Følner sequence Φ, such that $(a', a) \in \mathbf{gen}(\mu, \Phi)$ and

$$(\ell+m)\mu(\Sigma\times E)+\ell\sum_{j=0}^{q}\mu(S^{-j}E\times\Sigma) \geq (2\ell+m)(k+1)\cdot\overline{\mathrm{d}}(A)+\ell(q-k)-(\ell+m)k >$$

$$(2\ell+m)(k+1)\frac{k+1}{k+2}+\ell(q-k)-\ell(k+1)k = \ell(k+1)(k+1)+\ell(q-k)-\ell(k+1)k = \ell(q+1),$$

We now justify why Theorem 6.3.6 applies in order for us to recover an (ℓ, m)-Erdős progression $((a', a), (x_{10}, x_{11}), (x_{20}, x_{21})) \in (\Sigma \times \Sigma)^3$, with

$$(x_{10}, x_{11}, x_{20}, x_{21}) \in E_j \times F_j = S^{-(j-1)}E \times \Sigma \times \Sigma \times S^{-(j-1)}E,$$

for some $j \in \{1, 2, \ldots, \ell + m\}$. To see this, note that $E_j = S^{-(j-1)}E \times \Sigma$ and so $E_{j+(q+1)} = (S^{(q+1)} \times S)^{-1}E_j$, for $j \in \{1, \ldots, \ell + m - q - 1\}$. Moreover we showed above that

$$(\ell + m)\mu(\Sigma \times E) + \ell \sum_{j=0}^{q} \mu(S^{-j}E \times \Sigma) > \ell(q + 1),$$

which is precisely (6.3.1). Then, using Proposition 6.3.3 we find an infinite set $B \subset \mathbb{N}$ such that

$$B \subset \{n \in \mathbb{N} : S^{(q+1)\ell n}a' \in S^{-(j-1)}E\}$$

and

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 < b_2\} \subset \{n \in \mathbb{N} : S^n a \in S^{-(j-1)}E\},$$

for some $j \in \{1, \ldots, \ell + m\}$. From the defining properties of a', we see that the former becomes $B \subset (A - (j-1))/(\ell + m)$. Thus, unraveling the definitions, we see that these two inclusions together translate to

$$\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \subset A - (j - 1),$$

and so we conclude. □

6.4 The ergodic theory setup

6.4.1 An overview

A sufficient condition for a triple $(x_0, x_1, x_2) \in X^3$ to be an (ℓ, m)-Erdős progression is that (x_0, x_1) is a $(T^\ell \times T^m)$-generic point for some invariant measure and $(x_1, x_2) \in X \times X$ is in the support of that measure. This is a general fact which can easily be deduced from the definitions (see, for example, [43, Lemma 2.4]). A well-known consequence of the mean ergodic theorem is that for an ergodic system (Y, ν, S) and any Følner sequence, there is a subsequence Φ such that ν-almost every point $y \in Y$ is S-generic along Φ. Hence, if in the above setting μ is a T-invariant measure, we want to consider an ergodic decomposition of $\mu \times \mu$.

As in [41] we are interested in progressions with prescribed first coordinate $a \in X$ and so we will reduce to the case that the ergodic decomposition is continuous. However, our work is different here because we need typical points to be $(T^\ell \times T^m)$-generic for general $\ell, m \in \mathbb{N}$. Another important aspect of this problem is that the linear patterns we are looking for are still dynamically controlled by the Kronecker factor. Therefore, we find it useful to introduce a measure σ on $X \times X$ which gives full measure to the set of points (x_1, x_2) such that (a, x_1, x_2) projects to an (ℓ, m)-three term progression on the Kronecker, in a way similar to that done in [41] for the case $\ell = m = 1$.

6.4.2 Continuous ergodic decomposition

While in general the factor map from an ergodic system (X, μ, T) to its Kronecker (Z, ν, R) is only measurable, for our purposes we may assume that it is also a continuous surjection. Indeed, using Proposition 3.20 from [43] one can show that Theorem 6.3.2 follows from the next seemingly weaker result. The proof of this implication is the same as the proof that Theorem 3.2 implies Theorem 1.4 in [41] or the proof of Theorem 3.3.1 via Theorem 3.4.4 in Chapter 3.

Theorem 6.4.1. *Let (X, μ, T) be an ergodic system and assume there is a continuous factor map π to its Kronecker. Let $a \in \boldsymbol{gen}(\mu, T, \Phi)$, for some Følner sequence Φ and $E \subset X$ be an open set with $\mu(E) > 0$. Then, there exist $x_1, x_2 \in X$ and $t \in \mathbb{N}$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression such that $T^t x_2 \in E$.*

In a similar fashion, Theorems 6.3.6 and 6.3.7 follow from the next seemingly weaker results, respectively, where the system is assumed to have a continuous Kronecker factor map (essentially, the proof of Theorem 3.3.1 from Theorem 3.4.4 in Chapter 3 contains one of the analogous arguments in the case $m = \ell = 1$).

Theorem 6.4.2. *Let (X, μ, T) be an ergodic system and assume there is a continuous factor map π to its Kronecker and $a \in \boldsymbol{gen}(\mu, T, \Phi)$ for some Følner sequence Φ. Moreover, let $\ell, m \in \mathbb{N}$, $q = \lceil m/\ell \rceil$ and assume that $E_1, \dots, E_{\ell+m}, F_1, \dots, F_{\ell+m} \subset X$ are open sets such that $F_j = T^{-(j-1)}F_1$, $j = 1, \dots, \ell + m$, $E_{i+(q+1)} = T^{-1}E_i$, for $i = 1, \dots, \ell + m - q - 1$, and also*

$$(\ell + m)\mu(F_1) + \ell\left(\mu(E_1) + \cdots + \mu(E_{q+1})\right) > \ell(q + 1). \tag{6.4.1}$$

Then, for some $j \in \{1, \dots, \ell + m\}$, there exist $x_1, x_2 \in X$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and $(x_1, x_2) \in E_j \times F_j$.

Theorem 6.4.3. *Let (X, μ, T) be an ergodic system and assume there is a continuous factor map π to its Kronecker. Let $a \in \boldsymbol{gen}(\mu, T, \Phi)$ for some Følner sequence Φ and $E, F \subset X$ be an open sets with*

$$(\ell + m)\mu(F) + \ell\mu(E) > 2\ell + m - 1. \tag{6.4.2}$$

Then, there exist $x_1, x_2 \in X$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and $(x_1, x_2) \in E \times F$.

We fix $\ell, m \in \mathbb{N}$ and an ergodic system (X, μ, T) with a continuous factor map π to its Kronecker (Z, ν, R). We also fix a disintegration $z \mapsto \eta_z$ of μ over the Kronecker. Then, for every $(x_1, x_2) \in X \times X$ we define the measure

$$\lambda_{(x_1,x_2)} = \int_Z \eta_{\ell z+\pi(x_1)} \times \eta_{mz+\pi(x_2)}\, d\nu(z) \tag{6.4.3}$$

on $X \times X$. We stress that (6.4.3) is well-defined since, for each $(x_1, x_2) \in X \times X$ the measures $\eta_{\ell z+\pi(x_1)}$ and $\eta_{mz+\pi(x_2)}$ are defined for ν-almost every $z \in Z$. Note that, because of the ergodicity of R, the subgroups ℓZ and mZ of Z both have positive measure (see the proof of Lemma 6.4.7 for more details on this). We next examine some properties of this (a posteriori) disintegration of $\mu \times \mu$.

Proposition 6.4.4. *In the above setting, the map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ satisfies the following properties.*

(i). The map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is continuous.

(ii). The map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a disintegration of $\mu \times \mu$, meaning that

$$\int_{X\times X} \lambda_{(x_1,x_2)}\, d(\mu \times \mu)(x_1, x_2) = \mu \times \mu.$$

(iii). For $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$, the point (x_1, x_2) is $(T^\ell \times T^m)$-generic for $\lambda_{(x_1,x_2)}$ and $\lambda_{(x_1,x_2)}$ is $(T^\ell \times T^m)$-ergodic.

(iv). For every $(x_1, x_2) \in X \times X$, we have that $\lambda_{(x_1,x_2)} = \lambda_{(T^\ell x_1, T^m x_2)}$.

The rest of this subsection is devoted to the proof of Proposition 6.4.4. The first step is a result showing that in some sense the Kronecker is a characteristic factor.

Proposition 6.4.5. *Fix an ergodic system (X, μ, T) with Kronecker factor (Z, ν, R) and factor map $\pi : X \to Z$. Then, for any $\ell, m \in \mathbb{N}$ and $f, g \in L^\infty(X, \mu)$ we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\ell n} x_0) \cdot g(T^{mn} x_1) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(f|Z)(R^{\ell n}\pi(x_0)) \cdot \mathbb{E}(g|Z)(R^{mn}\pi(x_1)), \tag{6.4.4}$$

for $(\mu \times \mu)$-almost every $(x_0, x_1) \in X \times X$.

Proof. Both limits in (6.4.4) exist by the pointwise ergodic theorem, so we simply need to establish their equality in the L^2 norm. By the Jacobs-de Leeuw-Glicksberg decomposition (see [35, Theorem 2.24]) this reduces to showing that whenever either f or g is a weak-mixing function, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{\ell n} f \otimes T^{mn} g = 0, \tag{6.4.5}$$

in $L^2(\mu \times \mu)$. Assuming, without loss of generality, that f is the weak-mixing function and setting $u_n = T^{\ell n} f \otimes T^{mn} g$ this follows directly by the van der Corput lemma (originally proven in this version in [7, Theorem 1.4]), for

$$\left| \frac{1}{K} \sum_{k=1}^{K} \frac{1}{N} \sum_{n=1}^{N} \langle u_{n+k}, u_n \rangle \right| = \left| \frac{1}{K} \sum_{k=1}^{K} \int_X T^{\ell k} f \cdot \overline{f} \, d\mu \int_X T^{mk} g \cdot \overline{g} \, d\mu \right| \leq$$
$$\|g\|_\infty^2 \frac{1}{K} \sum_{k=1}^{K} \left| \int_X T^{\ell k} f \cdot \overline{f} \, d\mu \right|,$$

which goes to 0 as $K \to \infty$ by the definition of weak-mixing functions and the inequality

$$\frac{1}{K} \sum_{k=1}^{K} \left| \int_X T^{\ell k} f \cdot \overline{f} \, d\mu \right| \leq \frac{\ell}{\ell K} \sum_{k=1}^{\ell K} \left| \int_X T^{k} f \cdot \overline{f} \, d\mu \right|.$$

□

We are now in the position to prove the main result of this subsection.

Proof of Proposition 6.4.4. We begin by showing the map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a disintegration of $\mu \times \mu$. Let $f, g \in L^\infty(\mu)$. Then,

$$\begin{aligned}
&\int_{X^2} f \otimes g \; d\lambda_{(x_1,x_2)} d(\mu \times \mu)(x_1, x_2) \\
&= \int_{X^2} \int_Z \int_X f \; d\eta_{\ell z + \pi(x_1)} \int_X g \; d\eta_{mz+\pi(x_2)} \; d\nu(z) d\mu(x_1) d\mu(x_2) \\
&= \int_Z \left(\int_X \int_X f \; d\eta_{\ell z+\pi(x_1)} \; d\mu(x_1) \times \int_X \int_X g \; d\eta_{mz+\pi(x_2)} \; d\mu(x_2) \right) \; d\nu(z) \\
&= \int_Z \left(\int_X f \; d\mu \times \int_X g \; d\mu \right) \; d\nu(z) = \int f \otimes g \; d(\mu \times \mu),
\end{aligned}$$

because of (6.4.3) and the fact that for each $w \in Z$,

$$\mu = \int_Z \eta_z \; d\nu(z) = \int_Z \eta_z \; d(\pi\mu)(z) = \int_X \eta_{\pi(x)} \; d\mu(x) = \int_X \eta_{w+\pi(x)} \; d\mu(x).$$

Part (ii) follows by standard approximation arguments using Stone-Weierstrass' theorem and the Riesz-Markov-Kakutani representation theorem.

To prove that $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is continuous we need to show that for each $F \in C(X \times X)$, the map $(x_1, x_2) \mapsto \int_{X \times X} F \; d\lambda_{(x_1,x_2)}$ is continuous, because – implicitly – the topology we endow the space of Borel measures on $X \times X$ with is the weak* topology. To this end, by another application of Stone-Weierstrass' theorem, we can assume that $F \in C(X \times X)$ in the previous is of the form $f \otimes g$, for some $f, g \in C(X)$. Now, letting $f, g \in C(Z)$, these functions are also uniformly continuous (by compactness) and so the map

$$(v, w) \mapsto \int_Z f(\ell z + v) \; g(mz + w) \; d\nu(z)$$

is continuous. Thus, the density of $C(Z)$ in $L^2(Z, m)$ implies the continuity of the analogous map for $f, g \in L^2(Z, m)$.

Now, if $f, g \in C(X)$, we have that $\mathbb{E}(f|Z), \mathbb{E}(g|Z) \in L^2(Z, m)$ and so we see that

$$(x_1, x_2) \mapsto \int_Z \mathbb{E}(f|Z)(\ell z + \pi(x_1)) \; \mathbb{E}(g|Z)(mz + \pi(x_2)) \; d\nu(z)$$

is continuous as the composition of continuous maps. Noting that $\mathbb{E}[f|Z](z) = \int_X f \; d\eta_z$ for ν-almost every $z \in Z$, we see that

$$\int_{X \times X} f \otimes g \; d\lambda_{(x_1,x_2)} = \int_Z \mathbb{E}(f|Z)(\ell z + \pi(x_1)) \; \mathbb{E}(g|Z)(mz + \pi(x_2)) \; d\nu(z) \quad (6.4.6)$$

and so the continuity of $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ follows.

Property (iv) is immediate up to null sets because $\pi \circ T = R \circ \pi$ as π is a factor map and then, the established continuity of the decomposition implies it for all points $(x_1, x_2) \in X \times X$.

We are only left with proving that $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a $(T^\ell \times T^m)-$ergodic decomposition of $\mu \times \mu$, because then part (iii) follows as a consequence of the pointwise ergodic theorem (see, for example, [43, Corollary 2.9]). In other words, we have to show that for each bounded and measurable $F : X \times X \to \mathbb{C}$ it holds that

$$\int_{X\times X} F \; d\lambda_{(x_1,x_2)} = \mathbb{E}(F|\mathcal{I})(x_1, x_2),$$

for $(\mu \times \mu)-$almost every $(x_1, x_2) \in X \times X$, where $\mathcal{I}$ denotes the σ-algebra of $(T^\ell \times T^m)-$invariant sets on $X \times X$. By the ergodic theorem, this is equivalent to showing that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(T^{\ell n} x_1, T^{mn} x_2) = \int_{X\times X} F \; d\lambda_{(x_1,x_2)},$$

for $(\mu\times\mu)-$almost every $(x_1, x_2) \in X\times X$ and by standard approximation arguments this reduces to showing that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\ell n} x_1) \cdot g(T^{mn} x_2) = \int_{X\times X} f \otimes g \; d\lambda_{(x_1,x_2)},$$

for $(\mu \times \mu)-$almost every $(x_1, x_2) \in X \times X$ and every $f, g \in L^\infty(X, \mu)$. By (6.4.4), it suffices to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(f|Z)(R^{\ell n}\pi(x_1)) \cdot \mathbb{E}(g|Z)(R^{mn}\pi(x_2)) = \int_{X\times X} f \otimes g \; d\lambda_{(x_1,x_2)}, \quad (6.4.7)$$

for $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$. As the limit in (6.4.7) exists almost everywhere, and the dominated convergence theorem applies, it suffices to prove convergence in $L^2(\mu \times \mu)$. By (6.4.6) and the density of (tensor products of) continuous functions in $L^2(\nu \times \nu)$ it thus suffices to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi_1(R^{\ell n} z_1) \cdot \phi_2(R^{mn} z_2) = \int_Z \phi_1(\ell z + z_1) \cdot \phi_2(mz + z_2) \; d\nu(z), \quad (6.4.8)$$

for any $\phi_1, \phi_2 \in C(Z)$ and any $z_1, z_2 \in Z$.

With this reduction, the algebraic structure of rotations in compact abelian

groups (see, for example, [31, Chapter 4]) allows us to conclude. More precisely, given $\phi_1, \phi_2 \in C(Z)$ and $z_1, z_2 \in Z$, the function $\phi : Z \to \mathbb{C}$ given by

$$\phi(z) = \phi_1(\ell z + z_1) \cdot \phi_2(mz + z_2)$$

is continuous, and therefore by the unique ergodicity of group rotations we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(R^n(0)) = \int_Z \phi(z)\ d\nu(z) = \int_Z \phi_1(\ell z + z_1) \cdot \phi_2(mz + z_2)\ d\nu(z).$$

Finally, $\phi(R^n(0)) = \phi_1(R^{\ell n} z_1) \cdot \phi_2(R^{mn} z_2)$, for each $n \in \mathbb{N}$, and thus (6.4.8) follows. □

6.4.3 A measure on $(\ell, m)-$Erdős progressions and some properties

As above, (X, μ, T) is an ergodic system and (Z, ν, R) is its Kronecker factor with (continuous) factor map $\pi : X \to Z$, and R is a rotation by some $b \in Z$. Moreover, we let $a \in \mathbf{gen}(\mu, T, \Phi)$, for some Følner sequence Φ. Given $\ell, m \in \mathbb{N}$ we consider the measure

$$\sigma_a = \int_Z \eta_{\ell z+\pi(a)} \times \eta_{(\ell+m)z+\pi(a)}\ d\nu(z) \tag{6.4.9}$$

on $X \times X$. The first useful property of σ_a which relates to the disintegration $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ reads as follows.

Lemma 6.4.6. *For σ_a-almost every $(x_1, x_2) \in X \times X$ it holds that $\lambda_{(a,x_1)} = \lambda_{(x_1,x_2)}$.*

Proof. It is obvious from the definition of σ_a that the set P defined by

$$P = \{(x_1, x_2) \in X \times X \colon \pi(x_1) = \ell w + \pi(a) \text{ and } \pi(x_2) = (\ell + m)w + \pi(a), \text{ for some } w \in Z\}$$

has $\sigma_a(P) = 1$. We fix $(x_1, x_2) \in P$ and let $w \in Z$ be such that $\pi(x_1) = \ell w + \pi(a)$ and $\pi(x_2) = (\ell + m)w + \pi(a)$. The proof will be complete once we show that $\lambda_{(x_1,x_2)} = \lambda_{(a,x_1)}$. To this end, since ν is a shift invariant measure, making the change of variables $z \mapsto z - w$ we get that

$$\lambda_{(x_1,x_2)} = \int_Z \eta_{\ell z+\ell w+\pi(a)} \times \eta_{mz+(\ell+m)w+\pi(a)}\ d\nu(z) = \int_Z \eta_{\ell z+\pi(a)} \times \eta_{mz+\ell w+\pi(a)}\ d\nu(z),$$

which equals $\lambda_{(a,x_1)}$ since $\pi(x_1) = \ell w + \pi(a)$. □

The last result illuminates why this particular definition of σ_a is useful. The measure was defined so that it essentially only witnesses pairs $(x_1, x_2) \in X \times X$ whose

projection on the Kronecker give rise to 3-term (ℓ, m)-progressions in Z through $(\pi(a), \pi(x_1), \pi(x_2))$. This is apparent from the equalities $\pi(x_1) - \pi(a) = \ell w$ and $\pi(x_2) - \pi(x_1) = mw$, some $w \in Z$, which hold for any $(x_1, x_2) \in P$.

Then, a potential route to finding (ℓ, m)-Erdős progressions of the form $(a, x_1, x_2) \in X^3$ is laid out. It is sufficient to find a pair $(x_1, x_2) \in P$ with $(x_1, x_2) \in \mathtt{supp}\lambda_{(x_1,x_2)}$, and such that (a, x_1) is $(T^\ell \times T^m)$-generic for $\lambda_{(a,x_1)}$, which coincides with $\lambda_{(x_1,x_2)}$. To this end, we shall also need two relations between the measure μ and push-forwards of projections of σ_a.

Lemma 6.4.7. *Let $\pi_i : X \times X \to X$ denote the projection $(x_1, x_2) \mapsto x_i$ onto the i-th coordinate, where $i \in \{1, 2\}$. Then, $\frac{1}{\ell}\left(\pi_1\sigma_a + T\pi_1\sigma_a + \cdots + T^{\ell-1}\pi_1\sigma_a\right) = \mu$ and $\frac{1}{\ell+m}\left(\pi_2\sigma_a + T\pi_2\sigma_a + \cdots + T^{\ell+m-1}\pi_2\sigma_a\right) = \mu$.*

Proof. We only prove the first claim as the second one follows similarly. Let ℓZ denote the subgroup $\{\ell z = z + z + \cdots + z \text{ with } \ell \text{ summands}\colon z \in Z\}$ and let ξ denote its Haar measure. Ergodicity of R means that $\{R^n 0 : n \in \mathbb{N}\}$ is dense in Z, and thus $Z = (\ell Z) \cup (R(\ell Z)) \cup \cdots \cup (R^{\ell-1}(\ell Z))$. Therefore, there exists $w \in Z$ and $j \in \{0, 1, \ldots, \ell - 1\}$ such that $\pi(a) = R^j(\ell w) = \ell w + jb$, so that

$$\pi_1\sigma_a = \int_Z \eta_{\ell z + \pi(a)}\, d\nu(z) = \int_Z \eta_{\ell(z+w)+jb}\, d\nu(z) = \int_{\ell Z + jb} \eta_u\, d(R^j\xi)(u).$$

Finally, $T^i\eta_u = \eta_{R^i u}$ and $R^\ell\xi = \xi$, which implies that

$$\frac{1}{\ell}T^i\pi_1\sigma_a = \int_{\ell Z+jb} T^i\eta_u\, d\frac{1}{\ell}(R^j\xi)(u) = \int_{\ell Z+(j+i)b} \eta_u\, d\frac{1}{\ell}(R^{j+i}\xi)(u),$$

for each $i \in \{0, 1, \ldots, \ell - 1\}$, where $j + i$ is taken mod ℓ and thus

$$\frac{1}{\ell}\left(\pi_1\sigma_a + \cdots + T^{\ell-1}\pi_1\sigma_a\right) = \int_Z \eta_z\, d\frac{1}{\ell}(\xi + R\xi + \cdots + R^{\ell-1}\xi)(z) = \mu.$$

□

6.4.4 Support of the measure on $(\ell, m)-$Erdős progressions

We begin with some notational remarks. Recall our setting; We have an ergodic system (X, μ, T) with a continuous factor map π to its Kronecker (Z, ν, R), say R is the rotation by some $b \in Z$, and a generic point $a \in \mathbf{gen}(\mu, T, \Phi)$ for some Følner sequence Φ. We write $z \to \eta_z$ for the disintegretion of μ over π. Then if $k \in \mathbb{N}$ is any, we can define $X_i = \pi^{-1}(R^i(kZ))$, for $i = 0, 1, \ldots, k-1$ and consider the ergodic

components of μ for the transformation T^k given by

$$\mu_i = \int_Z \eta_{kz+ib}\, d\nu(z) = \int_{kZ+ib} \eta_u\, d(R^i\zeta)(u), \tag{6.4.10}$$

where ζ is the Haar measure on kZ. This subsection is devoted to showing that if

$$S = \{(x_1, x_2) \in X \times X : (x_1, x_2) \in \mathtt{supp}(\lambda_{(x_1,x_2)})\}, \tag{6.4.11}$$

then $\sigma_a(S) = 1$. We begin with an extension of [41, Proposition 3.10].

Remark 6.4.8. In the above setting, for any $x \in X_i$ we have that $\pi(x) = kw + ib$, some $w \in Z$, $i = 0, 1, \ldots, k-1$.

For the next result we use notation from [41]. In particular, $\mathcal{F}(X)$ denotes the family of non-empty and closed subsets of the compact metric space (X, d), endowed with the Hausdorff metric, denoted by $\mathbf{H}$.

Proposition 6.4.9. *Fix a system (X, μ, T) and a continuous factor map π to its Kronecker factor (Z, ν, R). Also fix $k \in \mathbb{N}$ and a disintegration $z \mapsto \eta_z$ over π. There is a sequence $\delta(j) \to 0$, such that for μ-almost every $x \in X$ (with $\pi(x) = kw + ib$ as in Remark 6.4.8) the following holds: for every neighbourhood U of x we have*

$$\lim_{j\to\infty} \frac{\nu\left(\{z \in Z : \eta_{kz+ib}(U) > 0\} \cap B(w, \delta(j))\right)}{\nu(B(w, \delta(j)))} = 1. \tag{6.4.12}$$

Remark. This result almost follows directly from [41, Proposition 3.10] by considering the finitely many ergodic components of (X, μ, T^k). Some additional information provided here is the fact that the sequence $(\delta(j))_{j\in\mathbb{N}}$ does not depend on the choice of the ergodic component.

Proof. We consider the maps $\Phi_i : Z \to (X)$ defined by $\Phi_i(z) = \mathtt{supp}(\eta_{kz+ib})$, for any $z \in Z$ and $i \in \{0, 1, 2, \ldots, k-1\}$. These maps are Borel measurable as the composition of three Borel measurable maps, $z \mapsto kz + ib$, $z \mapsto \eta_z$ and $\nu \mapsto \mathtt{supp}(\nu)$ (the latter is measurable in way of Lemma 3.8 of [41]).

Just like in the proof of Proposition 3.10 of [41], by Lusin's theorem, for each $j \in \mathbb{N}$ there is a closed set $Z_{i,j} \subset Z$ with $\nu(Z_{i,j}) > 1 - 2^{-j}$ and a $\delta(i, j) > 0$ so that for all $z_1, z_2 \in Z_{i,j}$,

$$d(z_1, z_2) \le \delta(i, j) \implies \mathbf{H}(\Phi_i(z_1), \Phi_i(z_2)) < \frac{1}{j}.$$

If we consider the sets

$$K_{i,j} = \{z \in Z_{i,j} \colon \nu\left(B(z, \delta(i,j)) \cap Z_{i,j}\right) > \left(1 - \frac{1}{j}\right) \nu\left(B(z, \delta(i,j))\right\}$$

and let $K_i = \bigcup_{M \geq 1} \bigcap_{j \geq M} K_{i,j}$ it follows by the same argument as in the proof of Proposition 3.10 in [41] that $\nu(K_i) = 1$.

Next, we let $L_i' = \{x \in X : x \in \mathtt{supp}(\eta_{\pi(x)})\} \cap \pi^{-1}(kK_i + ib)$. By the above we see that $\nu(kK_i) = \nu(kZ)$ and since $\mu(\{x \in X : x \in \mathtt{supp}(\eta_{\pi(x)})\}) = 1$ (see Lemma 3.9 in [41]) it follows that $\mu_i(L_i') = 1$. Thus, setting $L' = L_0' \cup L_1' \cup \cdots \cup L_{k-1}'$ we have $\mu(L') = 1$.

Fix $x \in L'$, and $i \in \{0, 1, \ldots, k-1\}$ so that $x \in L_i'$ and let U be an open neighbourhood of x. In this case $\pi(x) = kw + ib$, some $w \in K_i$, because $\pi(x) \in kK_i + ib$. Now, as $w \in K_i$ and U is open there is $j_0 \in \mathbb{N}$ such that $w \in K_{i,j}$ and $B(x, 1/j) \subset U$, for all $j \geq j_0$. We claim that

$$B(w, \delta(i,j)) \cap Z_{i,j} \subset H := \{z \in Z : \eta_{kz+ib}(U) > 0\}. \tag{6.4.13}$$

Indeed, let $w' \in B(w, \delta(i,j)) \cap Z_{i,j}$. As $d(w', w) < \delta(i,j)$ and w', w are continuity points for Φ_i, we see that $\mathbf{H}(\Phi_i(w), \Phi_i(w')) < 1/j$. Then, because $x \in \Phi_i(w)$, there exists $x' \in \Phi_i(w')$ with $d(x, x') < 1/j$. This of course implies that $x' \in U$ and so $x' \in U \cap \Phi_i(w')$. As $\Phi_i(w') = \mathtt{supp}(\eta_{kw'+ib})$ it follows that $\eta_{kw'+ib}(U) > 0$, that is $w' \in H$. As $w \in K_{i,j}$ it follows from (6.4.13) and by the construction of $Z_{i,j}$ that

$$\frac{\nu(H \cap B(w, \delta(i,j)))}{\nu(B(w, \delta(i,j)))} \geq 1 - \frac{1}{j},$$

for all $j \geq j_0$. This then implies that

$$\lim_{j \to \infty} \frac{\nu(H \cap B(w, \delta(i,j)))}{\nu(B(w, \delta(i,j)))} = 1.$$

Setting $\delta(j) = \min_{\{i=0,1,\ldots,k-1\}} \delta(i,j)$, for each $j \in \mathbb{N}$, gives (6.4.12). □

We are now in the position to prove the main result of this section.

Proposition 6.4.10. *Let* $S = \{(x_1, x_2) \in X \times X : (x_1, x_2) \in \mathtt{supp}(\lambda_{(x_1,x_2)})\}$ *as in* (6.4.11). *Then* $\sigma_a(S) = 1$.

Proof. Using Proposition 6.4.9 above for $k = \ell$ and $k = m$, we find a sequence $\delta(j) \to 0$ and two sets $L, L' \subset X$ such that each point $x' \in L'$ satisfies (6.4.12) with $k = \ell$ and each $x \in L$ satisfies (6.4.12) with $k = m$. Now, $\mu(L) = \mu(L') = 1$

and using Lemma 6.4.7 we see that $\sigma_a(L' \times L) = 1$. To see this simply note that $L' \times L = (L' \times X) \cap (X \times L)$ and $\pi_1\sigma_a(L') = \pi_2\sigma_a(L) = 1$. We have thus reduced matters to showing that $L' \times L \subset S$.

To this end, let $(x_0, x_1) \in L' \times L$ and U_0, U_1 be neighbourhoods of x_0, x_1 respectively. It suffices to verify that $\lambda_{(x_0,x_1)}(U) > 0$, where $U = U_0 \times U_1$. Writing $\pi(x_0) = \ell w_0 + i_0 b$ and $\pi(x_1) = m w_1 + i_1 b$ as in Remark 6.4.8, we have that

$$\lambda_{(x_0,x_1)} = \int_Z \eta_{\ell z + \ell w_0 + i_0 b} \times \eta_{mz + m w_1 + i_1 b} \, \mathrm{d}\nu(z)$$

and making the change of variables $z \mapsto z - w_0$ we see that

$$\lambda_{(x_0,x_1)}(U) = \int_Z \eta_{\ell z + i_0 b}(U_0) \times \eta_{mz + m(w_1 - w_0) + i_1 b}(U_1) \, \mathrm{d}\nu(z).$$

Now, as $x_0 \in L'$ and $x_1 \in L$, there is $\delta > 0$ such that

$$\frac{\nu\left(\{z \in Z : \eta_{\ell z + i_0 b}(U_0) > 0\} \cap B(w_0, \delta)\right)}{\nu(B(w_0, \delta))} \geq \frac{5}{6} \tag{6.4.14}$$

and also

$$\frac{\nu\left(\{z \in Z : \eta_{mz + i_1 b}(U_1) > 0\} \cap B(w_1, \delta)\right)}{\nu(B(w_1, \delta))} \geq \frac{5}{6}.$$

But $\{z \in Z : \eta_{mz + i_1 b}(U_1) > 0\} - (w_1 - w_0) = \{z \in Z : \eta_{mz + m(w_1 - w_0) + i_1 b}(U_1) > 0\}$ and so we get

$$\frac{\nu\left(\{z \in Z : \eta_{mz + m(w_1 - w_0) + i_1 b}(U_1) > 0\} \cap B(w_0, \delta)\right)}{\nu(B(w_0, \delta))} \geq \frac{5}{6}. \tag{6.4.15}$$

Finally, we consider the set G defined by

$$W = \{z \in Z : \eta_{\ell z + i_0 b}(U_0) > 0 \text{ and } \eta_{mz + m(w_1 - w_0) + i_1 b}(U_1) > 0\}.$$

It is clear from (6.4.14) and (6.4.15) that W contains at least half of the ball $B(w_0, \delta)$ and thus $\nu(W) > 0$. As for all $z \in W$ we have

$$\eta_{\ell z + i_0 b} \times \eta_{mz + m(w_1 - w_0) + i_1 b}(U_0 \times U_1) > 0$$

and $\nu(W) > 0$, it follows that $\lambda_{(x_0,x_1)}(U_0 \times U_1) > 0$, as desired. □

6.4.5 Proofs of correspondence principles

We move on to prove the correspondence principle-type of results stated in Lemmas 6.3.8 and 6.3.9. We start with the latter as it has a slightly simpler proof.

Recall that Σ denotes the space $\{0,1\}^{\mathbb{Z}}$ and is endowed with the product topology so that it is compact metrizable. We also let $S\colon \Sigma \to \Sigma$ denote the shift transformation given by $S(x(n)) = x(n+1)$, for any $n \in \mathbb{Z}$, $x = (x(n))_{n\in\mathbb{Z}} \in \Sigma$. We also recall the statements for convenience.

Lemma 6.3.9. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$ with $k = m/\ell$. Then, there exists an ergodic system $(\Sigma \times \Sigma, \mu, S \times S)$, an open set $E \subset \Sigma$, a pair of points $a'', a \in \Sigma$ and a Følner sequence Φ, such that $(a'', a) \in \boldsymbol{gen}(\mu, \Phi)$ and*

$$(\ell+m)\mu(\Sigma \times E) + \ell\mu(E \times \Sigma) \geq (2\ell+m)\left((k+1)\cdot \overline{\mathrm{d}}(A) - k\right).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ and $A/(\ell+m) = \{n \in \mathbb{N} : S^{\ell n} a'' \in E\}$.

Proof. Let $A \subset \mathbb{N}$. By definition, there exists a sequence (N_i) of positive integers such that

$$\overline{\mathrm{d}}(A) = \lim_{i\to\infty} \frac{|A \cap [1, N_i]|}{N_i}.$$

We let $a \in \Sigma = \{0,1\}^{\mathbb{Z}}$ be the indicator of A, that is,

$$a(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we let $a'' \in \Sigma$ be defined by

$$a''(n) = \begin{cases} a((\ell+m)i) & \text{if } n = \ell i, \text{ some } i \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

If $E = \{x \in \Sigma\colon x(0) = 1\}$, we observe that E is clopen in Σ, $A = \{n \in \mathbb{N}\colon S^n a \in E\}$ and also

$$\{n \in \mathbb{N} : S^{\ell n} a'' \in E\} = \{n \in \mathbb{N} : S^{(\ell+m)n} a \in E\} = A/(\ell+m).$$

Now let $N_i' = \lfloor N_i/(k+1) \rfloor$, for all $i \in \mathbb{N}$, $i \geq k+1$, and consider the sequence of Borel probability measures (μ_i) on $\Sigma \times \Sigma$ given by

$$\mu_i = \frac{1}{N_i'} \sum_{n=1}^{N_i'} \delta_{(S\times S)^n (a'', a)}.$$

Letting μ' be a weak* accumulation point of (μ_i) we obtain an $(S \times S)$-invariant measure. It follows by definition that for all $i \in \mathbb{N}$, $i \geq k+1$, we have

$$\mu_i(\Sigma \times E) = \frac{1}{N_i'} \sum_{n=1}^{N_i'} \delta_{S^n a}(E) = \frac{|A \cap [1, N_i']|}{N_i'}. \tag{6.4.16}$$

For any such i we also have that

$$\begin{aligned}
\mu_i(\Sigma \times E) = \frac{1}{N_i'} \sum_{n=1}^{N_i'} \delta_{S^n a}(E) &= \frac{1}{N_i'} \left(\sum_{n=1}^{N_i} \delta_{S^n a}(E) - \sum_{n=N_i'+1}^{N_i} \delta_{S^n a}(E) \right) \\
&\geq \frac{k+1}{N_i} \left(\sum_{n=1}^{N_i} \delta_{S^n a}(E) - \frac{k}{k+1} N_i + o_{N_i \to \infty}(N_i) \right) \\
&= (k+1) \frac{|A \cap [1, N_i]|}{N_i} - k + o_{N_i \to \infty}(1)
\end{aligned} \tag{6.4.17}$$

Taking limits in (6.4.17) as $i \to \infty$ we have, by the definition of μ', the fact that $E \subset X$ is clopen and the choice of (N_i) that

$$\mu'(\Sigma \times E) \geq (k+1) \cdot \overline{\mathrm{d}}(A) - k. \tag{6.4.18}$$

On the other hand, for any $i \in \mathbb{N}$, $i \geq k+1$, we have that

$$\begin{aligned}
\mu_i(E \times \Sigma) = \frac{1}{N_i'} \sum_{n=1}^{N_i'} \delta_{S^n a''}(E) &= \frac{1}{N_i'} \left(\sum_{n=1, n \notin \ell\mathbb{N}}^{N_i'} \delta_{S^n a''}(E) + \sum_{n=1}^{\lfloor N_i'/\ell \rfloor} \delta_{S^{\ell n} a''}(E) \right) \\
= \frac{1}{N_i'} \left(N_i' - \lfloor \frac{N_i'}{\ell} \rfloor + \sum_{n=1}^{\lfloor N_i'/\ell \rfloor} \delta_{S^{(\ell+m)n} a}(E) \right) &\geq 1 - \frac{1}{\ell} + \frac{1}{N_i'} \sum_{n=1}^{\lfloor N_i/(\ell+m) \rfloor - 2} \delta_{S^{(\ell+m)n} a}(E) \\
= 1 - \frac{1}{\ell} + (k+1) \frac{|A \cap (\ell+m)\mathbb{N} \cap [1, N_i]|}{N_i} &+ o_{N_i \to \infty}(1),
\end{aligned} \tag{6.4.19}$$

where in the inequality we used that $\ell + m = \ell(k+1)$. Now, observe that

$$|A \cap (\ell+m)\mathbb{N} \cap [1, N_i]| = |A \cap [1, N_i]| - |A \cap (\mathbb{N} \setminus (\ell+m)\mathbb{N}) \cap [1, N_i]|,$$

so that

$$\liminf_{i \to \infty} \frac{|A \cap (\ell+m)\mathbb{N} \cap [1, N_i]|}{N_i} \geq \overline{\mathrm{d}}(A) - 1 + \frac{1}{(\ell+m)}.$$

Using this and taking limits as $i \to \infty$ in (6.4.19) we see that

$$\mu'(E \times \Sigma) \geq 1 - \frac{1}{\ell} + (k+1)\left(\overline{\mathrm{d}}(A) - 1 + \frac{1}{\ell+m}\right) = (k+1)\cdot\overline{\mathrm{d}}(A) - k. \quad (6.4.20)$$

Combining (6.4.18) and (6.4.20) we have that

$$(\ell+m)\mu'(\Sigma \times E) + \ell\mu'(E \times \Sigma) \geq (2\ell+m)\left((k+1)\cdot\overline{\mathrm{d}}(A) - k\right).$$

Although μ' is not necessarily ergodic, we can use its ergodic decomposition to find an $(S \times S)$-ergodic component of it, call it μ, such that

$$(\ell+m)\mu(\Sigma \times E) + \ell\mu(E \times \Sigma) \geq (2\ell+m)\left((k+1)\cdot\overline{\mathrm{d}}(A) - k\right).$$

We conclude as in the proof of Lemma 3.3.3. □

Lemma 6.3.8. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$ with $k = m/\ell$ and let $q = \lceil k \rceil$, the ceiling of k. Then, there exist an ergodic system $(\Sigma \times \Sigma, \mu, S^{(q+1)} \times S)$, an open set $E \subset \Sigma$, a pair of points $a, a' \in \Sigma$ and a Følner sequence Φ, such that $(a', a) \in \boldsymbol{gen}(\mu, \Phi)$ and*

$$(\ell+m)\mu(\Sigma\times E)+\ell\sum_{j=0}^{q}\mu(S^{-j}E\times\Sigma) \geq (\ell+m)\left((k+1)\cdot\overline{\mathrm{d}}(A)-k\right)+\ell(k+1)\cdot\overline{\mathrm{d}}(A)+\ell(q-k).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ and $(A-j)/(\ell+m) = \{n \in \mathbb{N} : S^{(q+1)\ell n + j} a' \in E\}$, for each $j = 0, 1, \ldots, \ell+m-1$, where $(A-j)/(\ell+m) = \{n \in \mathbb{N} : n(\ell+m)+j \in A\}$.

Proof. Let $A \subset \mathbb{N}$. As before, let (N_i) be a sequence of integers such that

$$\overline{\mathrm{d}}(A) = \lim_{i\to\infty} \frac{|A \cap [1, N_i]|}{N_i}$$

and $a \in \Sigma = \{0,1\}^{\mathbb{Z}}$ be the indicator of A. This time we let $a' \in \Sigma$ be defined by

$$a'(n) = \begin{cases} a(\ell(k+1)i + j) & \text{if } n = \ell(q+1)i + j,\ i \in \mathbb{N},\ j \in \{0, 1, \ldots, \ell(k+1)-1\} \\ 1 & \text{otherwise.} \end{cases}$$

Observe that a' is well-defined because $\ell(k+1) \leq \ell(q+1)$. If $E = \{x \in \Sigma \colon x(0) = 1\}$, we observe that E is clopen in Σ, $A = \{n \in \mathbb{N} \colon S^n a \in E\}$ and, since $(\ell+m)n = \ell(k+1)n$,

$$\{n \in \mathbb{N} : S^{(q+1)\ell n + j} a' \in E\} = \{n \in \mathbb{N} : S^{(\ell+m)n+j} a \in E\} = (A-j)/(\ell+m),$$

for each $j \in \{0, 1, \ldots, \ell + m - 1\}$. Now let $N_i' = \lfloor N_i/(k+1) \rfloor$, for all $i \in \mathbb{N}$ and consider the sequence of Borel probability measures (μ_i) on $\Sigma \times \Sigma$ given by

$$\mu_i = \frac{1}{N_i'} \sum_{n=1}^{N_i'} \delta_{(S^{(q+1)} \times S)^n (a', a)},$$

if $i \geq k+1$, and let μ' be a weak* accumulation point of (μ_i). As before we have that

$$\mu'(\Sigma \times E) \geq (k+1) \cdot \overline{\mathrm{d}}(A) - k, \tag{6.4.21}$$

On the other hand, for any $i \in \mathbb{N}$, $i \geq k+1$, we have that

$$\begin{aligned} \sum_{j=0}^{q} \mu_i(S^{-j} E \times \Sigma) &= \frac{1}{N_i'} \sum_{j=0}^{q} \sum_{n=1}^{N_i'} \delta_{S^{(q+1)n} a'}(S^{-j} E) \\ &= \frac{k+1}{N_i} \left(\sum_{n=1}^{N_i'(q+1)} \delta_{S^n a'}(E) \right) + o_{N_i \to \infty}(1). \end{aligned} \tag{6.4.22}$$

We can relate this with the density of A because

$$\begin{aligned} &\sum_{n=1}^{N_i'(q+1)} \delta_{S^n a'}(E) \\ &= \sum_{n=1}^{\lfloor N_i/(\ell(k+1)) \rfloor} \sum_{j=0}^{\ell(k+1)-1} \delta_{S^{\ell(q+1)n+j} a'}(E) + \sum_{n=1}^{\lfloor N_i/(\ell(k+1)) \rfloor} \sum_{j=\ell(k+1)}^{\ell(q+1)-1} 1 + o_{N_i \to \infty}(N_i) \\ &= \sum_{n=1}^{\lfloor N_i/(\ell(k+1)) \rfloor} \sum_{j=0}^{\ell+m-1} \delta_{S^{(\ell+m)n+j} a}(E) + \frac{N_i}{\ell(k+1)} \left(\ell(q+1) - \ell(k+1) \right) + o_{N_i \to \infty}(N_i) \\ &= \sum_{n=1}^{N_i} \delta_{S^n a} + N_i \frac{q-k}{k+1} + o_{N_i \to \infty}(N_i) = |A \cap [1, N_i]| + N_i \frac{q-k}{k+1} + o_{N_i \to \infty}(N_i), \end{aligned}$$

where we have used, when convenient, the equality $\ell + m = \ell(k+1)$ and the fact that $q = \lceil k \rceil \geq k$. Using this in (6.4.22) and taking limits as $i \to \infty$ we see that

$$\sum_{j=0}^{q} \mu'(S^{-j} E \times \Sigma) \geq (k+1) \cdot \overline{\mathrm{d}}(A) + q - k. \tag{6.4.23}$$

It follows by combining (6.4.21) and (6.4.23) that

$$(\ell+m)\mu'(\Sigma\times E)+\ell\sum_{j=0}^{q}\mu'(S^{-j}E\times\Sigma)\geq$$
$$(\ell+m)\left((k+1)\cdot\overline{\mathrm{d}}(A)-k\right)+\ell(k+1)\cdot\overline{\mathrm{d}}(A)+\ell(q-k).$$

The rest of the proof follows the same arguments as the previous one. □

6.5 Proofs of main dynamical theorems

For the proof of our main result we need a lemma that guarantees $(T^\ell\times T^m)$-generic points, for almost all the measures $\lambda_{(a,x_1)}$ with prescribed first coordinate.

Lemma 6.5.1. *If (X,μ,T) is an ergodic system and $a\in \boldsymbol{gen}(\mu,\Phi)$ for some Følner sequence Φ, then for μ-almost every $x_1\in X$ we have that (a,x_1) is $(T^\ell\times T^m)$-generic for $\lambda_{(a,x_1)}$.*

Proof. The proof is an adaptation of the proof of [41, Lemma 3.12], but requires a few adjustments which are not immediately obvious.

Using property (iii) of Proposition 6.4.4 and Fubini's theorem it follows that for each one of a full measure set of points $x'\in \mathtt{supp}(\mu)$ there is a full measure set of points $x\in X$ so that $(x',x)\in \mathbf{gen}(\lambda_{(x',x)},T^\ell\times T^m,\Phi')$, where $\Phi'=(\{1,\dots,N\})_{N\in\mathbb{N}}$. We let $(G_j)_{j=1}^\infty$ be a dense subset of $C(X\times X)$ and for each $j\in\mathbb{N}$ we set $\tilde{G}_j(x,y)=\int_{X\times X}G_j\ d\lambda_{(x,y)}$. As the map $(x_1,x_2)\mapsto\lambda_{(x_1,x_2)}$ is continuous and $(T^\ell\times T^m)$-invariant, it follows that each $\tilde{G}_j$ is continuous and $(T^\ell\times T^m)$-invariant.

Now, as $a\in\mathbf{gen}(\mu,T,\Phi)$ we have that $\mathtt{supp}(\mu)\subset\overline{O_T(a)}\subset\bigcup_{j=0}^{\ell-1}\overline{O_{T^\ell}(T^ja)}$, where $O_S(a)$ is the (forward) orbit of a under a homeomorphism $S:X\to X$, i.e. $\{S^na\colon n\in\mathbb{N}\}$. Therefore, $\mu(\overline{O_{T^\ell}(a)})>0$ and so we can choose some x' as above with $x'\in\overline{O_{T^\ell}(a)}$. This allows us to find, for each $k\in\mathbb{N}$ and $N(k)\in\mathbb{N}$, an $s(k)\in\mathbb{N}$ such that $\max_{1\leq n\leq N(k)}\left\|G_j(T^{\ell n}x',\cdot)-G_j(T^{\ell(n+s(k))}a,\cdot)\right\|_\infty<2^{-k}$ and $\left\|\tilde{G}_j(x',\cdot)-\tilde{G}_j(T^{\ell s(k)}a,\cdot)\right\|_\infty<2^{-k}$ for all $j\leq k$.

Because (x',x) is $(T^\ell\times T^m)$-generic for $\lambda_{(x',x)}$ for μ-almost every $x\in X$, we can find for each $k\in\mathbb{N}$ some $N(k)\in\mathbb{N}$ such that

$$F_k(x)=\max_{1\leq j\leq k}\left|\frac{1}{N(k)}\sum_{n=1}^{N(k)}G_j(T^{\ell n}x',T^{mn}x)-\tilde{G}_j(x',x)\right|$$

satisfies $\|F_k\|_{L^1(\mu)} < 2^{-k}$. By the choice of $s(k)$ we have that

$$\tilde{F}_k(x) := \max_{1\leq j\leq k}\left|\frac{1}{N(k)}\sum_{n=1}^{N(k)} G_j(T^{\ell(n+s(k))}a, T^{mn}x) - \tilde{G}_j(T^{\ell s(k)}a, x)\right|$$

satisfies $\left\|\tilde{F}_k\right\|_{L^1(\mu)} < 3/2^k$. Then, by the $(T^\ell\times T^m)$-invariance of $\tilde{G}_j$, we can consider $\Psi_k = \{s(k)+1,\dots,s(k)+N(k)\}$ and then we see that

$$F'_k(x) := \tilde{F}_k(T^{ms(k)}x) = \max_{1\leq j\leq k}\left|\frac{1}{|\Psi_k|}\sum_{n\in\Psi_k} G_j(T^{\ell n}a, T^{mn}x) - \tilde{G}_j(a,x)\right|.$$

As μ is $T-$invariant it follows that $\|F'_k\|_{L^1(\mu)} = \left\|\tilde{F}_k\right\|_{L^1(\mu)}$ and so, if we define $F(x) := \sum_{k=1}^{\infty} F'_k(x)$ it follows that $\|F\|_{L^1(\mu)} < \infty$ and so F is finite μ-almost everywhere. Finally, for each $x_1 \in X$ such that $F(x_1) < \infty$ it must hold that $F'_k(x_1) \to 0$ as $k\to\infty$ and so $(a,x_1) \in \mathbf{gen}(\lambda_{(a,x_1)}, T^\ell\times T^m, \Psi)$. □

Combining some of the above results we can guarantee the existence of (ℓ,m)-Erdős progressions via the following analogue of Proposition 3.4.3. The proof is almost identical and so we omit it. For reference, the established properties of the measures σ_a and $\lambda_{(x_1,x_2)}$ used in the proof are those in Lemmas 6.4.7, 6.5.1, and Proposition 6.4.10.

Proposition 6.5.2. *Let (X,μ,T) be an ergodic system and assume there is a continuous factor map $\pi\colon X\to Z$ to its Kronecker factor. Let $a\in \boldsymbol{gen}(\mu,\Phi)$, for some Følner sequence Φ. Then for σ_a-almost every $(x_1,x_2)\in X\times X$, the point (a,x_1,x_2) is an (ℓ,m)-Erdős progression.*

We are now in the position to prove all our main dynamical results. Before presenting the proofs, we emphasise again that, according to our discussion in the beginning of Section 6.4.2, these theorems also imply Theorems 6.3.2, 6.3.6 and 6.3.7, respectively.

Proof of Theorem 6.4.1. Since for σ_a-almost every point $(x_1,x_2)\in X\times X$ the triple (a,x_1,x_2) is an (ℓ,m)-Erdős progression by Proposition 6.5.2, we simply need to show that $\sigma_a(X\times T^{-t}E) > 0$ for some $t\in\mathbb{N}$. The latter follows directly by the fact that $\pi_2\sigma_a\left(\bigcup_{t\in\mathbb{N}} T^{-t}E\right) > 0$. Indeed, as μ is ergodic we have that $\mu(\bigcup_{t\in\mathbb{N}} T^{-t}E) = 1$ and thus, in way of Lemma 6.4.7, we actually see that $\pi_2\sigma_a\left(\bigcup_{t\in\mathbb{N}} T^{-t}E\right) = 1$. □

Remark 6.5.3. In fact one can modify the above proof to show that there exist $t_1, t_2 \in \mathbb{N}$ such that $\sigma_a(E \times T^{-t_1}E) > 0$ and $\sigma_a(T^{-t_2}E \times T^{-t_2}E) > 0$. These in turn guarantee (ℓ, m)-Erdős progressions $(a, x_1, x_2), (a, x_1', x_2') \in X^3$ with

$$(x_1, T^{t_1}x_2), (T^{t_2}x_1', T^{t_2}x_2') \in E \times E$$

and then, Theorem 1.2.1 can be strengthened, so that for any set $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ the following hold:

(i) There exist an infinite set $B \subset \mathbb{N}$ and a shift $t \in \mathbb{N}$ such that $\ell B \subset A$ and

$$\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} + t \subset A.$$

(ii) There exist an infinite set $B \subset \mathbb{N}$ and a shift $t \in \mathbb{N}$ such that

$$\ell B \cup \{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 < b_2\} \subset A - t.$$

Proof of Theorem 6.4.2. Applying Proposition 6.5.2 once again, we only have to show that for some $j \in \{1, \dots, \ell + m\}$, $\sigma_a(E_j \times F_j) > 0$, because this would imply the existence of $x_1, x_2 \in X$ so that $(a, x_1, x_2) \in X^3$ is an (ℓ, m)-Erdős progression and $(x_1, x_2) \in E_j \times F_j$. This would follow from the inequality

$$\sigma_a(E_j \times X) + \sigma_a(X \times F_j) > 1, \tag{6.5.1}$$

because

$$E_j \times F_j = (E_j \times X) \cap (X \times F_j)$$

and σ_a is a probability measure. It suffices to show that (6.5.1) has to be satisfied with $j = 1$ provided that it fails for $j = 2, \dots, \ell + m$. In particular, we assume that

$$\sum_{j=2}^{\ell+m} (\sigma_a(E_j \times X) + \sigma_a(X \times F_j)) \leq \ell + m - 1,$$

which can be rewritten as

$$\sum_{j=2}^{\ell+m} (\pi_1\sigma_a(E_j) + \pi_2\sigma_a(F_j)) \leq \ell + m - 1. \tag{6.5.2}$$

Under this assumption, we have to show that $\pi_1\sigma_a(E_1) + \pi_2\sigma_a(F_1) > 1$, which by

Lemma 6.4.7 can be rewritten as

$$\pi_1\sigma_a(E_1) + (\ell+m)\mu(F_1) - \sum_{j=1}^{\ell+m-1} T^j\pi_2\sigma_a(F_1) > 1. \tag{6.5.3}$$

Recalling that $F_j = T^{-(j-1)}F_1$, for each $j = 1, \dots, \ell+m$, we see that (6.5.3) is equivalent to

$$\pi_1\sigma_a(E_1) + (\ell+m)\mu(F_1) - \sum_{j=2}^{\ell+m} \pi_2\sigma_a(F_j) > 1,$$

which by (6.5.2) would follow from

$$\pi_1\sigma_a(E_1) + (\ell+m)\mu(F_1) + \sum_{j=2}^{\ell+m} \pi_1\sigma_a(E_j) - (\ell+m-1) > 1. \tag{6.5.4}$$

Next, we note that (6.5.4) can be rewritten as

$$(\ell+m)\mu(F_1) + \sum_{j=1}^{\ell+m} \pi_1\sigma_a(E_j) > \ell+m. \tag{6.5.5}$$

We now recall that $E_{i+(q+1)} = T^{-1}E_i$, for $i = 1, \dots, \ell+m-q-1$, and if $q > m/\ell$ (i.e. whenever $m/\ell \notin \mathbb{Z}$) we also consider auxiliary sets $E_{\ell+m+1}, \dots, E_{\ell(q+1)}$ such that $E_{i+(q+1)} = T^{-1}E_i$, for $i = 1, \dots, (\ell-1)(q+1)$. Then,

$$\sum_{j=1}^{\ell(q+1)} \pi_1\sigma_a(E_j) = \sum_{j=1}^{q+1}\sum_{i=0}^{\ell-1} T^i\pi_1\sigma_a(E_j) = \ell\sum_{j=1}^{q+1}\mu(E_j)$$

in accordance with Lemma 6.4.7. We also make the trivial observation that

$$\sum_{j=1}^{\ell(q+1)} \pi_1\sigma_a(E_j) - \sum_{j=1}^{\ell+m} \pi_1\sigma_a(E_j) = \sum_{j=\ell(k+1)+1}^{\ell(q+1)} \pi_1\sigma_a(E_j) \le \ell(q-k),$$

since $\ell(k+1) = \ell+m$. Therefore, it follows that

$$(\ell+m)\mu(F_1) + \sum_{j=1}^{\ell+m} \pi_1\sigma_a(E_j) \ge (\ell+m)\mu(F_1) + \ell\sum_{j=1}^{q+1}\mu(E_j) - \ell(q-k)$$

and so (6.5.5) follows from (6.4.1). □

Proof of Theorem 6.4.3. We need to find an Erdős progression of the form $(a, x_1, x_2) \in$

X^3 with $(x_1, x_2) \in E \times F$ and as before, by Proposition 6.5.2 it suffices to show that $\sigma_a(E \times F) > 0$. Once again, this would follow from

$$\sigma_a(E \times X) + \sigma_a(X \times F) > 1.$$

To this end, we simply note that

$$\begin{aligned}
&\sigma_a(E \times X) + \sigma_a(X \times F) = \pi_1\sigma_a(E) + \pi_2\sigma_a(F) \\
&= \ell\mu(E) + (\ell + m)\mu(F) - \sum_{j=1}^{\ell-1} \pi_1\sigma_a(T^{-j}E) - \sum_{i=1}^{\ell+m-1} \pi_2\sigma_a(T^{-i}F) \\
&\geq \ell\mu(E) + (\ell + m)\mu(F) - (\ell - 1) - (\ell + m - 1) > 1,
\end{aligned}$$

where the second equality follows by Lemma 6.4.7 and the strict inequality by the assumption in (6.4.2). This concludes the proof. □

6.6 Examples for optimality

In this short section we will show that Theorems 1.3.7 and 6.2.2 are optimal. That is, the density thresholds presented in both these results cannot be improved. We do not include the details of the proof as they are easy to fill from the examples in Section 3.5.1.

Proposition 6.6.1 (cf. Proposition 1.3.8)**.** *Let $\ell, m \in \mathbb{N}$ and $k = m/\ell$. Then, there exist two sets $A, A' \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = (k+1)/(k+2) = 1 - 1/(k+2)$ and $\overline{\mathrm{d}}(A') = 1 - 1/\left(\ell(k+1)(k+2)\right)$ such that for any infinite set B and any integer $t \in \mathbb{N}$ we have that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \not\subset A$ and $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \not\subset A'$.*

Proof of Proposition 6.6.1. The set A in the statement is defined as

$$A = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(k+1)^{2n}, (k + 1 - 1/n) \cdot (k+1)^{2n}).$$

It is clear by the definition of A that $\overline{\mathrm{d}}(A) = \mathrm{d}_\Phi(A)$, where $\Phi = (\Phi_N)$ is the Følner sequence given by $N \mapsto [1, (k + 1 - 1/N)(k+1)^{2N}) \cap \mathbb{N}$ and one can readily check that $\overline{\mathrm{d}}(A) = (k+1)/(k+2)$.

Keeping A as above we consider $A' = A \cup \left(\bigcup_{j=1}^{\ell+m-1} (\ell + m)\mathbb{N} + j\right)$. In other words, $A' = A \cup ((\mathbb{N} \setminus A) \setminus ((\ell + m)\mathbb{N}))$. Then, $\overline{\mathrm{d}}(A') = 1 - 1/\left(\ell(k+1)(k+2)\right)$ and there is no infinite set $B \subset \mathbb{N}$ satisfying $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \subset A'$. □

6.7 Lower density results

We also want to briefly explore what further results we can get by considering the input of the information provided by a set's lower asymptotic density. The proofs in this section are straightforward adaptations of the arguments used thus far, so we omit repetitive details, but we include comments regarding all the non-obvious changes in the argumentation.

A possibility to capture this new input – one that has been tried and found to be fruitful in [40] – is to replace (6.4.18), which appears in both correspondence principles in Lemmas 6.3.8 and 6.3.9, by the always true inequality $\mu'(\Sigma\times E) \geq \underline{\mathrm{d}}(A)$. To see this recall that μ' is defined as a weak* limit of the sequence of measures (μ_i) and take the liminf as $i \to \infty$ in (6.4.16). Making this simple change in the proofs of the correspondence principles we recover the following results.

Lemma 6.7.1. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$ with $k = m/\ell$ and let $q = \lceil k \rceil$, the ceiling of k. Then, there exist an ergodic system $(\Sigma \times \Sigma, \mu, S^{(q+1)} \times S)$, an open set $E \subset \Sigma$, a pair of points $a, a' \in \Sigma$ and a Følner sequence Φ, such that $(a', a) \in \boldsymbol{gen}(\mu, \Phi)$ and*

$$(\ell + m)\mu(\Sigma \times E) + \ell \sum_{j=0}^{q} \mu(S^{-j}E \times \Sigma) \geq (\ell + m) \cdot \underline{\mathrm{d}}(A) + \ell(k + 1) \cdot \overline{\mathrm{d}}(A) + \ell(q - k).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ and $(A - j)/(\ell + m) = \{n \in \mathbb{N} : S^{(q+1)\ell n + j} a' \in E\}$, for each $j = 0, 1, \ldots, \ell + m - 1$, where $(A - j)/(\ell + m) = \{n \in \mathbb{N} : n(\ell + m) + j \in A\}$.

Lemma 6.7.2. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$ with $k = m/\ell$. Then, there exists an ergodic system $(\Sigma \times \Sigma, \mu, S \times S)$, an open set $E \subset \Sigma$, a pair of points $a'', a \in \Sigma$ and a Følner sequence Φ, such that $(a'', a) \in \boldsymbol{gen}(\mu, \Phi)$ and*

$$(\ell + m)\mu(\Sigma \times E) + \ell\mu(E \times \Sigma) \geq (\ell + m) \cdot \underline{\mathrm{d}}(A) + \ell \left((k + 1) \cdot \overline{\mathrm{d}}(A) - k\right).$$

It also holds that $A = \{n \in \mathbb{N} : S^n a \in E\}$ and $A/(\ell + m) = \{n \in \mathbb{N} : S^{\ell n} a'' \in E\}$.

Now that we have these additional correspondence principles we may expand on the arguments presented in Section 6.3.2, using Theorems 6.3.6, 6.3.7 and Proposition 6.3.3 to get the following combinatorial result.

Theorem 6.7.3. *Let $A \subset \mathbb{N}$ and $\ell, m \in \mathbb{N}$.*

1. *If $\underline{\mathrm{d}}(A) + \overline{\mathrm{d}}(A) > 1$, there is an infinite set $B \subset \mathbb{N}$ and some $t \in \{0, 1, \ldots, \ell + m - 1\}$ such that $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 \leq b_2\} + t \subset A$.*

2. *If* $\underline{\mathrm{d}}(A) + \overline{\mathrm{d}}(A) > 2 - 1/(\ell + m)$, *there is an infinite set* $B \subset \mathbb{N}$ *such that* $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 \leq b_2\} \subset A$.

We have the following immediate corollary of Theorem 6.7.3.

Corollary 6.7.4. *Let* $A \subset \mathbb{N}$ *and* $\ell, m \in \mathbb{N}$.

1. *If* $\underline{\mathrm{d}}(A) > 1/2$, *there is an infinite set* $B \subset \mathbb{N}$ *and some* $t \in \{0, 1, \ldots, \ell + m - 1\}$ *such that* $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 \leq b_2\} + t \subset A$.

2. *If* $\underline{\mathrm{d}}(A) > 1 - 1/(2(\ell + m))$, *there is an infinite set* $B \subset \mathbb{N}$ *such that* $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B,\ b_1 \leq b_2\} \subset A$.

Remark. Compare this result with the upper density analogues in Theorems 1.3.7 and 6.2.2. In particular, the threshold values for upper density depend on the parameters m and ℓ involved in the sumsets. It is surprising that this is no longer the case for threshold values of lower density, at least in the case of shifted sumsets. Of course, some kind of dependence in the case of unshifted patterns is enforced by the fact that infinite sumsets $mB + \ell B$ essentially "live" in $(m + \ell)\mathbb{N}$, as was utilised in the proof of Proposition 6.3.5.

We will conclude this discussion by explaining how Corollary 6.7.4 (and thus Theorem 6.7.3 too) is also optimal.

Proposition 6.7.5. *Let* $\ell, m \in \mathbb{N}$. *There exist two sets* $A, A' \subset \mathbb{N}$ *with* $\underline{\mathrm{d}}(A) = 1/2$ *and* $\underline{\mathrm{d}}(A') = 1 - 1/(2(\ell + m))$ *such that for any infinite set* B *and any integer* $t \in \mathbb{N}$ *we have that* $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \not\subset A$ *and* $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} \not\subset A'$.

Proof of Proposition 6.7.5. Once again, the details of the proof are easily filled following the examples of Section 3.5.1. Let $k = m/\ell$ and A_1, A_2 be the subsets of $\mathbb{N}$ defined by

$$A_1 = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(k+1)^{2n}, (k + 1 - 1/n) \cdot (k+1)^{2n})$$

and

$$A_2 = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(k + 1 + 1/n) \cdot (k+1)^{2n}, (k+1)^{2(n+1)}).$$

Observe that the set

$$\mathbb{N} \setminus (A_1 \cup A_2) = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} \left[(k + 1 - 1/n) \cdot (k+1)^{2n}, (k + 1 + 1/n) \cdot (k+1)^{2n}\right]$$

has zero density. The set $A \subset \mathbb{N}$ is defined by

$$A = (A_1 \cap 2\mathbb{N}) \cup (A_2 \cap (2\mathbb{N}+1)) .$$

Keeping A_1 and A_2 as they were defined above, $A' \subset \mathbb{N}$ is defined by

$$A' = (A_1 \setminus (2(\ell+m)\mathbb{N}) \cup (A_2 \setminus (2(\ell+m)\mathbb{N} + (\ell+m))) .$$

□

Remark 6.7.6. We stress that for both sets A, A' above we actually have that their natural densities are realised. This was to be expected because of Theorem 6.7.3. See also the discussion in the last paragraph of Section 3.5.1.

6.8 Remarks and questions about further extensions

A significant part of this work was focused on extensions of Theorem 1.2.1 by relaxing the restrictions imposed on the sumsets involved. In short, we managed to provide complete characterisation for the existence of the patterns $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$ for infinite $B \subset \mathbb{N}$ in shifts of a set $A \subset \mathbb{N}$, based on the values for the upper and lower asymptotic densities of A.

Another direction could be to consider patterns of the form $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\}$. We recall Remark 6.2.1, according to which an assumption of positive upper density is not sufficient to find shifts of such infinite sumsets in a subset of the integers (unless, of course, $\ell = m$). A natural question, then, is if this problem has a 'density solution'. We present two related constructions and then state the question precisely.

Proposition 6.8.1. *Let $\ell, m \in \mathbb{N}$ be distinct with $\ell > m$. There exist two sets $A, A' \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = \ell/(\ell+m)$ and $\overline{\mathrm{d}}(A') = 1 - m/(\ell+m)^2$ such that for any infinite set $B \subset \mathbb{N}$ and any $t \in \mathbb{N}$ we have $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} \not\subset A - t$ and $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} \not\subset A'$.*

Remark 6.8.2. It may be interesting to compare these bounds with the optimal bounds established for the sumsets $\{mb_1 + \ell b_2 : b_1, b_2 \in B, \text{ and } b_1 \leq b_2\}$ in Theorems 1.3.7, 6.2.2 and Proposition 6.6.1. In particular, by assumption we have that $k = m/\ell < 1$ and then $\ell/(\ell+m) = 1/(k+1)$, and this is greater than $(k+1)/(k+2)$, whenever $k(k+1) < 1$. Also, $1 - m/(\ell+m)^2$ is greater than $1 - 1/(\ell(k+1)(k+2)) = 1 - 1/((\ell+m)(k+2))$, again, precisely when $k(k+1) < 1$.

The surprising fact that springs from this observation is the following; Say, without loss of generality, that $\ell > m$. Then, the largest bounds (we can find) for the value of upper density for a set that doesn't contain sumsets of the form $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B, \text{ and } b_1 \neq b_2\}$ are greater than the optimal bounds for a set that doesn't contain sumsets $\{mb_1 + \ell b_2 \colon b_1, b_2 \in B, \text{ and } b_1 \leq b_2\}$ whenever $\ell/m = 1/k$ is greater than the Golden ratio! The relation between the bounds is reversed if $\ell/m = 1/k$ is less than the Golden ratio and the same comparisons hold for unshifted patterns.

We can also compare the bounds of Proposition 6.8.1 with the ones from the aforementioned results, for the sumsets $\{\ell b_1 + mb_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$. In this case, the comparison is not nearly as mystical, for the latter sumsets cannot be found in sets of upper density up to $(k^{-1}+1)/(k^{-1}+2)$ and $1 - 1/(m(k^{-1}+1)(k^{-1}+2))$, for the cases of shift and no shift, respectively. But both of these are larger than the respective bounds from Proposition 6.8.1, i.e. $\ell/(\ell+m)$ and $1 - m/(\ell+m)^2$.

Proof of Proposition 6.8.1. The details of the proof are once again omitted to ease exposition.

The set $A \subset \mathbb{N}$ satisfying the assumptions of the statement is defined by

$$A = \mathbb{N} \cap \bigcup_{n \in \mathbb{N}} [(\ell/m)^{2n}, (\ell/m - 1/n) \cdot (\ell/m)^{2n}).$$

The set $A' \subset \mathbb{N}$ is defined by

$$A' = A \cup \left(\bigcup_{j=1}^{\ell+m-1} (\ell+m)\mathbb{N} + j \right).$$

□

Remark. Observe that the set $A \subset \mathbb{N}$ above is thick and as such $\mathrm{d}^*(A) = 1$.

Despite the urge to make a conjecture, due to limited dynamical as well as combinatorial evidence – apart from the very interesting Remark 6.8.2, which hints that the bounds of Proposition 6.8.1 may not be unrelated to the optimal bounds – we constrain ourselves to asking the following question.

Question 6.8.3. Let $\ell, m \in \mathbb{N}$ be distinct with $\ell > m$ and let $A \subset \mathbb{N}$.

(i) If $\overline{\mathrm{d}}(A) > \ell/(\ell+m)$, does there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} + t \subset A$?

(ii) If $\overline{\mathrm{d}}(A) > 1 - m/(\ell+m)^2$, does there exist an infinite set $B \subset \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} \subset A$?

With the sets constructed in Proposition 6.8.1 as a basis, we can recover the following result.

Proposition 6.8.4. *Let $\ell, m \in \mathbb{N}$ be distinct with $\ell > m$. There exist two sets $A, A' \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) = 1/2$ and $\underline{\mathrm{d}}(A') = 1 - 1/(2(\ell+m))$ such that for any infinite set B and any integer $t \in \mathbb{N}$ we have that $\{mb_1+\ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\}+t \not\subset A$ and $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} \not\subset A'$.*

Proof. Let $A_1, A_2 \subset \mathbb{N}$ be defined by

$$A_1 = \mathbb{N} \cap \bigcup_{n\in\mathbb{N}} [(\ell/m)^{2n}, (\ell/m - 1/n) \cdot (\ell/m)^{2n})$$

and

$$A_2 = \mathbb{N} \cap \bigcup_{n\in\mathbb{N}} [(\ell/m + 1/n) \cdot (\ell/m)^{2n}, (\ell/m)^{2(n+1)}).$$

Then, the claimed properties are satisfied by the sets $A = (A_1 \cap 2\mathbb{N}) \cup (A_2 \cap (2\mathbb{N}+1))$ and $A' = (A_1 \setminus 2(\ell+m)\mathbb{N}) \cup (A_2 \setminus (2(\ell+m)\mathbb{N} + (\ell+m)))$. □

Therefore, we also ask the following.

Question 6.8.5. Let $\ell, m \in \mathbb{N}$ be distinct and let $A \subset \mathbb{N}$.

(i) If $\underline{\mathrm{d}}(A) > 1/2$, does there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} + t \subset A$?

(ii) If $\underline{\mathrm{d}}(A) > 1 - 1/(2(\ell+m))$, does there exist an infinite set $B \subset \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\} \subset A$?

Provided Question 6.8.3 has a positive answer one should also inquire about the completely unrestricted problem, that is, the existence of infinite sumsets $mB+\ell B = \{mb_1 + \ell b_2 : b_1, b_2 \in B\}$. To this end, we ask two more questions after making an observation which follows from current results.

Proposition 6.8.6. *Let $\ell, m \in \mathbb{N}$ be distinct with $m > \ell$ and let $k = m/\ell$. Then, there exist sets $A, A' \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = (k+1)/(k+2)$ and $\overline{\mathrm{d}}(A') = 1 - 1/\left(\ell(k+1)(k+2)\right)$ such that for any infinite set B and any integer $t \in \mathbb{N}$ we have that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} + t \not\subset A$ and $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \not\subset A'$.*

Proof. All we need to observe is that both sumsets $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$ and $\{\ell b_1 + mb_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$ are contained in $\{mb_1+\ell b_2 : b_1, b_2 \in B\}$,

for any $B \subset \mathbb{N}$, and so we can reduce to Proposition 6.6.1. Indeed, if a set fails to contain either of the former sumsets, then it will also fail to contain $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\}$. Now, as $k > 1$, we have that

$$\max\{(k+1)/(k+2), (k^{-1}+1)/(k^{-1}+2)\} = (k+1)/(k+2),$$

and so it follows by Proposition 6.6.1 that there exists a set $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) = (k+1)/(k+2)$ and such that for any infinite $B \subset \mathbb{N}$ and $t \in \mathbb{N}$ we have that $\{mb_1 + \ell b_2 : b_1, b_2 \in B \text{ and } b_1 \leq b_2\} + t \not\subset A$. In particular, it follows that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \not\subset A - t$. The existence of A' is proved following the exact same argument and using the set A' provided by Proposition 6.6.1. □

Repeating the same observation for values of lower density and Proposition 6.7.5 we have the following proposition.

Proposition 6.8.7. *Let $\ell, m \in \mathbb{N}$. There exist two sets $A, A' \subset \mathbb{N}$ with $\underline{\mathrm{d}}(A) = 1/2$ and $\underline{\mathrm{d}}(A') = 1 - 1/(2(\ell+m))$ such that for any infinite set B and any integer $t \in \mathbb{N}$ we have that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \not\subset A - t$ and $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \not\subset A'$.*

It would not be surprising if either of the next questions had a negative answer (particularly Question 6.8.8), but apart from our inability to improve on the previous constructions there is no evidence suggesting the opposite.

Question 6.8.8. Let $\ell, m \in \mathbb{N}$ be distinct with $m > \ell$ and let $k = m/\ell$. Let also $A \subset \mathbb{N}$.

(i) If $\overline{\mathrm{d}}(A) > (k+1)/(k+2)$, does there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} + t \subset A$?

(ii) If $\overline{\mathrm{d}}(A) > 1 - 1/(\ell(k+1)(k+2))$, does there exist an infinite set $B \subset \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \subset A$?

Question 6.8.9. Let $\ell, m \in \mathbb{N}$ be distinct and $A \subset \mathbb{N}$.

(i) If $\underline{\mathrm{d}}(A) > 1/2$ does there exist an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} + t \subset A$?

(ii) If $\underline{\mathrm{d}}(A) > 1 - 1/(2(\ell+m))$ does there exist an infinite set $B \subset \mathbb{N}$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B\} \subset A$?

As the above exposition on unrestricted sumsets $mB + \ell B$ focused on the case of distinct ℓ and m, we want to highlight that, since $\{mb_1 + mb_2 \colon b_1, b_2 \in B \text{ and } b_1 \leq b_2\} = mB + mB$, the special unrestricted sumsets of the form $mB + mB$ are already

covered in Theorems 1.3.7, 6.2.2, Corollary 6.7.4 and Propositions 6.6.1, 6.7.5. In particular, for some $A \subset \mathbb{N}$ and $m \in \mathbb{N}$ we have the following implications, all of which are optimal.

(i) If $\overline{\mathrm{d}}(A) > 2/3$, there exist some infinite set $B \subset \mathbb{N}$ and some $t \in \{0, 1, \ldots, 2m-1\}$ such that $mB + mB + t \subset A$.

(ii) If $\overline{\mathrm{d}}(A) > 1 - 1/(6m)$, there exists some infinite set $B \subset \mathbb{N}$ such that $mB + mB \subset A$.

(iii) If $\underline{\mathrm{d}}(A) > 1/2$, there exist some infinite set $B \subset \mathbb{N}$ and some $t \in \{0, 1, \ldots, 2m-1\}$ such that $mB + mB + t \subset A$.

(iv) If $\underline{\mathrm{d}}(A) > 1 - 1/(4m)$, there exists some infinite set $B \subset \mathbb{N}$ such that $mB + mB \subset A$.

We note that the bound in (i) does not depend on m, because the optimal bound established in Theorem 1.3.7 explicitly depends only on the ratio $k = m/\ell$, which in this case is always 1. This suggests that perhaps (i) above already follows from Theorem 1.3.2, and this is indeed the case. The proof of this fact is not very complicated, but serves no other purpose as to be included here. Similarly, using Theorem 1.3.1, one can prove that for any set $A \subset \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 5/6$, there exists $B \subset \mathbb{N}$ infinite and some $t \in \{0, 1, \ldots, m-1\}$ such that $mB + mB + t \subset A$ – note the difference for the range of the shift – but of course, density larger than $5/6$ is no longer sufficient for us to remove potential shifts, as the bounds in part (ii) above are optimal.

We finish with a proof of the last claim, showing that the range of shifts can be shortened. Observe that the set $A \subset \mathbb{N}$ can be written as the disjoint union

$$A = m(A/m) \cup (m((A-1)/m) + 1) \cup \cdots \cup (m((A-(m-1))/m) + (m-1)),$$

where $(A-i)/m = \{n \in \mathbb{N} : nm + i \in A\}$ and then $m((A-i)/m) + i = A \cap (m\mathbb{N} + i)$, for each $i \in \{0, 1, \ldots, m-1\}$. Moreover, $m \cdot \overline{\mathrm{d}}(m((A-i)/m)) \leq \overline{\mathrm{d}}((A-i)/m)$. Hence,

$$\overline{\mathrm{d}}(A) \leq \sum_{i=0}^{m-1} \overline{\mathrm{d}}(m((A-i)/m)) \leq \sum_{i=0}^{m-1} (\frac{1}{m}) \overline{\mathrm{d}}((A-i)/m).$$

As $\overline{\mathrm{d}}(A) > 5/6$, the previous implies that $\overline{\mathrm{d}}((A-i)/m) > 5/6$ for some $i \in \{0, 1, \ldots, m-1\}$. Then, by Theorem 1.3.1, there exists an infinite set $B \subset \mathbb{N}$ such that $B + B \subset (A-i)/m$, which means that $m(B+B) + i \subset A$, and $i \in \{0, 1, \ldots, m-1\}$.

We conclude with some related conjectures in view of the results presented in this chapter as well as those of Chapter 5, and particularly Theorem 1.3.3. The bounds of the first one are suggested by Theorem 6.7.3. The (conceived proof of the) third one seems to be dependant on (the conceived proof of the) second one, and the details in the statements of the last two are slightly more speculative at the moment, although most likely correct.

For simplicity, we fix $\ell, m \in \mathbb{N}$ such that $k = m/\ell \in \mathbb{N}$. Note that $\ell(k+1) = m+\ell$. We expect similar results to hold in the case that $m/\ell \notin \mathbb{N}$, by replacing this quantity with its ceiling (as was done in this chapter).

Conjecture 6.8.10. *Let $A \subset \mathbb{N}$ be a set, $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$, and $\Phi/(k+1) = (\Phi_N/(k+1))_{N\in\mathbb{N}}$, where $\Phi_N/(k+1) = \{n \in \mathbb{N} : n \cdot (k+1) \in \Phi_N\}$. Then, the following hold:*

(i) $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/(k+1)}(A) > 1$, *there exists an infinite set $B \subset \mathbb{N}$ and some integer $0 \leq t \leq \ell + m - 1$ such that $\{mb_1 + \ell b_2 : b_1, b_2 \in B,\ b_1 \leq b_2\} + t \subset A$.*

(ii) If $\mathrm{d}_\Phi(A) + \mathrm{d}_{\Phi/(k+1)}(A) > 2 - \frac{1}{m+\ell}$, *there exists an infinite set $B = \{b_n : n \in \mathbb{N}\} \subset \mathbb{N}$ such that $\{mb_i + \ell b_j : i \leq j\} \subset A$.*

(iii) If $\mathrm{d}_\Phi(A\cap(m+\ell)G) + \mathrm{d}_{\Phi/(k+1)}(A\cap(m+\ell)G) > \frac{1}{m+\ell}$, *there exists an infinite set $B = \{b_n : n \in \mathbb{N}\} \subset \mathbb{N}$ such that $\{mb_i + \ell b_j : i \leq j\} \subset A$.*

Indeed, following an argument similar to the proof of Proposition 5.3.1, statements (i), (ii) and (iii) of Conjecture 6.8.10 should be equivalent. Then, its validity should follow from Theorem 6.3.7 via a correspondence principle that strengthens Lemma 6.3.9, analogous to Lemma 5.4.1 used in Chapter 5. The precise details of this last step are not immediately obvious and should be checked.

For a countable abelian group G, we let $D_{\ell,m} : G \to G$ denote the map $g \mapsto (m+\ell)g$, and we also let $s = [G : (m+\ell)G]$ and $r = |\ker(D_{\ell,m})|$.

Conjecture 6.8.11. *Let $(G, +)$ be a countable abelian group with $s < \infty$. Let $A \subset G$ such that $\mathrm{d}^*(A) > 0$. Then, there exist an infinite set $B = \{b_n : n \in \mathbb{N}\} \subset G$ and some $t \in G$ such that $t + \{mb_i + \ell b_j : i < j\} \subset A$.*

Conjecture 6.8.11 should hold more generally for all countable amenable groups that are $(m+\ell)$-*absolutely continuous* (which has the obvious meaning according to [11, Definition 1.4]) and any set $A \subset G$ which has positive upper density with respect to some *left Følner sequence* (see [11, Definition 1.3]).

We also believe the following extension of Theorem 1.3.3 to be true and the conjectured bounds to be optimal in the sense of Theorem 1.3.5.

Conjecture 6.8.12. *Let $(G,+)$ be a countable abelian group with $s<\infty$ and $r<\infty$. Let $A\subset G$ and Φ be any Følner sequence in G. Pass to a subsequence of Φ if necessary so that the densities exist. Then the following hold:*

(1) If $\mathrm{d}_{\Phi}(A)+\mathrm{d}_{\Phi/(k+1)}(A)>1$, there exist an infinite set $B=\{b_n : n\in\mathbb{N}\}\subset G$ and some $t\in G$ such that $\{mb_i+\ell b_j : i\leq j\}+t\subset A$.

(2) If $\mathrm{d}_{\Phi}(A)+\mathrm{d}_{\Phi/(k+1)}(A)>\frac{2s-1}{s}$, there exists an infinite set $B=\{b_n : n\in\mathbb{N}\}\subset G$ such that $\{mb_i+\ell b_j : i\leq j\}\subset A$.

(3) If $\mathrm{d}_{\Phi}(A\cap(m+\ell)G)+\mathrm{d}_{\Phi/(k+1)}(A\cap(m+\ell)G)>\frac{1}{s}$, there exists an infinite set $B=\{b_n : n\in\mathbb{N}\}\subset G$ such that $\{mb_i+\ell b_j : i\leq j\}\subset A$.

Chapter 7

Density analogues of Hindman's finite sums theorem

7.1 Overview

This chapter is based on the results of [24] and it concerns density analogues of Hindman's finite sums theorem. Our starting point and motivation is the main result of [45] which is a truncated density version of Hindman's finite sums theorem.

For any set A of natural numbers with positive upper Banach density, we show the existence of an infinite set B and sequences $(t_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$ of natural numbers such that $\left\{\sum_{n\in F} n : F \subset B + s_k, 1 \leq |F| \leq k\right\} + t_k \subset A$, for every $k \in \mathbb{N}$. This strengthens the aforementioned density finite sums theorem of Kra, Moreira, Richter, and Robertson [45].

We further show, given such a set A, the existence of an infinite set B and a sequence $(t_k)_{k\in\mathbb{N}}$ of natural numbers such that $\left\{\sum_{n\in F} n : F \subset B, |F| = k\right\} + t_k \subset A$, for every $k \in \mathbb{N}$. As a corollary, we obtain a sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers such that $B_1 + \cdots + B_k \subset A$, for every $k \in \mathbb{N}$. We also establish the optimality of our main theorems by providing counterexamples to potential further generalizations, and thereby addressing questions of the aforementioned authors.

Finally, it is noteworthy that, in the context of density analogues to Hindman's finite sums theorem, the results of this chapter are the first – and as of now, only – ones to guarantee sumsets with an unbounded number of summands generated by a fixed infinite set, contained in sets of positive density.

7.2 Introduction

7.2.1 Main results

In 1974, Hindman [26] proved the celebrated finite sums theorem, showing that for any finite partition of the natural numbers, $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_k$, there exists an infinite set $B \subset \mathbb{N}$ such that

$$\Big\{ \sum_{n\in F} n \colon F \subset B,\ 1 \leq |F| < \infty \Big\} \subset C_i, \quad \text{for some } i = 1, \ldots, k. \tag{7.2.1}$$

Erdős [14, 16] was interested in extending Hindman's result to a density analogue, namely by finding infinite sumset patterns similar to those in (7.2.1), in any set $A \subset \mathbb{N}$ whose upper (Banach) density is positive. Obvious parity obstructions and a construction of Straus (recall our discussion in Section 1.1.2) rendered unclear what could potentially be a density analogue of Hindman's theorem.

A series of breakthroughs in the last decade revealed that infinite sumset patterns of increasing complexities can be found in sets of positive density. These results verified conjectures predicted by Erdős and gave rise to further natural questions and conjectures (most of which were formulated in [44]). In particular, using structure theory of multiple ergodic averages, Kra, Moreira, Richter and Roberson proved a truncated density version of Hindman's theorem, which we recall for convenience.

Theorem D. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, for every $k \in \mathbb{N}$, there exists an infinite set $B_k \subset \mathbb{N}$ and an integer $t_k \geq 0$ such that*

$$\Big\{ \sum_{n\in F} n \colon\ F \subset B_k \text{ with } 1 \leq |F| \leq k \Big\} + t_k \subset A.$$

Again, Straus's example shows that one cannot choose the set B_k and the shift t_k independently of k. However, the same authors asked if a sequence of shifts to the dense sets would allow for an unbounded number of summands generated by a fixed infinite set.

Question 7.2.1 ([44, Question 2.11])**.** Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Does there exist an infinite set $B \subset \mathbb{N}$ and a sequence of shifts $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\Big\{ \sum_{n\in F} n \colon\ F \subset B \text{ with } 1 \leq |F| \leq k \Big\} + t_k \subset A, \text{ for every } k \in \mathbb{N}\ ?$$

We answer Question 7.2.1 in the negative (see Proposition 7.3.2) by constructing a set of density arbitrarily close to 1 for which even a weaker version of the conclusion

fails. However, our main results, Theorems 1.2.3 and 1.2.4, extend Theorem D in two different directions, achieving in each case a certain degree of independence between the infinite set B and the number of summands $k \in \mathbb{N}$.

Theorem 1.2.3. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$ and $\ell \geq 0$ an integer. Then, there exist an infinite set $B_\ell \subset \mathbb{N}$ and a sequence $(t_{k,\ell})_{k\in\mathbb{N}} \subset \mathbb{N}$, such that*

$$\left\{ \sum_{n\in F} n \colon \ F \subset B_\ell \text{ with } k \leq |F| \leq k+\ell \right\} + t_{k,\ell} \subset A, \text{ for every } k \in \mathbb{N}.$$

In Section 7.3, we show that Theorem 1.2.3 is optimal in the sense that one cannot choose the set B_ℓ independently of ℓ. We also show that the sequence of shifts $(t_k)_{k\in\mathbb{N}}$ cannot, in general, be bounded. Moreover, by the proof of Theorem 1.2.3, we can choose the sequence $(t_{k,\ell})_{k\in\mathbb{N}}$ to be increasing.

Theorem 1.2.4. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, there exist an infinite set B and sequences $(s_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, $(t_k)_{k\in\mathbb{N}}$ such that*

$$\left\{ \sum_{n\in F} n \colon \ F \subset B + s_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

Letting $B_k = B + s_k$, for every $k \in \mathbb{N}$, we immediately see how this implies Theorem D. Our construction in Section 7.3 shows that the shift sequences $(t_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$ are needed, as well as the necessity of them being unbounded. The second auxiliary sequence in Theorem 1.2.4, as opposed to only shifting the set of positive density, is natural in view of the construction that addresses Question 7.2.1. Furthermore, we prove a combination of the conclusions of Theorems 1.2.3 and 1.2.4 (see Proposition 7.7.8 for more details).

The main result of [43] shows that for any $k \in \mathbb{N}$, any set of positive density contains a sumset of the form $B_1 + \cdots + B_k$, where $B_1, \ldots, B_k$ are infinite. As a corollary of our main theorems (see Section 7.4.1), we strengthen this by removing the dependence of the infinite sets on k. This positively answers [43, Question 8.1] and verifies [44, Conjecture 2.10].

Corollary 1.2.5. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, there exists an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers such that*

$$B_1 + B_2 + \cdots + B_k \subset A, \text{ for every } k \in \mathbb{N}.$$

7.2.2 Dynamical reformulation and key ingredients

In this section, we summarize our approach to prove Theorem 1.2.3 and Theorem 1.2.4, highlighting the new tools developed to address them. We make use of notation and terminology that is defined in Chapter 2.

Inspired by the aforementioned literature [22, 30, 41, 43, 45], after using Furstenberg's correspondence principle, we reinterpret the combinatorial statements into an ergodic language. Concerning Theorem 1.2.3, the dynamical result that implies it is the following.

Theorem 7.2.2. *Let (X, μ, T) be an ergodic measure-preserving system with a point $a \in X$ such that $\mu\left(\overline{\{T^n a : n \in \mathbb{N}\}}\right) = 1$. Let also $\ell \geq 0$ be fixed and $E \subset X$ be an open set with $\mu(E) > 0$. Then, there exist an infinite set $B \subset \mathbb{N}$ and sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$T^{(t_k+b_1+\cdots+b_i)}a \in E, \tag{7.2.2}$$

for all $k \in \mathbb{N}$, all $k \leq i \leq k + \ell$, and $b_1, b_2, \ldots, b_i$ distinct elements in B.

To prove a similar dynamical statement, the authors of [45] introduce a family of measures on finite product spaces, which they call *progressive measures* (see Definition 7.5.2). The positivity of finite cartesian products of open sets with respect to such measures is used to recover Theorem D. However, since we are looking for sumset patterns with an unbounded number of summands, we need to deal with infinite cartesian products of open sets and in general, these sets have zero measure with respect to any useful measure.

We then first adapt the definition of progressive measures to measures on $X^{\mathbb{N}_0}$ and build one, denoted by $\sigma \in (X^{\mathbb{N}_0})$, by lifting a natural measure ξ on $Z_\infty^{\mathbb{N}}$, where (Z_∞, m, T) is the infinite-step pronilfactor. This is a natural approach, and it turns out that the marginals of this measure σ in the first $k + 1$ coordinates do not necessarily coincide with the measures constructed in [45] by lifting measures from the k-step pronilfactor Z_{k-1} (see Appendix B for relevant examples). One can actually recover the proof of Theorem D by considering σ lifted from Z_∞, but for this result the original measures of [45] give an optimal control.

Another obstacle is that we cannot obtain our main results from statements about convergence of orbits in dynamical systems. In previous works, the introduction of Erdős progressions (see [45, Definition 2.1]) was critical for obtaining sumset-type configurations; see, for instance, [2, 3, 10, 11, 38, 40, 43, 45, 49]. In contrast, the use of Erdős progressions could only lead to weaker versions of Theorems 1.2.3 and 1.2.4, in particular versions with nested infinite sets, as we explain

in Proposition 7.3.8 (see also the remark following this proposition). Instead, expanding on the ideas already present in [22], we construct the sumsets inductively, using properties of the measure σ.

For the following discussion and to illustrate the properties of σ, fix $k = 4$, and denote by σ_4 the projection on the first 5 coordinates. We use that the first marginal of σ is the Dirac mass of the predetermined point $a \in X$ in Theorem 7.2.2 and the following:

1. An *initialization property*; for an open set $E \subset X$, if $\mu(E) > 0$, then

$$\sigma_4(X \times T^{-t}E \times T^{-t}E \times T^{-t}E \times T^{-t}E) > 0,$$

   for some $t \in \mathbb{N}$ (see Proposition 7.6.1).

2. *Left-progressiveness* (see Definition 7.5.2); if $U_1, U_2, U_3 \subset X$ are open sets and $\sigma_4(X \times U_1 \times U_2 \times U_3 \times X) > 0$, then

$$\sigma_4(T^{-n}U_1 \times (U_1 \cap T^{-n}U_2) \times (U_2 \cap T^{-n}U_3) \times U_3 \times X) > 0,$$

   for infinitely many $n \in \mathbb{N}$.

3. *Right-progressiveness* (see Definition 7.5.2); if $U_1, U_2, U_3 \subset X$ are open sets and $\sigma_4(X \times U_1 \times U_2 \times U_3 \times X) > 0$, then

$$\sigma_4(X \times U_1 \times (U_2 \cap T^{-n}U_1) \times (U_3 \cap T^{-n}U_2) \cap T^{-n}U_3) > 0,$$

   for infinitely many $n \in \mathbb{N}$.

The first two properties were already exploited in [45], where the second was called *progressiveness*. This notion is not sufficient for our purposes, and we refer to it as left-progressiveness, because we need to exploit its symmetrical version described in **3**. It is this last property that enables us to enlarge the distance to the first coordinate, which ultimately allows us to increase the number of summands in the sumsets of Theorem 7.2.2.

To put things into perspective, for an open set $E \subset X$ with $\sigma_4(X^2 \times E \times X^2) > 0$, a first application of left-progressiveness allows us to find $b_1 \in \mathbb{N}$ such that

$$\sigma_4(X \times T^{-b_1}E \times E \times X \times X) > 0, \tag{7.2.3}$$

and with a second application of the same property, we find $b_2 > b_1$ such that

$$\sigma_4(T^{-(b_1+b_2)}E \times \left(T^{-b_1}E \cap T^{-b_2}E\right) \times E \times X \times X) > 0.$$

Since the first marginal of σ is δ_a, we get that $T^{b_1+b_2}a \in E$. Continuing this process, we get a sumset $\{b_i + b_j : b_i, b_j \in B \text{ distinct}\}$ inside the set of return times $\{n \in \mathbb{N} : T^n a \in E\}$ of a to E. However, if we apply right-progressiveness beforehand in (7.2.3), we find $t \in \mathbb{N}$ such that

$$\sigma_4(X \times T^{-b_1}E \times (E \cap T^{-t-b_1}E) \times T^{-t}E \times X) > 0.$$

In particular, $\sigma_4(X \times X \times T^{-b_1}\tilde{E} \times \tilde{E} \times X) > 0$ where $\tilde{E} = T^{-t}E$ and $b_1 \in \mathbb{N}$ is the same as in (7.2.3). Therefore applying left-progressiveness multiple times as before, we find that not only $\{b_i + b_j : b_i, b_j \in B \text{ distinct}\}$ is in the set $\{n \in \mathbb{N} : T^n a \in E\}$, but also $t + \{b_i + b_j + b_m : b_i, b_j, b_m \in B \text{ distinct}\}$.

The previous discussion gives a sense of how to obtain Theorem 7.2.2 (and therefore Theorem 1.2.3) from the initialization property, left-progressiveness, and right-progressiveness. Nevertheless, when it comes to Theorem 1.2.4, this method is insufficient. The dynamical reformulation of this theorem is the following.

Theorem 7.2.3. *Let (X, μ, T) be an ergodic measure-preserving system with a point $a \in X$ such that $\mu\left(\overline{\{T^n a : n \in \mathbb{N}\}}\right) = 1$. Let also $E \subset X$ be an open set with $\mu(E) > 0$. Then, there exist an infinite set $B \subset \mathbb{N}$ and sequences $(s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$T^{(t_k+is_k+b_1+\cdots+b_i)}a \in E, \tag{7.2.4}$$

for all $k \in \mathbb{N}$, all $1 \leq i \leq k$, and $b_1, b_2, \ldots, b_i$ distinct elements in B.

In order to prove Theorem 7.2.3 we need to use an additional property of our measure $\sigma \in (X^{\mathbb{N}_0})$ which we term *multiple right-progressiveness*. This notion is derived from appropriated double ergodic averages. To prove it, we first show an analogue of Furstenberg's multiple recurrence theorem and then that for these related ergodic averages, the infinite-step pronilfactor is characteristic; the details can be found in Section 7.7.

We refer to Definition 7.7.4 for more details, but when, for instance, $k = 4$, multiple right-progressiveness implies that for open sets $U_1, U_2 \subset X$, if

$$\sigma_4(X \times U_1 \times U_2 \times X \times X) > 0,$$

then there are $t, s \in \mathbb{N}$ such that

$$\sigma_4(X \times U_1 \times (U_2 \cap T^{-(t+s)}U_1) \times (T^{-(t+s)}U_2 \cap T^{-(t+2s)}U_1) \times (T^{-(t+2s)}U_2 \cap T^{-(t+3s)}U_1)) > 0.$$

This property recovers both right-progressiveness and the initialization property (see Remark 7.7.5). Multiple right-progressiveness not only serves to translate sets to the right, but it also allows us to consider an increasing number of patterns with increasing length, by paying the cost of adding an extra shift $s \in \mathbb{N}$, which corresponds to the second shift appearing in Theorem 1.2.4.

7.2.3 Other directions

Given an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers, and $\alpha = \{\alpha_1, \dots, \alpha_k\} \subset \mathbb{N}$, some finite subset of $\mathbb{N}$, we write

$$B_\alpha = \sum_{i\in\alpha} B_i = B_{\alpha_1} + \cdots + B_{\alpha_k}.$$

In this notation, Corollary 1.2.5 can be rewritten as follows:

Corollary 1.2.5. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, there exists an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers such that $B_{\alpha(k)} \subset A$, for every $\alpha(k) = \{1, \dots, k\} \subset \mathbb{N}$, for every $k \in \mathbb{N}$.*

Indeed, it follows from the proof of Corollary 1.2.5 from Theorem 1.2.3 (given at the end of Section 7.4.1) that for any $\ell \in \mathbb{N}$, we could use the family of finite subsets of $\mathbb{N}$ given by $\{\alpha(k) = \{\ell + 1, \dots, \ell + k\} : k \in \mathbb{N}\}$ in place of $\{\alpha(k) = \{1, \dots, k\} : k \in \mathbb{N}\}$ and the result would still hold.

A natural question that follows from these observations is what other families of finite sets can be used as indices for an extension of Corollary 1.2.5 to hold.

Problem 7.2.4. *Describe all the families of finite subsets of $\mathbb{N}$, $\{\alpha(k) : k \in \mathbb{N}\}$, for which the following holds:*

Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. Then, there exists an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of infinite sets of natural numbers such that $B_{\alpha(k)} \subset A$, for every $k \in \mathbb{N}$.

Note that some restrictions have to be placed on the family $\{\alpha(k) : k \in \mathbb{N}\}$ for such a result to hold. For example, if $\{\alpha(k) : k \in \mathbb{N}\}$ is the family of all subsets of $\mathbb{N}$ of cardinality m, for some $m \in \mathbb{N}$, then the conclusion of Problem 7.2.5 forces A to be an IP-set (see Remark 4.6.9 for the definition).

Given that our approach deduces Corollary 1.2.5 from Theorem 1.2.3 (or Theorem 1.2.4) it makes sense to ask for extensions of these results in the direction of Problem 7.2.5. For simplicity, we make this precise only for Theorem 1.2.3.

Problem 7.2.5. *Describe all the families of finite subsets of* $\mathbb{N}$, $\{\alpha(k) : k \in \mathbb{N}\}$, *for which the following holds:*

Let $A \subset \mathbb{N}$ *be a set with* $\mathrm{d}^*(A) > 0$. *Then, there exist an infinite set* $B \subset \mathbb{N}$ *and a sequence* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, *such that*

$$\left\{ \sum_{n\in F} n\colon \ F \subset B \text{ with } |F| \in \alpha(k) \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

Another direction of inquiry is related to extending our results in more general countable abelian group settings. We state a question that aims to extend Theorem 1.2.3 for $\ell = 0$ to initiate this discussion.

Problem 7.2.6. *Suppose* $(G, +)$ *is a countably infinite abelian group, such that* $[G : kG] < \infty$, *for every* $k \in \mathbb{N}$. *Does the following hold?*

Let $A \subset G$ *be a set with* $\mathrm{d}^*(G) > 0$. *Then, there exist an infinite set* $B \subset G$ *and a sequence* $(t_k)_{k\in\mathbb{N}} \subset G$, *such that*

$$\left\{ \sum_{n\in F} n\colon \ F \subset B \text{ with } |F| = k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

The assumption that $[G : kG] < \infty$ for all $k \in \mathbb{N}$ seems necessary in view of the extension of Straus's example in abelian groups due to Ackelsberg [1], that was discussed in Chapter 4. Indeed, as explained in [3],the counterexample to [3, Proposition 1.12] can be extended directly to k-fold sumsets. Since the conclusion of Problem 7.2.6 gives, in particular, k-fold sumsets for every $k \in \mathbb{N}$ in every set of positive density, this forces the assumption that $[G : kG] < \infty$ for all $k \in \mathbb{N}$.

In analogy to how Theorem 1.2.3 generalizes Theorem D, Problem 7.2.6 aims to generalize the conjectured analogue of Theorem D to abelian groups, which is still open.

Conjecture 7.2.7 (cf. [3, Conjecture 1.13])**.** *Let* $k \in \mathbb{N}$. *Let* $(G, +)$ *be a countably infinite abelian group such that* $[G : k!G] < \infty$. *If* $A \subset G$ *has positive upper Banach density, there exists an infinite set* $B \subset G$ *and some* $t \in G$ *such that*

$$\left\{ \sum_{n\in F} n : F \subset B \text{ with } 1 \le |F| \le k \right\} + t \subset A.$$

The main stumbling blocks towards a proof of Conjecture 7.2.7, and hence, towards an answer to Problem 7.2.6 also, are laid out in [3, Section 1], right after its formulation. Essentially, the structure theory of Host and Kra as well as key equidistribution results for orbits in nilsystems, are both unavailable for G-systems of arbitrarily high order, when G is an infinitely generated abelian group.

We conclude with a problem that seems to be within reach with current methods. In analogy to how Theorem 1.2.1 generalizes Theorem C, Hernández in [22] characterized all the integer weighted sumsets for which Theorem D still holds. More precisely:

Theorem 7.2.8 (cf. [22, Theorem 1.3])**.** *Let $d, k, r \in \mathbb{N}$ and $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$. There exists an infinite set $B \subset \mathbb{N}$ and some $t \in \mathbb{N}_0$ such that*

$$\left\{ kb_1 + w_1 b_2 + \cdots + w_d b_{d+1} : b_1, \ldots, b_{d+1} \in B \text{ and } b_1 > b_2 > \cdots > b_{d+1} \right\},$$

for all $w_1, \ldots, w_d \in \{-r, \ldots, r\}$ and $k \neq -\sum_{i=1}^{j} w_i$, for all $j \in \{1, \ldots, d\}$.

Restricting, once again for simplicity, to the statement of Theorem 1.2.3, one may wish to find a weighted version, paralleling the relationship between Theorem D and Theorem 7.2.8. However, a direct analogue of Theorem 7.2.8 fails even when $\ell = 0$.

Proposition 7.2.9. *For every $\varepsilon > 0$, there exists a set $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 1 - \varepsilon$, and such that for any infinite set $B \subset \mathbb{N}$ and sequence of shifts $(t_k)_{k \in \mathbb{N}}$, it does not hold that*

$$\left\{ b_1 + \sum_{j=2}^{k} w_j b_j : b_1, \ldots, b_k \in B \text{ and } b_1 > \cdots > b_k \text{ and } w_1, \ldots, w_k \in \{1, 2\} \right\} + t_k \subset A.$$

Proof. The set A defined in (7.3.1) works and this can be shown using a similar argument as in the proof of Proposition 7.3.2. □

In view of Proposition 7.2.9 we ask the following question, which should be viewed as a first step towards the open-ended problem of finding the correct weighted versions of Theorems 1.2.3 and 1.2.4.

Question 7.2.10. Let $r \in \mathbb{N}$ be given. Is it true that for any $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$, and any sequence $(w_k)_{k \in \mathbb{N}} \subset \{1, \ldots, r\}$, there exists an infinite set $B \subset \mathbb{N}$ and

sequence $(t_k)_{k\in\mathbb{N}}$ (which are allowed to depend on (w_k)) such that

$$\left\{ \sum_{j=1}^{k} w_j b_j : b_1, \dots, b_k \in B \text{ and } b_1 > \cdots > b_k \right\} + t_k \subset A?$$

7.3 Negative results and optimality of the main theorems

In [44], several possible results serving the purpose of a potential analogue to Hindman's finite sums theorem were conjectured or asked. We begin with a weaker version of Question 7.2.1 stated in the introduction.

Question 7.3.1 ([44, Question 2.12])**.** Let $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$. Does there exist a nested sequence of infinite sets $B_1 \supset B_2 \supset B_3 \supset \dots$, where B_{k+1} is cofinite in B_k for each $k \in \mathbb{N}$, and a sequence of shifts $(t_k)_{k\in\mathbb{N}}$ such that, for any $k \in \mathbb{N}$ it holds that

$$\left\{ \sum_{n\in F} n : F \subset B_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A \ ?$$

Note that a positive answer to Question 7.2.1 would clearly imply a positive answer to Question 7.3.1. But one could – ostensibly – weaken these statements even further and in the setting of Question 7.3.1 simply ask for a nested sequence of infinite sets $B_1 \supset B_2 \supset B_3 \supset \dots$, without demanding they be cofinite in B_1. We remark that in fact this last statement is equivalent to that of Question 7.3.1 and a proof of this equivalence is included in Proposition 7.3.5 below. The next result, which follows by a modification of Straus's example mentioned in the introduction, disproves this statement.

Proposition 7.3.2. *Given any $\varepsilon > 0$ there exists a set $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 1 - \varepsilon$ such that for any sequence $(k_i)_{i\in\mathbb{N}}$, for any nested sequence $B_1 \supset B_2 \supset \dots$, of infinite sets, and any sequence $(t_i)_{i\in\mathbb{N}}$, it does not hold that*

$$\left\{ \sum_{n\in F} n : F \subset B_i \text{ with } k_i \leq |F| \leq k_i + i \right\} + t_i \subset A, \text{ for every } i \in \mathbb{N}.$$

Before proving Proposition 7.3.2, we list some consequences.

Remark 7.3.3. The set $A \subset \mathbb{N}$ provided by Proposition 7.3.2 is such that:

1. Taking $k_i = 1$ for each $i \in \mathbb{N}$, then for any nested sequence $B_1 \supset B_2 \supset B_3 \supset$

$\ldots$, of infinite sets and any sequence of shifts $(t_k)_{k\in\mathbb{N}}$, it does not hold that

$$\left\{ \sum_{n\in F} n : F \subset B_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

In view of the discussion prior to the statement of Proposition 7.3.2, this answers Questions 7.2.1 and 7.3.1. Note that it also shows the necessity of the auxiliary shift-sequence $(s_k)_{k\in\mathbb{N}}$ in the statement of Theorem 1.2.4.

2. If we let $k_i = i$ for each $i \in \mathbb{N}$, then for any infinite set $B \subset \mathbb{N}$ and any sequence $(t_{k,\ell})_{k,\ell\in\mathbb{N}}$ it does not hold that

$$\left\{ \sum_{n\in F} n : F \subset B \text{ with } k \leq |F| \leq k+\ell \right\} + t_{k,\ell} \subset A,$$

for each $k, \ell \in \mathbb{N}$. This shows that Theorem 1.2.3 cannot be improved in the sense of allowing the infinite set B to be independent of ℓ.

Proof of Proposition 7.3.2. Let $\varepsilon > 0$ be fixed and consider an increasing sequence of prime numbers $(p_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} n/p_n < \varepsilon$. Then, we define the set

$$A = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} \left(\bigcup_{j=0}^{n-1} p_n\mathbb{N} + j \right). \tag{7.3.1}$$

By the choice of (p_n) it readily follows that $\mathrm{d}^*(A) > 1-\varepsilon$. Now, assume there exists an infinite sequence of sets $B_1 \supset B_2 \supset B_3 \supset \ldots$ and a sequence of shifts $(t_i)_{i\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$ it holds that

$$\left\{ \sum_{n\in F} n : F \subset B_i \text{ with } k_i \leq |F| \leq k_i + i \right\} + t_i \subset A.$$

Notice that, if $r > i$, the fact that $B_r \subset B_i$ implies

$$\left\{ \sum_{n\in F} n : F \subset B_r \text{ with } k_i \leq |F| \leq k_i + i \right\} + t_i \subset A.$$

Fix $i, m, n, r \in \mathbb{N}$ with $t_i < n < p_n < r < m$ and $i < r < m$ so that $B_m \subset B_r \subset B_i$.

Assume first that there is an infinite set $\tilde{B}_m \subset B_m \cap p_n\mathbb{N}$. In particular,

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B}_m \text{ with } |F| = k_i \right\} \subset p_n\mathbb{N}.$$

Then, by assumption and the above discussion, it must be the case that

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B}_m \text{ with } |F| = k_i \right\} + t_i \subset A.$$

However, $t_i < n$ and thus

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B}_m \text{ with } |F| = k_i \right\} + t_i \subset \bigcup_{j=0}^{n-1} p_n\mathbb{N} + j \subset \mathbb{N} \setminus A,$$

which is a contradiction.

Therefore, by the pigeonhole principle, we may assume there exists $c_n \in \{1, \dots, p_n - 1\}$ and an infinite set $\tilde{B}_m \subset B_m \cap (p_n\mathbb{N} + c_n)$. As before, the preceding discussion implies that

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B}_m \text{ with } k_r \leq |F| \leq k_r + r \right\} + t_r \subset A.$$

Now, p_n is a prime and $c_n \not\equiv 0 \pmod{p_n}$, hence c_n is a generator in $\mathbb{Z}_{p_n}$. As $r > p_n$ this means that there exists some $d \in \{k_r, \dots, k_r + r\}$ such that $dc_n + t_r \equiv 0 \pmod{p_n}$. We have thus established that

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B}_m \text{ with } |F| = d \right\} + t_r \subset A \cap (p_n\mathbb{N} + dc_n + t_r) \subset A \cap p_n\mathbb{N} = \emptyset,$$

which is again a contradiction. □

Proposition 7.3.4. *There exists a set $A \subset \mathbb{N}$ with $\mathrm{d}^*(A) > 0$ such that the statement of Theorem 1.2.4 fails if either of the sequences $(t_k)_{k\in\mathbb{N}}$ or $(s_k)_{k\in\mathbb{N}}$ is bounded.*

Remark. The analogous is true for the statement of Theorem 1.2.3.

Proof. Take $A \subset \mathbb{N}$ as in (7.3.1). Suppose Theorem 1.2.4 holds with a bounded sequence of shifts $(t_k)_{k\in\mathbb{N}}$ and some sequence $(s_k)_{k\in\mathbb{N}}$. Let $n > q = \max\{t_k : k \in \mathbb{N}\}$. Assume that for some $k > p_n$ there exists an infinite set $\tilde{B}+s_k \subset (B+s_k)\cap(p_n\mathbb{N}+d_n)$ for some $d_n \in \{1, \dots, p_n - 1\}$. Then, as d_n is a generator in Z_{p_n} and $k > p_n$ we can find $d \in \{1, \dots, k\}$ so that $d \cdot d_n + t_k \equiv 0 \pmod{p_n}$. We thus reach a contradiction because

$$\left\{ \sum_{n\in F} n : F \subset \tilde{B} + s_k \text{ with } |F| = d \right\} + t_k \subset p_n\mathbb{N} + d \cdot d_n + t_k \subset p_n\mathbb{N} \subset \mathbb{N} \setminus A.$$

Hence, for every $k > p_n$, there is an infinite set $\tilde{B}+s_k \subset (B+s_k)\cap p_n\mathbb{N}$. This leads to a contradiction once again, since $\{\sum_{n\in F} n : F \subset \tilde{B}+s_k \text{ with } |F|=k\}+t_k \subset A$ and

$$\left\{\sum_{n\in F} n : F \subset \tilde{B}+s_k \text{ with } |F|=k\right\}+t_k \subset p_n\mathbb{N}+t_k \subset \bigcup_{j=0}^{q} p_n\mathbb{N}+j \subset \bigcup_{j=0}^{n} p_n\mathbb{N}+j \subset \mathbb{N}\backslash A.$$

To see that Theorem 1.2.4 does not in general hold with a finite sequence $(s_k)_{k\in\mathbb{N}}$, note that, if it did we would find by the pigeonhole principle an infinite set B, an integer $s \in \mathbb{N}$ and sequences $(n_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}}$ so that

$$\left\{\sum_{n\in F} n : F \subset B+s \text{ with } 1 \leq |F| \leq n_k\right\}+t_k \subset A, \text{ for every } k \in \mathbb{N}.$$

This would contradict our negative answer to Question 7.2.1. □

We next show that imposing a cofiniteness assumption for nested versions of infinite sumsets results, as, for instance, in Question 7.3.1, is superfluous in the following sense.

Proposition 7.3.5. *If $A \subset \mathbb{N}$, then the following are equivalent:*

(i) There exists a nested sequence of infinite sets $B_1 \supset B_2 \supset B_3 \supset \ldots$, where B_{k+1} is cofinite in B_k, for each $k \in \mathbb{N}$, and a sequence of shifts $(t_k)_{k\in\mathbb{N}}$ such that, for any $k \in \mathbb{N}$

$$\left\{\sum_{n\in F} n : F \subset B_k \text{ with } 1 \leq |F| \leq k\right\}+t_k \subset A \tag{7.3.2}$$

(ii) There exists a nested sequence of infinite sets $B_1 \supset B_2 \supset B_3 \supset \ldots$, and a sequence of shifts $(t_k)_{k\in\mathbb{N}}$ such that, (7.3.2) holds for any $k \in \mathbb{N}$.

Proof. It is clear that (i) implies (ii). For the converse, suppose $(B_i)_{i\in\mathbb{N}}$ satisfies (ii). Let $b_1 = \min\{b : b \in B_1\}$ and then, let $b_2 = \min\{b : b \in B_2 \setminus \{b_1\}\}$. Inductively, we can choose $b_{i+1} = \min\{b : b \in B_{i+1} \setminus \{b_1, \ldots, b_i\}\}$. With that we have built a sequence $(b_i)_{i\in\mathbb{N}}$ which is infinite because every set B_i is infinite. For each $i \in \mathbb{N}$, define $C_i = \{b_j : j \geq i\}$ and observe that, since $C_i \subset B_i$, the sequence $C_1 \supset C_2 \supset C_3 \supset \ldots$ fulfills (7.3.2). Finally, by construction, $C_i\backslash C_{i+1} = \{b_{i+1}\}$, hence C_{i+1} is cofinite in C_i, concluding the proof. □

Remark 7.3.6. The same proof can be performed for other types of sumsets results. In the present work, our "positive" results are proven in non-nested versions, that

is, the strongest possible. We do not know, however, if Theorem 1.2.3 and Theorem 1.2.4 are equivalent to their respective nested versions. It could be the case, for example, that every element $b \in B_1$ appears only in finitely many B_i which a priori makes the construction of a single infinite set B impossible.

We finish this section by showing that infinite analogues of Erdős progressions (see [45, Definition 2.1]) can be utilized to obtain weaker versions of our main results, Theorem 1.2.3 and Theorem 1.2.4. The sumsets obtained via this method are generated by a nested sequence of infinite sets.

Definition 7.3.7. Let X be a compact metric space and $T\colon X \to X$ a homeomorphism. A sequence $(x_k)_{k\in\mathbb{N}_0}$ in X is called an infinite Erdős progression if there exists an increasing sequence $(c_n)_{n\in\mathbb{N}}$ of natural numbers such that

$$\lim_{n\to\infty} T^{c_n} x_k = x_{k+1}, \text{ for every } k \in \mathbb{N}_0.$$

Proposition 7.3.8. *Let X be a compact metric space, $T\colon X \to X$ a homeomorphism and $(U_k)_{n\in\mathbb{N}}$ a sequence of open subsets of X. Assume there exists an infinite Erdős progression $(x_k)_{k\in\mathbb{N}_0}$ such that $x_k \in U_k$ for each $k \in \mathbb{N}$. Then, there exists a sequence of infinite sets $B_1, B_2, \dots$ such that $B_{k+1} \subset B_k$ and*

$$\Big\{ \sum_{n\in F} n\colon F \subset B_k,\ |F| = k \Big\} \subset \Big\{ m \in \mathbb{N}\colon T^m x_0 \in U_k \Big\},$$

for every $k \in \mathbb{N}$.

Proof. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence such that the convergence in Definition 7.3.7 is satisfied. Fixing $k \in \mathbb{N}$, the proof of [45, Lemma 2.2] gives an infinite set $B_k \subset \{c_n : n \in \mathbb{N}\}$ such that

$$\Big\{ \sum_{n\in F} n : F \subset B_k,\ |F| = k \Big\} \subset \Big\{ m \in \mathbb{N}\colon T^m x_0 \in U_k \Big\}.$$

Then, writing $B_k = \{b_{k,1} < b_{k,2} < \cdots\}$, since $(b_{k,n})_{n\in\mathbb{N}}$ is a subsequence of $(c_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} T^{b_{k,n}} x_m = x_{m+1}, \text{ for every } m \in \mathbb{N}.$$

Another application of [45, Lemma 2.5] then gives an infinite set $B_{k+1} \subset \{b_{k,n} : n \in \mathbb{N}\}$ so that

$$\Big\{ \sum_{n\in F} n : F \subset B_{k+1},\ |F| = k+1 \Big\} \subset \Big\{ m \in \mathbb{N}\colon T^m x_0 \in U_{k+1} \Big\}.$$

Continuing inductively we obtain the result. In fact, the proof of Proposition 7.3.5 shows that we can actually choose the sets $B_1, B_2, \dots$, so that $B_k \setminus B_{k+1}$ is a finite set for each $k \in \mathbb{N}$. □

Remark. Indeed, it is also the case that Proposition 7.3.8 is, in general, optimal. Namely, there are infinite Erdős progressions that can only be translated to sumsets generated by nested sequences.

Let $(x_k)_{k\in\mathbb{N}_0}$ denote an infinite Erdős progression with $(c_n)_{n\in\mathbb{N}}$ being the sequence such that $\lim_{n\in\mathbb{N}} T^{c_n} x_k = x_{k+1}$. For $\varepsilon > 0$ fixed, define the sequence $(n_k)_{k\in\mathbb{N}}$ by $n_k = \min\{c_n : d(T^{c_n} x_k, x_{k+1}) < \varepsilon\}$.

Assume that the Erdős progression is chosen in a way that for some $\varepsilon > 0$, the sequence $(n_k)_{k\in\mathbb{N}}$ is increasing (all we need is for it to be unbounded). Then, there cannot exist an infinite $B \subset \{c_n : n \in \mathbb{N}\}$ such that $B^{\oplus k} \subset \{n : d(T^n x_0, x_k) < \varepsilon/2\}$, for every $k \in \mathbb{N}$.

Indeed, if b_1 is the smallest element in $B = \{b_1 < b_2 < \cdots\}$, simply choose $k \in \mathbb{N}$ so that $n_{k-1} > b_1$. Then, we should have

$$d(T^{b_1+b_2+\ldots+b_k} x_0, x_k) < \varepsilon/2,$$

for all $b_2 < \cdots < b_k$ in B. But if $b_2, \dots, b_k$ are large enough (each one in relation to the previous) then

$$d(T^{b_1+b_2+\ldots+b_k} x_0, T^{b_1} x_{k-1}) < \varepsilon/2.$$

By the choice of n_{k-1} we have that $d(T^{b_1} x_{k-1}, x_k) \geq \varepsilon$, which would lead to a contradiction.

7.4 Dynamical interpretation

7.4.1 Translation to dynamics

The proofs of our main combinatorial results begin by translating them to statements in ergodic theory. A fundamental step in order to facilitate this reformulation is a version of Furstenberg's correspondence principle introduced in [18]. For a proof of this version, we refer the reader to [43].

Proposition 7.4.1. *Let $A \subset \mathbb{N}$ be a set of with $\mathrm{d}^*(A) > 0$. Then, there exists an ergodic system (X, μ, T), a point $a \in X$ with $\mu\left(\overline{\{T^n a : n \in \mathbb{N}\}}\right) = 1$ and a clopen set $E \subset X$ such that $A = \{n \in \mathbb{N} : T^n a \in E\}$ and $\mu(E) > 0$.*

We show the deduction of our main combinatorial results, Theorems 1.2.3 and 1.2.4, from the dynamical theorems mentioned in the introduction.

Proof that Theorem 7.2.3 implies Theorem 1.2.4. Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$. By the correspondence principle (Proposition 7.4.1) we find an ergodic system (X, μ, T), a point $a \in X$ with $\mu\left(\overline{\{T^n a : n \in \mathbb{N}\}}\right) = 1$ and a clopen set $E \subset X$ such that $A = \{n \in \mathbb{N} : T^n a \in E\}$ and $\mu(E) > 0$. Theorem 7.2.3 then guarantees the existence of an infinite set $B \subset \mathbb{N}$ and sequences $(s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$T^{(t_k + i s_k + b_1 + \cdots + b_i)} a \in E,$$

for every $k \in \mathbb{N}$ and for every $1 \leq i \leq k$, and $b_1, b_2, \ldots, b_i$ distinct elements of B. Since $A = \{n \in \mathbb{N} : T^n a \in E\}$ we have that

$$b_1 + \cdots + b_i + i s_k + t_k \in A,$$

for every $k \in \mathbb{N}$ and for every $1 \leq i \leq k$, and $b_1, b_2, \ldots, b_i$ distinct elements of B. This translates to

$$\left\{ \sum_{n\in F} n \colon \ F \subset B + s_k \text{ with } 1 \leq |F| \leq k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N},$$

which is precisely the statement of Theorem 1.2.4. □

The proof that Theorem 7.2.2 implies Theorem 1.2.3 is analogous to the previous and as such we omit it. Let us also show that Theorem 1.2.3, for example, implies Corollary 1.2.5.

Proof that Theorem 1.2.3 implies Corollary 1.2.5. Let $B \subset \mathbb{N}$ and $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ be the infinite set and the sequence obtained by Theorem 1.2.3 for $\ell = 0$, so that

$$\left\{ \sum_{b\in F} b : F \subset B, |F| = k \right\} + t_k \subset A, \text{ for every } k \in \mathbb{N}. \tag{7.4.1}$$

Let $B = \{b_k : k \in \mathbb{N}\}$ be an enumeration of the set B, let $\{p_k : k \in \mathbb{N}\}$ be an enumeration of the primes and for each $k \in \mathbb{N}$ define $B_k = \{b_{p_k^n} + t_k - t_{k-1} \colon k \in \mathbb{N}\}$, where $t_0 = 0$. By (7.4.1) we have that

$$\sum_{i=1}^{k} (b_{p_i^{n_i}} + t_i - t_{i-1}) = \Big(\sum_{i=1}^{k} b_{p_i^{n_i}} \Big) + t_k \subset A,$$

for every $k \in \mathbb{N}$ and any $n_1, \ldots, n_k \in \mathbb{N}$. By definition, this implies that $B_1 + B_2 + \cdots + B_k \subset A$, for every $k \in \mathbb{N}$, concluding the proof. □

We saw that the dynamical results, Theorems 7.2.2 and 7.2.3, imply respectively Theorems 1.2.3 and 1.2.4 – and hence, also, Corollary 1.2.5. Their proofs are then the main focus of the rest of this paper.

7.5 Progressive measures and infinite-step nilsystems

A property that we use throughout the article is that of unique ergodicity of orbit closures in infinite-step pronilsystems.

Proposition 7.5.1. *Let (Z, m, T) be an infinite-step pronilsystem, $d \geq 1$ and $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ a Følner sequence in $\mathbb{Z}^d$. Then, for every $k \geq 2$, any $x_1, \dots, x_k \in Z$ and any $S_i = T^{a_{i,1}} \times T^{a_i,2} \times \cdots \times T^{a_{i,k}}$, where $a_{i,1}, \dots, a_{i,k}$ are arbitrary integers, for $i = 1, \dots, d$, the measure*

$$\rho = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{(n_1,\dots,n_d)\in\Phi_N} (S_1^{n_1} \cdots S_d^{n_d})\delta_{(x_1,\dots,x_k)},$$

where the limit is in the weak topology, is well-defined. Moreover ρ is independent from the Følner sequence Φ.*

Proof. Let Z_s be the s-step pronilfactor of Z with factor map $\pi_s \colon Z \to Z_s$. It follows by [32] that $\pi_k \colon Z \to Z_s$ is continuous. Let Ω_s be the orbit closure of $(\pi_s(x_1), \pi_s(x_2), \dots, \pi_s(x_k)) \in Z_s^k$ under the $\mathbb{Z}^d$-action given by the maps $S_1, \dots, S_d$. By properties of pronilsystems, $(\Omega_s, S_1, \dots, S_d)$ is minimal and uniquely ergodic. If Ω_∞ is the orbit closure of $(x_1, x_2, \dots, x_k) \in Z^k$ under the $\mathbb{Z}^d$-action given by the maps $S_1, \dots, S_d$, then Ω_∞ is the inverse limit of $(\Omega_s)_{s\in\mathbb{N}}$ and therefore is minimal and uniquely ergodic which concludes the proposition. □

7.5.1 Progressive measures

We now define the notion of progressiveness for measures on finite (and infinite) product spaces. This property can be viewed as a variation of recurrence. Later, we introduce a more general property, which can analogously be viewed as variation of multiple recurrence. These properties are the key ingredients that allow us to prove our main dynamical theorems.

Definition 7.5.2. Let (X, T) be a topological system and let $k \in \mathbb{N}$. We say that a Borel probability measure $\tau \in (X^{k+1})$ is *left-progressive* if for all open sets $U_1, \dots, U_k \subset X$ with

$$\tau(X \times U_1 \times \cdots \times U_k) > 0,$$

there exist infinitely many $n \in \mathbb{N}$ such that

$$\tau((X \times U_1 \times \cdots \times U_k) \cap (T \times \cdots \times T)^{-n}(U_1 \times \cdots \times U_k \times X)) > 0.$$

We say that τ as above is *right-progressive* if for all open sets $U_1, \ldots, U_{k-1} \subset X$ with

$$\tau(X \times U_1 \times \cdots \times U_{k-1} \times X) > 0,$$

there exist infinitely many $n \in \mathbb{N}$ such that

$$\tau((X \times U_1 \times \cdots \times U_{k-1} \times X) \cap (T \times \cdots \times T)^{-n}(X \times X \times U_1 \times \cdots \times U_{k-1})) > 0.$$

Finally, a measure is *progressive* if it is both left- and right-progressive.

For some measures on product spaces that we consider, the first marginal has a distinguished role from the rest. As a useful convention we index such measures starting from the 0-th coordinate.

Remark. In [45, Definition 3.1], the authors introduced a notion of progressive measures that coincides with what we refer to as left-progressive.

We naturally extend this notion to measures $\tau \in (X^{\mathbb{N}_0})$.

Definition 7.5.3. Let (X, T) be a topological system. We say that a Borel probability measure $\tau \in (X^{\mathbb{N}_0})$ is *progressive* if $P_k\tau$ is progressive for all $k \geq 2$, where $P_k \colon X^{\mathbb{N}_0} \to X^{k+1}$ is the projection onto the first $k+1$ coordinates (including the 0-th coordinate).

We now proceed to define a Borel measure σ on the infinite product space $X^{\mathbb{N}_0}$, with base space coming from a given measure-preserving system (X, μ, T), which turns out to be progressive. In order to achieve this we first define a measure ξ on $Z_\infty^{\mathbb{N}}$, where (Z_∞, m, T) is the infinite-step pronilfactor of (X, μ, T) and then lift this measure on $X^{\mathbb{N}_0}$. We remark that the definition of σ crucially depends on a single predetermined point $a \in X$. However, the factor map $\pi : X \to Z_\infty$, which is used to lift ξ to σ, is a priori only measurable and thus defined almost everywhere (hence, potentially, not on a). Fortunately, we can assume that the factor map is actually continuous, a manoeuvre known among aficionados that was also exploited in [45]. This reduction is formally proven at the end of this section.

Definition 7.5.4. We say that (X, μ, T) has a *topological infinite-step pronilfactor* if there is a continuous factor map $\pi_\infty : X \to Z_\infty$, where (Z_∞, m, T) denotes the infinite-step pronilfactor of (X, μ, T).

Let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor (Z_∞, m, T). Let also $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$ and $a \in \mathtt{gen}(\mu, \Phi)$. Similarly to [45], our goal is to construct a measure $\xi \in (Z_\infty^{\mathbb{N}})$ given by

$$\xi = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{\infty} \delta_{T^{in}\pi_\infty(a)}, \tag{7.5.1}$$

where the limit (assuming it exists) is in the weak* topology, then lift it and build a progressive measure $\sigma \in (X^{\mathbb{N}_0})$ (see (7.5.2)). For that we first need to ensure that the limit in (7.5.1) is well-defined.

Lemma 7.5.5. *Let (Z, m, T) be an infinite-step pronilsystem, $a \in Z$ and $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ a Følner sequence in $\mathbb{N}$. Then for every $k \geq 2$, the measure*

$$\xi_k = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^n a} \times \delta_{T^{2n} a} \times \cdots \times \delta_{T^{kn} a}$$

is well-defined. Moreover ξ_k is independent from the Følner sequence Φ.

Proof. This is a corollary of Proposition 7.5.1 with $d = 1$, $S_1 = T \times T^2 \times \cdots \times T^k$. □

By a classical approximation argument we deduce that the limit in (7.5.1) exists.

Corollary 7.5.6. *The measure given by* (7.5.1) *is well-defined.*

Proof. This follows at once from Lemma 7.5.5 using the definition of weak* convergence, the observation that $P_k^* \xi_n = \xi_k$, for each $k, n \in \mathbb{N}$, with $k \leq n$, where P_k^* denotes the projection onto the first k coordinates, and the fact that $\{f_1 \otimes \cdots \otimes f_k : k \in \mathbb{N}, f_1, \ldots, f_k \in C(Z_\infty)\}$ is densely embedded in $C(Z_\infty^{\mathbb{N}})$. □

With that we can define the Borel probability measure $\sigma \in (X^{\mathbb{N}_0})$ through the use of Kolmogorov's extension theorem as follows.

Definition 7.5.7. For $k \in \mathbb{N}$, we define the measure $\sigma_k \in (X^{k+1})$ via the formula

$$\int_{X^{k+1}} f_0 \otimes f_1 \otimes \cdots \otimes f_k \, \mathrm{d}\sigma_k = f_0(a) \int_{Z_\infty^k} \mathbb{E}(f_1 \mid Z_\infty) \otimes \cdots \otimes \mathbb{E}(f_k \mid Z_\infty) \, \mathrm{d}P_k^* \xi, \tag{7.5.2}$$

for all $f_0, f_1, \ldots, f_k \in C(X)$, where P_k^* denotes the projection onto the coordinates $\{1, \ldots, k\}$ (excluding the 0-th coordinate). It follows by Stone-Weierstrass' theorem that each σ_k is well-defined via (7.5.2). The sequence of measures $(\sigma_k)_{k\in\mathbb{N}}$ satisfies the compatibility condition of Kolmogorov's extension theorem, so we denote by $\sigma \in (X^{\mathbb{N}_0})$ the unique extension of this sequence.

Remark. By definition, for any $k \in \mathbb{N}$, $P_k\sigma = \sigma_k$, where P_k denotes the projection onto the first $k+1$ coordinates, including the 0-th coordinate. It follows immediately from the definition that $P_k\sigma$ is invariant under $Id \times T \times \cdots \times T^k$. Moreover, for each $k \in \mathbb{N}$ and for each $i = 1, \ldots, k$ we have that the marginals $P_k\sigma_i$ of $P_k\sigma$ satisfy $P_k\sigma_i \leq i\mu$, i.e., $P_k\sigma_i(A) \leq i\mu(A)$ for any Borel set $A \subset X$. This follows using the unique ergodicity of orbits on (Z_∞, T) and the T^i-invariance of the marginals $P_k^*\xi_i$ of $P_k^*\xi$ as in [45, Lemma 4.6].

In order to prove Theorems 7.2.2 and 7.2.3 we first work under the assumption of topological infinite-step pronilfactors. This allows us to define the measure σ as above, and using this we prove the following theorems.

Theorem 7.5.8. *Let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence, let $a \in \mathtt{gen}(\mu, \Phi)$ and let σ be the measure defined in Definition 7.5.7. Then, for $\ell \in \mathbb{N}$ fixed and any open set $E \subset X$ with $\mu(E) > 0$ there exist sequences $(b_n)_{n\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$0 < \sigma_{k+\ell+1}\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+\ell+1}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right),$$

for every $k \in \mathbb{N}$, where $B_k = \{b_1, \ldots, b_k\}$.

Theorem 7.5.9. *Let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence, let $a \in \mathtt{gen}(\mu, \Phi)$ and let σ be the measure defined in Definition 7.5.7. Then, for every open set $E \subset X$ with $\mu(E) > 0$ there exist sequences $(b_n)_{n\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$\sigma_{k+1}\left(\bigcap_{i=1}^{k}\prod_{j=0}^{k+1}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\right) > 0,$$

for every $k \in \mathbb{N}$, where $B_k = \{b_1, \ldots, b_k\}$.

To prove that these results suffice to recover Theorems 7.2.2 and 7.2.3, the only necessary ingredient is the fact that for a general measure preserving system (X, μ, T) with a priori measurable pronilfactors, one can pass to an extension with continuous factor map onto X and with topological infinite-step topological pronilfactor that agrees with that of X. This technical result was established in [43, Proposition 5.7 and Lemma 5.8] for pronilfactors of finite-step. The proof extends to the case of infinite-step pronilfactors needed for our setting.

Proposition 7.5.10. *Let (X, μ, T) be an ergodic system and $a \in X$ be a transitive point. Then there exists an ergodic system $(\tilde{X}, \tilde{\mu}, \tilde{T})$ with topological infinite-step pronilfactor, a Følner sequence Ψ in $\mathbb{N}$, a point $\tilde{a} \in \mathtt{gen}(\tilde{\mu}, \Psi)$ and a continuous factor map $\pi\colon \tilde{X} \to X$ with $\pi(\tilde{a}) = a$. Moreover $(\tilde{X}, \tilde{\mu}, \tilde{T})$ and (X, μ, T) are measurably isomorphic and $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a joining of X and Z_∞.*

Proof. By [43, Lemma 5.7] we can find $(\tilde{X}, \tilde{\mu}, \tilde{T})$ which is measurably isomorphic to (X, μ, T), $\pi\colon \tilde{X} \to X$ is continuous, $\pi(\tilde{a}) = a$ and such that the maps $\tilde{\pi}_k\colon \tilde{X} \to Z_k(\tilde{X})$ are continuous and $Z_k(\tilde{X})$ is isomorphic to $Z_k(X)$ for every $k \in \mathbb{N}$. It follows by the definition of infinite-step pronilfactors that $Z_\infty(\tilde{X})$ is measurably isomorphic to $Z_\infty(X)$. By the continuity of the factor maps $\tilde{\pi}_k$ it follows that $Z_\infty(\tilde{X})$ is a topological inverse limit of $((Z_k(\tilde{X}), T))_{k\in\mathbb{N}}$. Hence by the universal property of topological inverse limits, there exists a continuous factor map $\tilde{\pi}_\infty\colon \tilde{X} \to Z_\infty(\tilde{X})$, concluding the proof. □

Proof that Theorem 7.5.9 implies Theorem 7.2.3. Fix an ergodic system (X, μ, T), a Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$, a point $a \in \mathtt{gen}(\mu, \Phi)$ and an open set $E \subset X$ with $\mu(E) > 0$. By Proposition 7.5.10 there is an ergodic extension $(\tilde{X}, \tilde{\mu}, \tilde{T})$ with a continuous factor map $\pi : \tilde{X} \to X$ that has topological infinite-step pronilfactor and there is a Følner sequence $\tilde{\Phi} = (\tilde{\Phi}_N)_{N\in\mathbb{N}}$ and a point $\tilde{a} \in \mathtt{gen}(\tilde{\mu}, \tilde{\Phi})$ such that $\pi(\tilde{a}) = a$. Let $\tilde{\sigma}$ be the measure on $\tilde{X}^{\mathbb{N}_0}$ given by Definition 7.5.7 and σ on $X^{\mathbb{N}_0}$ be its pushforward under π^∞ (equivalently, σ on $X^{\mathbb{N}_0}$ is the unique extension with projections the compatible measures σ_k on X^{k+1} defined as the pushforwards of $\tilde{\sigma}_k$ by π^{k+1} on $\tilde{X}^{k+1}$). Finally, let $\tilde{E} = \pi^{-1}(E)$. Applying Theorem 7.5.9 to $(\tilde{X}, \tilde{\mu}, \tilde{T})$ we find strictly increasing sequences $(b_n)_{n\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\tilde{\sigma}_{k+1}\left(\bigcap_{i=1}^{k}\prod_{j=0}^{k+1}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus\ell-j}\right)}\tilde{E}\right) > 0,$$

for every $k \in \mathbb{N}$, where $B_k = \{b_1, \dots, b_k\}$. Note that $\tilde{\sigma}_{k+1}(\{\tilde{a}\} \times X^{k+1}) = 1$. By the definition of $\tilde{\sigma}$ and the fact that π is a factor map it follows that

$$\sigma_{k+1}\left(\bigcap_{i=1}^{k}\prod_{j=0}^{k+1}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus\ell-j}\right)}E\right) > 0, \tag{7.5.3}$$

for every $k \in \mathbb{N}$, and also that $\sigma_{k+1}(\{a\} \times X^{k+1}) = 1$, for every $k \in \mathbb{N}$. In particular,

considering only the 0-th coordinate in (7.5.3) we see that

$$a \in \bigcap_{i=1}^{k} \left(\bigcap_{\ell=1}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell}\right)} E \right),$$

for every $k \in \mathbb{N}$. It follows that, for each $k \in \mathbb{N}$, $1 \leq i \leq k$, and $1 \leq \ell \leq i$,

$$a \in T^{-(t_i+\ell s_i+B_k^{\oplus \ell})} E.$$

As this holds for every $k \in \mathbb{N}$ we actually get that

$$a \in T^{-(t_i+\ell s_i+B^{\oplus \ell})} E,$$

for all $i \in \mathbb{N}$ and $1 \leq \ell \leq i$, where $B = (b_n)_{n\in\mathbb{N}}$, which is the statement of Theorem 7.2.3. □

The proof that Theorem 7.5.8 implies Theorem 7.2.2 is very similar and thus is omitted.

7.6 Theorem 7.5.8 via progressive measures.

7.6.1 The measure σ is progressive

In this section we recover some statements analogous to the key intermediate results used in the proof of the main theorem in [45]. These are exactly the tools we use to prove that the measure σ defined above is progressive. While some proofs are quite similar, we unfortunately cannot always cite the needed results from [45], since the measures we study are defined via the infinite-step pronilfactor, whereas the analogous measures therein were defined via finite-step pronilfactors.

Throughout this section, we let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor and $\sigma \in \mathcal{M}(X^{\mathbb{N}_0})$ be the measure defined in Definition 7.5.7. The first result we need is the initialization step of the inductive proofs of Theorem 7.5.8 and Theorem 7.5.9.

Proposition 7.6.1 (Initialization step)**.** *For any $k \in \mathbb{N}$ and any open set $E \subset X$ with $\mu(E) > 0$, there exist infinitely many $n \in \mathbb{N}$ such that*

$$\sigma_k(X \times T_\Delta^n(E \times \cdots \times E)) > 0.$$

A proof of this result relies on an analogue of [45, Theorem 5.2] in our setting. The proof of this is almost identical to the one in [45], except for the justification of

exchanging the order of limits in the averages analogous to (5.3) and (5.4) therein, which in our context is possible via Proposition 7.5.1. However, we can also recover Proposition 7.6.1 from a new key property of our measure σ, that is, multiple right-progressiveness, which is introduced in Section 7.7.1. This self-contained proof of Proposition 7.6.1 can be found in Remark 7.7.5 below.

We use the following result which extends [45, Lemma 6.11] for infinite-step pronilsystems.

Lemma 7.6.2. *Let (X, μ, T) be an ergodic system, $a \in \mathtt{gen}(\mu, \Phi)$ for some Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ and denote by (Z_∞, m, T) the infinite-step pronilfactor of this system. Assume (Y, ν, S) is an infinite-step pronilsystem. Then, for every $g \in C(X)$, $y \in Y$ and $F \in C(Y)$ we have that*

$$\limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} g(T^n a) F(S^n y) \right| \leq \|\mathbb{E}(g|Z_\infty)\|_{L^1(m)} \cdot \|F\|_\infty . \tag{7.6.1}$$

Proof. We first refine $(\Phi_N)_{N\in\mathbb{N}}$ so that it realises the limsup on the LHS of (7.6.1) as a limit and then find a further subsequence $(\Phi_{N_j})_{j\in\mathbb{N}}$ so that (a, y) is generic along (Φ_{N_j}) for some $(T\times S)$-invariant measure ρ on $X \times Y$. If ν is an S-invariant measure on the pronilsystem Y such that $y \in \mathtt{gen}(\nu, \Phi)$, since $a \in \mathtt{gen}(\mu, \Phi)$, we see that the first and second marginals of ρ are μ and ν, respectively. Letting $\tilde{g} = \mathbb{E}(g|Z_\infty)$ we only have to prove that

$$\int_{X\times Y} (g - \tilde{g}) \otimes F \ \mathrm{d}\rho = 0 \tag{7.6.2}$$

Although ρ may not be ergodic, it suffices to prove (7.6.2) assuming ρ is an ergodic measure and then using its ergodic decomposition.

Using Proposition 2.6.2 we have that $\|(g - \tilde{g}) \otimes \mathbb{1}\|_{U^k(X\times Y,\rho,T\times S)} = 0$ for all $k \in \mathbb{N}$. By the same proposition we then see that $(g - \tilde{g}) \otimes \mathbb{1}$ is orthogonal to the infinite-step pronilfactor of $(X \times Y, \rho, T \times S)$. On the other hand, (Y, ν, S) is a factor of $(X \times Y, \rho, T \times S)$, and so $\mathbb{1} \otimes F$ is measurable with respect to the maximal factor of $(X \times Y, \rho, T \times S)$ that is an infinite-step pronilsystem, namely its infinite-step pronilfactor. The functions $(g - \tilde{g}) \otimes \mathbb{1}$ and $\mathbb{1} \otimes F$ are thus orthogonal in $L^2(\rho)$ and (7.6.2) follows. □

Because each σ_k, for $k \geq 2$, is invariant under $Id \times T \times \cdots \times T^k$ and its marginals satisfy $\sigma_{ki} \leq i\mu$, for $i = 1, \ldots, k$, we can apply [45, Theorem 6.6] directly.

Proposition 7.6.3. *Let $k \in \mathbb{N}$, with $k \geq 2$ and let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be Følner sequence in $\mathbb{N}$. Then, for any $f_1, \ldots, f_{k-1} \in L^\infty(\mu)$ and any bounded sequence $b : \mathbb{N} \to \mathbb{C}$ we*

have that

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} b(n)\cdot(\mathbb{1}\otimes T^n f_1\otimes\cdots\otimes T^n f_{k-1}\otimes\mathbb{1})\right\|_{L^2(\sigma_k)}$$
$$\leq C_k\cdot\|b\|_\infty\cdot\min\{\|f_i\|_{U^k(X,\mu,T)}:1\leq i\leq k-1\},$$

where $C_k>0$ is a constant that only depends on k.

This, in combination with Lemma 7.6.2, together imply the analogue of [45, Theorem 6.2] for σ_k, by repeating the proof in [45, Section 6.4] (we also use Proposition 7.5.1 that allows us to interchange the order of analogous limits). Namely, we have the following.

Proposition 7.6.4. *Let $k\in\mathbb{N}$ with $k\geq 2$. For any function $F\in C(X^k)$ we have that*

$$\lim_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(F\otimes\mathbb{1})-\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(\mathbb{1}\otimes F)\right\|_{L^2(\sigma_k)}=0.$$

Finally, [45, Theorem 6.1] also applies for each measure σ_k, $k\in\mathbb{N}$, (more precisely, for the projection of σ_k to its last k-coordinates) because the only assumption there is that the measure is $(T\times T^2\times\cdots\times T^k)-$invariant.

Proposition 7.6.5. *Let $k\in\mathbb{N}$ with $k\geq 2$. Let $F\in C(X^k)$ be a nonnegative function and $\Phi=(\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$. Then*

$$\int\mathbb{1}\otimes F d\sigma_k>0\Longrightarrow\liminf_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int T_\Delta^n(\mathbb{1}\otimes F)\cdot(\mathbb{1}\otimes F)\,\mathrm{d}\sigma_k>0.$$

Remark 7.6.6. In fact, it follows from the proof of [45, Theorem 6.1], and in particular from the use of Furstenberg's multiple recurrence theorem, that in the notation of the proposition, for any $\varepsilon>0$ there is $\delta>0$ so that

$$\int\mathbb{1}\otimes F d\sigma_k>\varepsilon\Longrightarrow\liminf_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int T_\Delta^n(\mathbb{1}\otimes F)\cdot(\mathbb{1}\otimes F)\,\mathrm{d}\sigma_k>\delta.$$

As shown in [45, Section 6.1], these last two results together imply that each σ_k, $k\geq 2$, is left-progressive. Since $P_k\sigma=\sigma_k$, we have shown the following.

Proposition 7.6.7. *The measure σ defined via* (7.5.2) *is left-progressive.*

The proof of Proposition 7.6.7 (this is implicit in [45]) actually gives the following.

Proposition 7.6.8. *Let $F \in C(X^k)$ be a nonnegative function with $\int \mathbb{1} \otimes F \, \mathrm{d}\sigma_k > 0$. Then,*

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int (T_\Delta^n F(a, x_1, \dots, x_{k-1}) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes F(x_1, \dots, x_k)) \, \mathrm{d}\sigma_k(x) > 0.$$

Next we show that σ is right-progressive.

Proposition 7.6.9. *The measure σ defined via (7.5.2) is right-progressive.*

Proof. Let $k \in \mathbb{N}$ with $k \geq 2$ be fixed. Let $U_1, \dots, U_{k-1} \subset X$ be open sets with

$$\sigma_k(X \times U_1 \times U_2 \times \cdots \times U_{k-1} \times X) > 0.$$

As $V = U_1 \times \cdots \times U_{k-1} \subset X^{k-1}$ is an open set, we can find a function $G \in C(X^{k-1})$ such that $0 \leq G \leq \mathbb{1}_V$ and $\int_{X^{k+1}} \mathbb{1} \otimes G \otimes \mathbb{1} \, \mathrm{d}\sigma_k > 0$. We thus have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \sigma_k((X \times V \times X) \cap T_\Delta^{-n}(X \times X \times V)) \\
&\qquad \geq \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^{k+1}} (\mathbb{1} \otimes G \otimes \mathbb{1}) \cdot T_\Delta^n(\mathbb{1} \otimes \mathbb{1} \otimes G) \, \mathrm{d}\sigma_k \\
&\qquad\qquad = \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^{k+1}} (\mathbb{1} \otimes G \otimes \mathbb{1}) \cdot T_\Delta^n(\mathbb{1} \otimes G \otimes \mathbb{1}) \, \mathrm{d}\sigma_k,
\end{aligned}$$

where the equality follows by an application of Proposition 7.6.4 for the continuous function $\mathbb{1} \otimes G \in C(X^k)$ and the measure σ_k. Applying Proposition 7.6.5 for the continuous function $\mathbb{1} \otimes G \otimes \mathbb{1} \in C(X^{k+1})$ it follows that the last expression is positive. Hence, there exist infinitely many $n \in \mathbb{N}$ such that

$$\sigma_k((X \times U_1 \times \cdots \times U_{k-1} \times X) \cap T_\Delta^{-n}(X \times X \times U_1 \times \cdots \times U_{k-1})) > 0$$

and this concludes the proof. □

We remark in passing that the proof of Proposition 7.6.9 actually shows the following result, although we do not need to use it.

Proposition 7.6.10. *Let $F \in C(X^{k-1})$ be a nonnegative function with $\int \mathbb{1} \otimes F \otimes \mathbb{1} \, \mathrm{d}\sigma_k > 0$. Then,*

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int \mathbb{1} \otimes \mathbb{1} \otimes (T_\Delta^n F(x_2, \dots, x_k)) \cdot (\mathbb{1} \otimes F(x_1, \dots, x_{k-1}) \otimes \mathbb{1}) \, \mathrm{d}\sigma_k(x) > 0.$$

As a corollary of Propositions 7.6.7 and 7.6.9 we deduce the following.

Theorem 7.6.11. *The measure σ defined via* (7.5.2) *is progressive.*

7.6.2 Proof of Theorem 7.5.8

We now prove Theorem 7.5.8 using the results of the previous subsection and an inductive argument. We repeat the statement of the theorem for the convenience of the reader.

Theorem 7.5.8. *Let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence, let $a \in \mathtt{gen}(\mu, \Phi)$ and let σ be the measure defined in Definition 7.5.7. Then, for $\ell \in \mathbb{N}$ fixed and any open set $E \subset X$ with $\mu(E) > 0$ there exist sequences $(b_n)_{n\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such*

$$0 < \sigma_{k+\ell+1}\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+\ell+1}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right),$$

for every $k \in \mathbb{N}$, where $B_k = \{b_1, \dots, b_k\}$.

The diagram sketching the proof Theorem 7.5.8 is the following.

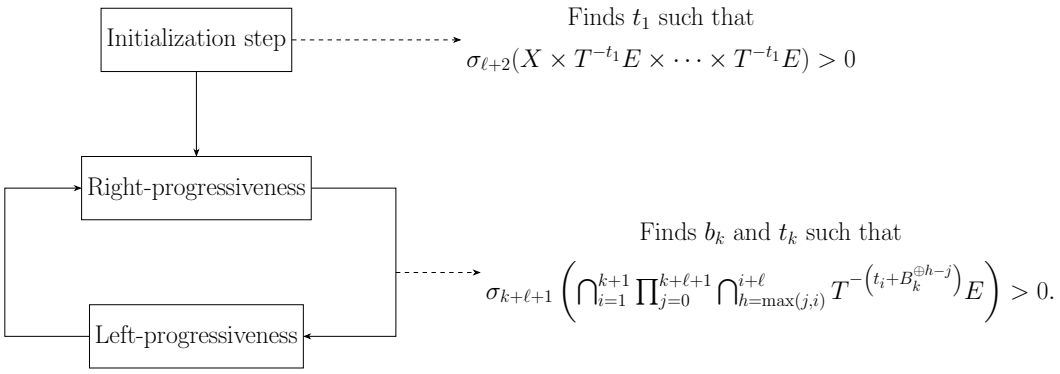


Figure 7.1: Diagram of the proof of Theorem 7.5.8

Proof. By Proposition 7.6.1 we can find $t_1 \in \mathbb{N}$ such that

$$\sigma_{\ell+1}(X \times T^{-t_1}E \times \cdots \times T^{-t_1}E) > 0.$$

As σ is right-progressive we can find t_2' such that if $t_2 = t_2' + t_1$, it holds that

$$\sigma_{\ell+2}((X \times T^{-t_1}E \times \cdots \times T^{-t_1}E \times X) \cap (X \times X \times T^{-t_2}E \times \cdots \times T^{-t_2}E)) > 0.$$

Using left-progressiveness we obtain $b_1 \in \mathbb{N}$ such that the $\sigma_{\ell+2}$ measure of

$$\left(T^{-(t_1+b_1)}E \times \left(T^{-(t_1+b_1)}E \cap T^{-t_1}E\right) \times \cdots \times \left(T^{-(t_1+b_1)}E \cap T^{-t_1}E\right) \times T^{-t_1}E \times X\right)$$
$$\bigcap \left(X \times T^{-(t_2+b_1)}E \times \left(T^{-(t_2+b_1)} \cap T^{-t_2}E\right) \times \cdots \times \left(T^{-(t_2+b_1)}E \cap T^{-t_2}E\right) \times T^{-t_2}E\right)$$

is positive. Recall the notational conventions from Section 2.7 and observe that this can be rewritten as

$$\sigma_{\ell+2}\left(\bigcap_{i=1}^{2}\prod_{j=0}^{\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t_i+B_1^{\oplus h-j}\right)}E\right) > 0,$$

where $B_1 = \{b_1\}$, using for example that $B_1^{\oplus h} = \emptyset$ and so $T^{-(t_2+B_1^{\oplus h})}E = X$, for $h \geq 2$. In this fashion we construct the sequences of the statement inductively.

Assume we have constructed $b_1 < \cdots < b_k$ and $t_1 < \cdots < t_{k+1} \in \mathbb{N}$ such that if $B_k = \{b_1, \ldots, b_k\}$, we have that

$$\sigma_{k+\ell+1}\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+\ell+1}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right) > 0. \tag{7.6.3}$$

We note in particular that

$$\sigma_{k+\ell+1}\left(\prod_{j=0}^{k+\ell+1}\bigcap_{h=\max(j,k+1)}^{k+1+\ell} T^{-\left(t_{k+1}+B_k^{\oplus h-j}\right)}E\right) > 0. \tag{7.6.4}$$

As σ is a right-progressive measure we can apply the defining property in Definition 7.5.2, only considering the set isolated in (7.6.4), in order to find t'_{k+2} such that if $t_{k+2} = t'_{k+2} + t_{k+1}$, then the $\sigma_{k+\ell+2}$ measure of the set

$$\left[\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right)\right] \cap$$
$$\left[X \times \prod_{j=0}^{k+\ell+1}\left(\bigcap_{h=\max(j,k+1)}^{k+1+\ell} T^{-\left(t_{k+2}+B_k^{\oplus h-j}\right)}E\right)\right]$$

is positive. The projection onto the 0-th coordinate of this set equals

$$\bigcap_{i=1}^{k+1}\bigcap_{h=i}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h}\right)} = \bigcap_{i=1}^{k+2}\bigcap_{h=i}^{i+\ell} T^{-\left(t_i+B_k^{\oplus h}\right)}$$

and the projection onto the rest $k+\ell+2$ coordinates becomes

$$\begin{aligned}
&\bigcap_{i=1}^{k+1}\prod_{j=1}^{k+\ell+2}\left(\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\cap\bigcap_{h=\max(j-1,k+1)}^{k+1+\ell}T^{-\left(t_{k+2}+B_k^{\oplus h-j+1}\right)}E\right)\\
&=\bigcap_{i=1}^{k+1}\prod_{j=1}^{k+\ell+2}\left(\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\cap\bigcap_{h=\max(j,k+2)}^{k+2+\ell}T^{-\left(t_{k+2}+B_k^{\oplus h-j}\right)}E\right)\\
&=\bigcap_{i=1}^{k+2}\prod_{j=1}^{k+\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E.
\end{aligned}$$

In total, this gives that

$$\sigma_{k+\ell+2}\left(\bigcap_{i=1}^{k+2}\prod_{j=0}^{k+\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right)>0.$$

By left-progressiveness, we obtain $b_{k+1}>b_k$ such that the $\sigma_{k+\ell+2}$ measure of

$$\begin{aligned}
&\left(\bigcap_{i=1}^{k+2}\prod_{j=0}^{k+\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\right)\cap\left(\bigcap_{i=1}^{k+2}\prod_{j=0}^{k+\ell+2}\bigcap_{h=\max(j+1,i)}^{i+\ell}T^{-\left(t_i+b_{k+1}+B_k^{\oplus h-j-1}\right)}E\right)\\
&=\bigcap_{i=1}^{k+2}\prod_{j=0}^{k+\ell+2}\left(\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_k^{\oplus h-j}\right)}E\cap\bigcap_{h=\max(j+1,i)}^{i+\ell}T^{-\left(t_i+b_{k+1}+B_k^{\oplus h-j-1}\right)}E\right)
\end{aligned}$$

is positive. If we let $B_{k+1}=\{b_1,\dots,b_k,b_{k+1}\}$, we have thus established that

$$0<\sigma_{k+\ell+2}\left(\bigcap_{i=1}^{k+2}\prod_{j=0}^{k+\ell+2}\bigcap_{h=\max(j,i)}^{i+\ell}T^{-\left(t_i+B_{k+1}^{\oplus h-j}\right)}E\right),$$

concluding the induction. □

7.7 Theorem 7.5.9 via multiple right-progressive measures

7.7.1 The measure σ is multiple right-progressive

For a metric space X, a homeomorphism $T\colon X\to X$ and a vector $u=(u_1,\dots,u_k)\in\mathbb{N}^k$ we write $T^u=T^{u_1}\times\cdots\times T^{u_k}$. Based on Furstenberg-Katznelson's multiple recurrence theorem [20, Theorem A], we extend [45, Theorem 6.3] to a multiple recurrence statement.

Theorem 7.7.1. *Let X be a compact metric space, let $T_1, \dots, T_\ell \colon X \to X$ be commuting homeomorphisms for some $\ell \in \mathbb{N}$ and $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence. Let also $k \in \mathbb{N}$, $v_1, \dots, v_\ell \in \mathbb{N}^k$ and $\nu \in (X^k)$ be a Borel probability measure invariant under $T_j^{v_j}$, for $j = 1, \dots, \ell$. For every measurable set $A = A_1 \times \cdots \times A_k \subset X^k$ with $\nu(A) > 0$, and every $u_1, \dots, u_\ell \in \mathbb{N}^k$, it holds that*

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \nu(A \cap (T_1^{u_1})^{-n} A \cap \cdots \cap (T_\ell^{u_\ell})^{-n} A) > 0 \tag{7.7.1}$$

Remark. In this work we are merely concerned with the case $T_1 = \cdots = T_\ell = T$ for a fixed $T \colon X \to X$, $v_i = (1, 2, \dots, k)$ and $u_i = (i, \dots, i) \in \mathbb{N}^k$ for all $i = 1, \dots, \ell$. But we believe the general statement is of independent interest. Notice that when $v_i = u_i$ for all $i = 1, \dots, \ell$, the statement is a special case of Furstenberg-Katznelson's multiple recurrence theorem [20]. We also remark that the assumption that A is a product set is necessary, as shown in [45, Example 6.4] for the single recurrence statement proven there.

Proof. This proof is similar to the one of [45, Theorem 6.3]. We let $v_i = (v_{i,1}, \dots, v_{i,k}) \in \mathbb{N}^k$ and $u_i = (u_{i,1}, \dots, u_{i,k}) \in \mathbb{N}^k$, for $i = 1, \dots, \ell$, be vectors as in the statement of the theorem.

Let $c = LCM(v_{i,j} \colon 1 \le i \le \ell, 1 \le j \le k)$ be the least common multiple of the coordinates of $v_1, \dots, v_\ell$ and set $w_{i,j} = c\, \dfrac{u_{i,j}}{v_{i,j}} \in \mathbb{N}$, for $1 \le i \le \ell$ and $1 \le j \le k$. Observe that

$$(T_i^{v_i})^{-w_{i,j}}(X^{j-1} \times A_j \times X^{k-j}) = (T_i^{u_i})^{-c}(X^{j-1} \times A_j \times X^{k-j}),$$

for all $i = 1, \dots, \ell$ and $j = 1, \dots, k$. Thus, for any $n \in \mathbb{N}$ we see that

$$\begin{aligned}(T_i^{u_i})^{-cn} A &= \bigcap_{j=1}^{k} (T_i^{u_i})^{-cn}(X^{j-1} \times A_j \times X^{k-j}) \\ &= \bigcap_{j=1}^{k} (T_i^{v_i})^{-w_{i,j}n}(X^{j-1} \times A_j \times X^{k-j}) \supset \bigcap_{j=1}^{k} (T_i^{v_i})^{-w_{i,j}n} A.\end{aligned}$$

For the Følner sequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ given by $\Psi_N = \Phi_N / c = \{n \in \mathbb{N} \colon cn \in \Phi_N\}$,

we have

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\nu(A\cap(T_1^{u_1})^{-n}A\cap\cdots\cap(T_\ell^{u_\ell})^{-n}A)\\ \geq\liminf_{N\to\infty}\frac{1}{c|\Psi_N|}\sum_{n\in\Psi_N}\nu(A\cap(T_1^{u_1})^{-cn}A\cap\cdots\cap(T_\ell^{u_\ell})^{-cn}A)\\ \geq\liminf_{N\to\infty}\frac{1}{c|\Psi_N|}\sum_{n\in\Psi_N}\nu\Big(A\cap\Big(\bigcap_{j=1}^{k}(T_1^{v_1})^{-w_{1,j}n}A\Big)\cap\cdots\cap\Big(\bigcap_{j=1}^{k}(T_\ell^{v_\ell})^{-w_{\ell,j}n}A\Big)\Big).\end{aligned}$$

Since ν is invariant under $T_1^{v_1},\dots,T_\ell^{v_\ell}$ (and therefore by $(T_i^{v_i})^{w_{i,j}}$ for all $i=1,\dots,\ell$ and $j=1,\dots,k$), the last average is positive by Furstenberg-Katznelson's multiple recurrence theorem, so we conclude. □

Remark. Notice that if we consider the special case where $T_1=\cdots=T_\ell=T$ for a fixed $T\colon X\to X$ in the previous proof, it suffices to apply Furstenberg's multiple recurrence.

Throughout this section, we let (X,μ,T) be an ergodic system with topological infinite-step pronilfactor (Z_∞,m,T), and $\sigma\in\mathcal{M}(X^{\mathbb{N}_0})$ be the measure defined in Definition 7.5.7 for some generic point $a\in\mathtt{gen}(\mu,\Phi)$. We use Theorem 7.7.1 to obtain the following result.

Corollary 7.7.2. *Let $k,\ell\in\mathbb{N}$ with $k\geq 2$. Let $f_1,\dots,f_k\in C(X)$ be nonnegative functions and $\Phi=(\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$. If $F=\mathbb{1}\otimes f_1\otimes\cdots\otimes f_k$ and*

$$\int F\,\mathrm{d}\sigma_k>0,$$

then we have that

$$\liminf_{M\to\infty}\liminf_{N\to\infty}\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int F\cdot\prod_{i=1}^{\ell}T_\Delta^{n+im}F\,\mathrm{d}\sigma_k>0.$$

Proof. Denote $F^*=f_1\otimes\cdots\otimes f_k$. Letting $\mathtt{supp}(f_i)=\overline{\{x:f_i(x)>0\}}$, for $i=1,\dots,k$, and setting $A=\mathtt{supp}(f_1)\times\cdots\times\mathtt{supp}(f_k)$, it follows by the assumption that $\int F^*\,\mathrm{d}P_k^*\sigma>0$ and hence that $P_k^*\sigma(A)>0$. This allows us to apply Theorem 7.7.1 for the set $A\subset X^k$ defined above and the measure $P_k^*\sigma$, which is invariant under $T\times T^2\times\cdots\times T^k$. In particular, with $k,\ell\in\mathbb{N}$ as above and $u_i=(i,\dots,i)$ for each $i\in[\ell]$, and $T_1=\cdots=T_\ell=T$, we obtain

$$\liminf_{M\to\infty}\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}P_k^*\sigma(A\cap T_\Delta^{-m}A\cap\cdots\cap T_\Delta^{-\ell m}A)>0.$$

As $f_1, \dots, f_k$ are continuous functions, this implies that

$$\liminf_{M\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \int F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{im} F^* \, \mathrm{d}P_k^* \sigma > 0.$$

Equivalently, for some $\varepsilon > 0$, the set

$$P_\varepsilon = \{m \in \mathbb{N} : \int F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{im} F^* \, \mathrm{d}P_k^* \sigma > \varepsilon\}$$

is such that $\underline{\mathrm{d}}_\Phi(P_\varepsilon) > 0$. Then, it follows by Proposition 7.6.5 (see also Remark 7.6.6) that there exists $\varepsilon' > 0$ such that for each $m \in P_\varepsilon$,

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{im} F^* \cdot T_\Delta^{n} F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{n+im} F^* \, \mathrm{d}P_k^* \sigma > \varepsilon',$$

which in turn implies that

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{n+im} F^* \, \mathrm{d}P_k^* \sigma > \varepsilon'',$$

for some $\varepsilon'' > 0$. As $\underline{\mathrm{d}}_\Phi(P_\varepsilon) > 0$ we conclude that

$$\liminf_{M\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int F^* \cdot \prod_{i=1}^{\ell} T_\Delta^{n+im} F^* \, \mathrm{d}P_k^* \sigma > 0,$$

which implies the conclusion of the theorem. □

The main result of this section is the following proposition which is an indirect consequence of Theorem 7.7.1.

Proposition 7.7.3. *Let $k, \ell \in \mathbb{N}$ with $k \geq 2$. Let $f_1, \dots, f_k \in C(X)$ be nonnegative functions and $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$. If $F = \mathbb{1} \otimes f_1 \otimes \cdots \otimes f_k$ and*

$$\int F d\sigma_k > 0,$$

then we have that

$$\liminf_{M\to\infty} \liminf_{N\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int (F \otimes \mathbb{1}^{\otimes\ell}) \cdot \prod_{i=1}^{\ell} (T_\Delta^{n+im} (\mathbb{1}^{\otimes i} \otimes F \otimes \mathbb{1}^{\otimes \ell - i})) \, \mathrm{d}\sigma_{k+\ell} > 0.$$

Observe that for $\ell = 1$ this implies Proposition 7.6.10. This result allows us to

utilize the following property that we naturally term multiple right-progressiveness.

Definition 7.7.4. Let (X, T) be a topological system. We say that a Borel probability measure $\tau \in \mathcal{M}(X^{\mathbb{N}_0})$ is *multiple right-progressive* if for each $k, \ell \in \mathbb{N}$ and any open sets $U_1, \dots, U_k \subset X$ with

$$\tau_k(X \times U_1 \times \cdots \times U_k) > 0,$$

there exist infinitely many $n, m \in \mathbb{N}$ such that

$$\tau_{k+\ell}\left((X \times U \times X^{\ell}) \cap T_{\Delta}^{-(n+m)}(X^2 \times U \times X^{\ell-1}) \cap \cdots \cap T_{\Delta}^{-(n+\ell m)}(X^{\ell+1} \times U)\right) > 0,$$

where $U = U_1 \times \cdots \times U_k$ and τ_k and $\tau_{k+\ell}$ are the projections onto the first $(k+1)$ and $(k+\ell+1)$ coordinates of τ, respectively.

Remark. We want to emphasize that Proposition 7.7.3 implies that σ is multiple right-progressive. Moreover, note that for $\ell = 1$, this property coincides with right-progressiveness.

Remark 7.7.5. The initialization step, i.e. Proposition 7.6.1, for the proofs of Theorem 7.5.8 and Theorem 7.5.9, can be obtained as a consequence of multiple right-progressiveness and left-progressiveness. Indeed, by multiple right-progressiveness for the case $k = 1$, $\ell \in \mathbb{N}$ and $U_1 = E$ an open set with $\mu(E) > 0$, we obtain infinitely many $n, m \in \mathbb{N}$ such that

$$\sigma_{\ell+1}(X \times E \times T^{-(n+m)}E \times T^{-(n+2m)}E \times \cdots \times T^{-(n+\ell m)}E) > 0.$$

Disregarding the second coordinate and using left-progressiveness, we find $\tilde{n}$ with

$$\begin{aligned} 0 &< \sigma_{\ell}(X \times T^{-(\tilde{n}+m)}E \times T^{-(\tilde{n}+2m)}E \times \cdots \times T^{-(\tilde{n}+\ell m)}E) \\ &= \sigma_{\ell}((I_d \times \tilde{T})^{-m}(X \times T^{-\tilde{n}}E \times T^{-\tilde{n}}E \times \cdots \times T^{-\tilde{n}}E)) \\ &= \sigma_{\ell}(X \times T_{\Delta}^{-\tilde{n}}(E \times \cdots \times E)), \end{aligned}$$

obtaining the conclusion of Proposition 7.6.1.

The proof of Proposition 7.7.3 for infinite-step pronilsystems is straightforward from the definition of σ. In fact, in this special case we do not need to average along the transformation T_{Δ}^{n}, and the result is a corollary of Theorem 7.7.1. Our goal then is to show that the infinite-step pronilfactor is characteristic. To prove this, it would be enough to show the related ergodic averages are controlled by uniformity seminorms. As we show in Appendix B.0.2 these seminorms do not always control

the single average, hence the need to introduce a second averaging over T_Δ^n. However, for the double averages in Proposition 7.7.3 we have the following.

Lemma 7.7.6. *Let (X,μ,T) be an ergodic system, $k,\ell\in\mathbb{N}$ with $k\geq 2$, let $\tau\in(X^k)$ be a $(T\times T^2\times\cdots\times T^k)$-invariant measure with marginals $\tau_i\leq C\mu$ for some $C>0$. Moreover, let $f_1,\dots,f_k\in C(X)$ and $\Phi=(\Phi_N)_{N\in\mathbb{N}}$ a Følner sequence in $\mathbb{N}$. If $F=f_1\otimes\cdots\otimes f_k$, then*

$$\limsup_{M\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\prod_{i=1}^{\ell}T_\Delta^{n+im}F\right\|_{L^2(\tau)}\leq C_{k,\ell}\min_{1\leq j\leq k}\{\|f_j\|_{U^{k\cdot\ell}(X,\mu,T)}\},$$

where $C_{k,\ell}>0$ is a constant that only depends on k and ℓ.

Proof. Let $c=LCM(2,\dots,k)$. Considering the Følner sequence $\Psi=(\Psi_N)_{N\in\mathbb{N}}$ given by

$$\Psi_N=\Phi_N/c=\{n:cn\in\Phi_N\},\text{ for }N\in\mathbb{N},$$

and applying [45, Lemma 6.8], we get that

$$\limsup_{M\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\prod_{i=1}^{\ell}T_\Delta^{n+im}F\right\|_{L^2(\tau)}\leq$$
$$\frac{1}{c^2}\sum_{r,r'=0}^{c-1}\limsup_{M\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\left(\prod_{j=1}^{k}\prod_{i=1}^{\ell}(T^{cn+icm+r+ir'}f_j)^{[j]}\right)\right\|_{L^2(\tau)},$$

where for a function $f\in C(X)$ we denote $f^{[j]}(x_1,\dots,x_k)=f(x_j)$, for $(x_1,\dots,x_k)\in X^k$. Fixing $r,r'\in\{0,\dots,c-1\}$ and letting $S=T\times T^2\times\cdots\times T^k$, and $c_j=c/j$ for $j=1,\dots,k$, we get that

$$\limsup_{M\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\left(\prod_{j=1}^{k}\prod_{i=1}^{\ell}(T^{cn+icm+r+ir'}f_j)^{[j]}\right)\right\|_{L^2(\tau)}$$
$$=\limsup_{M\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\left(\prod_{j=1}^{k}S^{c_jn}\left(\prod_{i=1}^{\ell}S^{ic_jm}(T^{r+ir'}f_j)^{[j]}\right)\right)\right\|_{L^2(\tau)}.$$

By [63, Theorem 1.1], the previous double limsup is actually a limit, and it is equal

to

$$\lim_{N\to\infty}\lim_{M\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\left(\prod_{j=1}^{k}S^{c_jn}\left(\prod_{i=1}^{\ell}S^{ic_jm}(T^{r+ir'}f_j)^{[j]}\right)\right)\right\|_{L^2(\tau)}. \tag{7.7.2}$$

Notice that for an integer $u > kc$, the function $(i,j) \in \{1,\dots,\ell\}\times\{1,\dots,k\} \mapsto uc_j + ic_j$ is injective. For a fixed such $u \in \mathbb{N}$, making the change of variables $n \mapsto n + um$, we obtain that the quantity in (7.7.2) becomes

$$\begin{aligned}
&\lim_{N\to\infty}\lim_{M\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\left(\prod_{j=1}^{k}S^{c_jn}\left(\prod_{i=1}^{\ell}S^{(uc_j+ic_j)m}(T^{r+ir'}f_j)^{[j]}\right)\right)\right\|_{L^2(\tau)}\\
&\leq \lim_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\lim_{M\to\infty}\left\|\frac{1}{|\Psi_M|}\sum_{m\in\Psi_M}\left(\prod_{j=1}^{k}S^{c_jn}\left(\prod_{i=1}^{\ell}S^{(uc_j+ic_j)m}(T^{r+ir'}f_j)^{[j]}\right)\right)\right\|_{L^2(\tau)}\\
&\leq C'_{k,\ell}\lim_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}\min\left\{\left\|S^{c_jn}(T^{r+ir'}f_j)^{[j]}\right\|_{U^{k\cdot\ell}(X^k,\tau,S)} : 1\leq j\leq k, 1\leq i\leq\ell\right\}\\
&= C'_{k,\ell}\min\left\{\left\|(T^{r+ir'}f_j)^{[j]}\right\|_{U^{k\cdot\ell}(X^k,\tau,S)} : 1\leq j\leq k, 1\leq i\leq\ell\right\},
\end{aligned}$$

where the last inequality and the constant $C'_{k,\ell} > 0$ come from [45, Lemma 6.7 (iii)]. Finally, by [45, Lemma 6.7(iv)], there exists a constant $C''_{k,\ell} > 0$ such that

$$\left\|(T^{r+ir'}f_j)^{[j]}\right\|_{U^{k\cdot\ell}(X^k,\tau,S)} = \left\|T^{r+ir'}f_j\right\|_{U^{k\cdot\ell}(X,\tau_j,T^j)} \leq C''_{k,\ell}\,\|f_j\|_{U^{k\cdot\ell}(X,\mu,T)}$$

and so we conclude the result by defining $C_{k,\ell} = C'_{k,\ell}\cdot C''_{k,\ell} > 0$. □

The fact that we can project to the infinite-step pronilfactor allows us to prove the following result.

Lemma 7.7.7. *If $k,\ell \in \mathbb{N}$ with $k \geq 2$, $F \in C(X^k)$ and $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence, then*

$$\begin{aligned}
\lim_{M\to\infty}\limsup_{N\to\infty}\Bigg\|&\frac{1}{|\Phi_M|}\sum_{m\in\Phi_N}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\prod_{i=1}^{\ell}T_\Delta^{n+im}(\mathbb{1}\otimes F\otimes\mathbb{1}^\ell)-\\
&\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\prod_{i=1}^{\ell}T_\Delta^{n+im}(\mathbb{1}^{1+i}\otimes F\otimes\mathbb{1}^{\ell-i})\Bigg\|_{L^2(\sigma_{k+\ell})} = 0.
\end{aligned}$$

Proof. We follow the proof of [45, Theorem 6.2]. By an approximation argument, we can assume $F = f_1\otimes\cdots\otimes f_k$ for $f_1,\dots,f_k \in C(X)$. Set $G = 1\otimes g_1\otimes\cdots\otimes g_k$

where $g_i = \mathbb{E}(f_i|Z_\infty)$ for each $i \in [k]$. Using Lemma 7.7.6 and approximating g_i by continuous functions since the marginals of ξ satisfy $\xi_i \leq im_{Z_\infty}$, we see that it is enough to prove the statement in Z_∞ with $F = f_1 \otimes \cdots \otimes f_k$ for $f_1, \ldots, f_k \in C(Z_\infty)$ and $a \in Z_\infty$. Using these reductions, by unique ergodicity of orbit closures in a pronilsystem, and the definition of $\xi_{k+\ell}$ it suffices therefore to show that

$$\left\| \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \left[\prod_{i=1}^{\ell} T_\Delta^{n+im}(F \otimes \mathbb{1}^\ell) - \prod_{i=1}^{\ell} T_\Delta^{n+im}(\mathbb{1}^i \otimes F \otimes \mathbb{1}^{\ell-i}) \right] \right\|_{L^2(\xi_{k+\ell})}^2$$

vanishes as $N, M \to \infty$. The last expression becomes

$$\begin{aligned}
&\lim_{J\to\infty} \frac{1}{|\Phi_J|} \sum_{j \in \Phi_J} \left| \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n+im} F(T^j a, \ldots, T^{kj} a) \right. \\
&\left. - \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n+im} F(T^{(i+1)j} a, \ldots, T^{(i+k)j} a) \right|^2 \\
&= \lim_{J\to\infty} \frac{1}{|\Phi_J|} \sum_{j \in \Phi_J} \left| \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{j+im} a, \ldots, T^{kj+im} a) \right. \\
&\left. - \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{(i+1)j+im} a, \ldots, T^{(i+k)j+im} a) \right|^2
\end{aligned}$$

We are thus left with showing that

$$\begin{aligned}
\lim_{N,M\to\infty} \lim_{J\to\infty} \frac{1}{|\Phi_J|} \sum_{j \in \Phi_J} &\left| \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{j+im} a, \ldots, T^{kj+im} a) \right. \\
&\left. - \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{(i+1)j+im} a, \ldots, T^{(i+k)j+im} a) \right|^2 = 0.
\end{aligned}$$

Arguing as in Proposition 7.5.1, the unique ergodicity allows to exchange the order of the limits, so that the above becomes

$$\begin{aligned}
\lim_{N,J\to\infty} \lim_{M\to\infty} \frac{1}{|\Phi_J|} \sum_{j \in \Phi_J} &\left| \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{j+im} a, \ldots, T^{kj+im} a) \right. \\
&\left. - \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{\ell} T_\Delta^{n} F(T^{(i+1)j+im} a, \ldots, T^{(i+k)j+im} a) \right|^2 = 0.
\end{aligned}$$

Making the change of variables $m \mapsto m - j$ allows us to conclude, because

$$\left| \sum_{m\in\Phi_M} \prod_{i=1}^{\ell} T_\Delta^n F(T^{j+im}a, \dots, T^{kj+im}a) - \sum_{m\in\Phi_M+j} \prod_{i=1}^{\ell} T_\Delta^n F(T^{j+im}a, \dots, T^{kj+im}a) \right|^2,$$

equals $\mathrm{o}_{M\to\infty}(|\Phi_M|)$, for any $n, j \in \mathbb{N}$, by the asymptotic invariance of Følner sequences. □

We are finally in a position to prove Proposition 7.7.3.

Proof of Proposition 7.7.3. We want to show that

$$\liminf_{M\to\infty} \liminf_{N\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int (F\otimes \mathbb{1}^{\otimes\ell}) \cdot \prod_{i=1}^{\ell} (T_\Delta^{n+im}(\mathbb{1}^{\otimes i}\otimes F\otimes \mathbb{1}^{\otimes\ell-i}))\, \mathrm{d}\sigma_{k+\ell}$$

is positive. We can apply Lemma 7.7.7 and the above averages become

$$\liminf_{M\to\infty} \liminf_{N\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int (F\otimes \mathbb{1}^{\otimes\ell}) \cdot \prod_{i=1}^{\ell} (T_\Delta^{n+im}(F\otimes \mathbb{1}^{\otimes\ell}))\, \mathrm{d}\sigma_{k+\ell}.$$

Note that we can rewrite these averages as

$$\liminf_{M\to\infty} \liminf_{N\to\infty} \frac{1}{|\Phi_M|} \sum_{m\in\Phi_M} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int F \cdot \prod_{i=1}^{\ell} T_\Delta^{n+im} F\, \mathrm{d}\sigma_k$$

and the latter expression is positive by Corollary 7.7.2. □

7.7.2 Proof of Theorem 7.5.9

Before presenting its proof, we restate Theorem 7.5.9 for the convenience of the reader.

Theorem 7.5.9. *Let (X, μ, T) be an ergodic system with topological infinite-step pronilfactor. Let $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence, let $a \in \mathtt{gen}(\mu, \Phi)$ and let σ be the measure defined in Definition 7.5.7. Then, for every open set $E \subset X$ with $\mu(E) > 0$ there exist sequences $(b_n)_{n\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$\sigma_{k+1}\left(\bigcap_{i=1}^{k} \prod_{j=0}^{k+1} \bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus\ell-j}\right)} E \right) > 0 \tag{7.7.3}$$

for every $k \in \mathbb{N}$, where $B_k = \{b_1, \dots, b_k\}$.

The proof of Theorem 7.5.9 can be summarized in the following diagram.

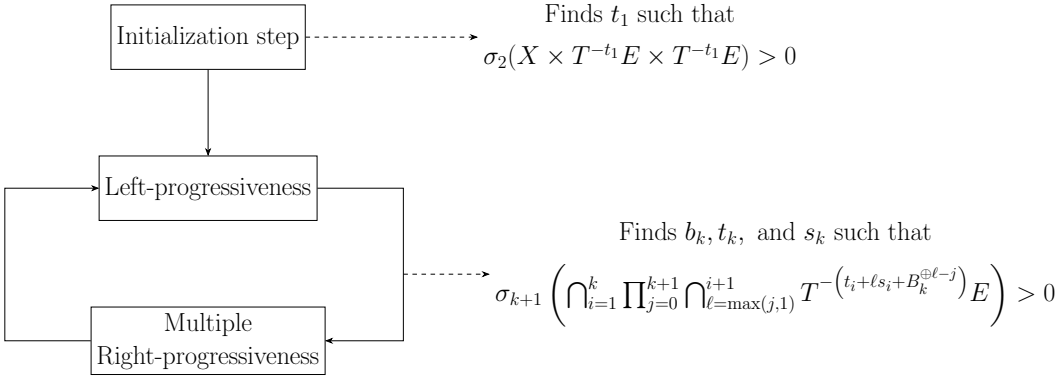


Figure 7.2: Diagram of the proof of Theorem 7.5.9

Proof. Recall that σ is progressive (see Theorem 7.6.11) as in Definition 7.5.3. Take $s_1 = 0$. As $\mu(E) > 0$, by Proposition 7.6.1, there is $t_1 \in \mathbb{N}$ such that

$$\sigma_2(X \times T^{-t_1}E \times T^{-t_1}E) > 0.$$

By left-progressiveness of σ_2, there is $b_1 \in \mathbb{N}$ such that

$$\sigma_2(T^{-(t_1+b_1)}E \times (T^{-(t_1+b_1)}E \cap T^{-t_1}E) \times T^{-t_1}E) > 0.$$

If we set $B_1 = \{b_1\}$, the previous is equivalent to

$$\sigma_2\left(\bigcap_{i=1}^{1}\prod_{j=0}^{2}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_1^{\oplus \ell-j}\right)}E\right) > 0,$$

in accordance with the notation in (7.7.3). If we let $\tilde{E}_1 = X \times T^{-(t_1+b_1)}E \times T^{-t_1}E$ and $E_1 = T^{-(t_1+b_1)}E \times (T^{-(t_1+b_1)}E \cap T^{-t_1}E) \times T^{-t_1}E$, then we can use the multiple right-progressiveness of σ to find $s_2 > s_1$ and $t' \in \mathbb{N}$ such that

$$\sigma_5((E_1 \times X^3) \cap T_\Delta^{-(t'+s_2)}(X \times \tilde{E}_1 \times X^2) \cap T_\Delta^{-(t'+2s_2)}(X^2 \times \tilde{E}_1 \times X) \cap T_\Delta^{-(t'+3s_2)}(X^3 \times \tilde{E}_1)) > 0.$$

If we let $\tilde{t}_2 = t' + t_1$, observe that the set

$$T_\Delta^{-(t'+s_2)}(X \times \tilde{E}_1 \times X^2) \cap T_\Delta^{-(t'+2s_2)}(X^2 \times \tilde{E}_1 \times X) \cap T_\Delta^{-(t'+3s_2)}(X^3 \times \tilde{E}_1)$$

is actually equal to the set

$$(X^2 \times T^{-(\tilde{t}_2+s_2+b_1)}E \times T^{-(\tilde{t}_2+s_2)}E \times X^2) \cap (X^3 \times T^{-(\tilde{t}_2+2s_2+b_1)}E \times T^{-(\tilde{t}_2+2s_2)}E \times X)$$
$$\bigcap (X^4 \times T^{-(\tilde{t}_2+3s_2+b_1)}E \times T^{-(\tilde{t}_2+3s_2)}E).$$

Thus, we have that the σ_5 measure of the intersection of $(E_1 \times X^3)$ with the set

$$X^2 \times T^{-(\tilde{t}_2+s_2+b_1)}E \times (T^{-(\tilde{t}_2+2s_2+b_1)}E \cap T^{-(\tilde{t}_2+s_2)}E) \times$$
$$(T^{-(\tilde{t}_2+3s_2+b_1)}E \cap T^{-(\tilde{t}_2+2s_2)}E) \times T^{-(\tilde{t}_2+3s_2)}E$$

is positive. By left-progressiveness (applied twice, only to the latter product set), we can find t_2' so that if $t_2 = t_2' + \tilde{t}_2$, then the σ_3 measure of the intersection of $(E_1 \times X)$ with

$$T^{-(t_2+s_2+b_1)}E \times (T^{-(t_2+2s_2+b_1)}E \cap T^{-(t_2+s_2)}E) \times (T^{-(t_2+3s_2+b_1)}E \cap T^{-(t_2+2s_2)}E) \times T^{-(t_2+3s_2)}E$$

is positive. Applying left-progressiveness once again to the last intersection and using the definition of E_1, we find $b_2 > b_1$ such that the σ_3 measure of

$$\begin{aligned}
&((T^{-(t_1+b_1)}E \cap T^{-(t_1+b_2)}E \cap T^{-(t_1+b_2+b_1)}E) \times \\
&(T^{-(t_1+b_2)}E \cap T^{-(t_1+b_1)}E \cap T^{-t_1}E) \times T^{-t_1}E \times X) \cap \\
&((T^{-(t_2+s_2+b_1)}E \cap T^{-(t_2+2s_2+b_1+b_2)}E \cap T^{-(t_2+s_2+b_2)}E) \times \\
&(T^{-(t_2+2s_2+b_1)}E \cap T^{-(t_2+s_2)}E \cap T^{-(t_2+3s_2+b_1+b_2)}E \cap T^{-(t_2+2s_2+b_2)}E) \times \\
&(T^{-(t_2+3s_2+b_1)}E \cap T^{-(t_2+2s_2)}E \cap T^{-(t_2+3s_2+b_2)}E) \times T^{-(t_2+3s_2)}E\Big)
\end{aligned}$$

is positive. If we let $B_2 = \{b_1, b_2\}$, the previous is equivalent to the following:

$$\sigma_3\left(\bigcap_{i=1}^{2}\prod_{j=0}^{3}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_2^{\oplus \ell-j}\right)}E\right) > 0.$$

In this fashion we construct the sequences of the statement inductively. Assume that we have created $b_1 < \cdots < b_k$, $t_1 < \cdots < t_k$ and $s_1 < \cdots < s_k$ such that

$$\sigma_{k+1}\left(\bigcap_{i=1}^{k}\prod_{j=0}^{k+1}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\right) > 0, \tag{7.7.4}$$

where $B_k = \{b_1, \dots, b_k\}$. Call

$$E_k = \prod_{j=0}^{k+1} T^{-\left(t_k+(k+1)s_k+B_k^{\oplus k+1-j}\right)} E = \prod_{j=0}^{k+1} T^{-\left(\tilde{t}_k+B_k^{\oplus k+1-j}\right)} E,$$

where $\tilde{t}_k = t_k + (k+1)s_k$. Notice that

$$E_k \supset \prod_{j=0}^{k+1} \bigcap_{\ell=\max(j,1)}^{k+1} T^{-\left(t_k+\ell s_k+B_k^{\oplus \ell-j}\right)} E \supset \bigcap_{i=1}^{k} \prod_{j=0}^{k+1} \bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)} E.$$

Using multiple right-progressiveness in (7.7.4) with E_k solely, we find $s_{k+1} > s_k$ and $t' \in \mathbb{N}$ such that the σ_{2k+3} measure of the intersection of

$$\left(\bigcap_{i=1}^{k} \prod_{j=0}^{k+1} \bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)} E \right) \times X^{k+2}$$

with

$$T_\Delta^{-(t'+s_{k+1})}(X \times E_k \times X^{k+1}) \cap \cdots \cap T_\Delta^{-(t'+(k+2)s_{k+1})}(X^{k+2} \times E_k)$$

is positive. We notice that the set

$$X^{k+1} \times \prod_{j=0}^{k+2} \bigcap_{\ell=\max(j,1)}^{k+2} T^{-\left(\tilde{t}_k+t'+\ell s_{k+1}+B_k^{\oplus \ell-j}\right)} E$$

contains the set

$$T_\Delta^{-(t'+s_{k+1})}(X \times E_k \times X^{k+1}) \cap \cdots \cap T_\Delta^{-(t'+(k+2)s_{k+1})}(X^{k+2} \times E_k). \tag{7.7.5}$$

as the set in the $(k+2+j)$-th coordinate of the set in (7.7.5) is

$$\bigcap_{\ell=\max(j,1)}^{k+2} T^{-\left(\tilde{t}_k+t'+\ell s_{k+1}+B_k^{\oplus \ell-j}\right)} E,$$

for each $j \in \{0, \dots, k+2\}$. To see this, notice, for example, that the $(k+2)$-th coordinate of the set in (7.7.5) is given by intersecting the $(k+2-\ell)$-th coordinates of E_k under $T^{(-t'+\ell s_{k+1})}$ for $\ell = 1, \dots, k+1$, and so it equals

$$\bigcap_{\ell=1}^{k+1} T^{-(t'+\ell s_{k+1})} T^{-\left(\tilde{t_k}+B_k^{\oplus k+1-(k+2-\ell-1)}\right)} E = \bigcap_{\ell=1}^{k+2} T^{-\left(\tilde{t_k}+t'+\ell s_{k+1}+B_k^{\oplus \ell}\right)} E,$$

since $B_k^{k+2} = \emptyset$.

Hence, by left-progressiveness applied $k+1$ times and disregarding the last coordinates, we can find $t_{k+1} > t_k$ (which groups each $\tilde{t}_k, t'$ and the shift given by the $(k+1)$ applications of left-progressiveness) such that the σ_{k+2} measure of

$$\left(\bigcap_{i=1}^{k}\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\right)\cap\left(\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j,1)}^{k+2} T^{-\left(t_{k+1}+\ell s_{k+1}+B_k^{\oplus \ell-j}\right)}E\right)$$
$$=\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\right)$$

is positive. Now, by left-progressiveness once again, we obtain $b_{k+1} > b_k$ such that the σ_{k+2} measure of

$$\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\right)\cap$$
$$\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j+1,1)}^{i+1} T^{-\left(t_i+\ell s_i+b_{k+1}+B_k^{\oplus \ell-j-1}\right)}E\right)$$
$$=\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+2}\left(\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\cap\bigcap_{\ell=j+1}^{i+1} T^{-\left(t_i+\ell s_i+b_{k+1}+B_k^{\oplus \ell-j-1}\right)}E\right)\right)$$

is positive. If we set $B_{k+1} = B_k \cup \{b_{k+1}\}$, we then have that

$$0 < \sigma_{k+2}\left(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+2}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_{k+1}^{\oplus \ell-j}\right)}E\right)$$

concluding the induction. We have constructed sequences $(b_n)_{n\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$ such that (7.7.3) holds for each $k \in \mathbb{N}$ and thus the proof is complete. □

It is possible to find a set $B \subset \mathbb{N}$ that satisfies the statements of both Theorem 1.2.3 and Theorem 1.2.4 at the same time. As Fig. 7.1 and Fig. 7.2 show, the proofs of these theorems only differ in the iterative application of multiple right-progressiveness to find the shifts t_k, s_k for Theorem 1.2.4, and right-progressiveness to find the shift t_k for Theorem 1.2.3. Thus, we can use both multiple right-progressiveness and right-progressiveness, in addition to left-progressiveness, to have,

in each iteration, three shifts t_k, t'_k and s_k such that

$$\sigma_{k+\ell+1}\Bigg[\Bigg(\bigcap_{i=1}^{k+1}\prod_{j=0}^{k+\ell+1}\bigcap_{h=\max(j,i)}^{i+\ell} T^{-\left(t'_i+B_k^{\oplus h-j}\right)}E\Bigg)\cap \\ \Bigg(\bigcap_{i=1}^{k}\Bigg(\prod_{j=0}^{k+1}\bigcap_{\ell=\max(j,1)}^{i+1} T^{-\left(t_i+\ell s_i+B_k^{\oplus \ell-j}\right)}E\Bigg)\times X^{\ell}\Bigg)\Bigg] > 0,$$

which would imply that the same infinite set $B \subset \mathbb{N}$ satisfies the conclusions of Theorems 1.2.4 and 1.2.3 simultaneously. Hence, we obtain the following result.

Proposition 7.7.8. *Let $A \subset \mathbb{N}$ be a set with $\mathrm{d}^*(A) > 0$ and $\ell \geq 0$ an integer. Then, there exist an infinite set $B \subset \mathbb{N}$ and sequences $(\tilde{t}_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$, $(s_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$\Bigg\{\sum_{n\in F} n\colon\ F \subset B \text{ with } k \leq |F| \leq k+\ell\Bigg\} + \tilde{t}_k \subset A, \text{ for every } k \in \mathbb{N},$$

and

$$\Bigg\{\sum_{n\in F} n\colon\ F \subset B + s_k \text{ with } 1 \leq |F| \leq k\Bigg\} + t_k \subset A, \text{ for every } k \in \mathbb{N},$$

simultaneously.

Appendix A

Properties of abelian groups and their Følner sequences

In this appendix, which is particularly useful for Chapters 4 and 5, we collect several results regarding Følner sequences in abelian groups that we need. Let G denote a countable abelian group with $\ell = [G : 2G] < \infty$ and $r = |\ker(D)| < \infty$.

Lemma A.0.1. *Let Φ, Ψ be Følner sequences in G. Then the following hold:*

(i) $\Phi \cup \Psi = (\Phi_N \cup \Psi_N)_{N\in\mathbb{N}}$ *is a Følner sequence in G.*

(ii) If $0 < \liminf_{N\to\infty} \frac{|\Psi_N|}{|\Phi_N|} \leq \limsup_{N\to\infty} \frac{|\Psi_N|}{|\Phi_N|} < +\infty$ *and,* $\liminf_{N\to\infty} \frac{|\Phi_N \cap \Psi_N|}{|\Psi_N|} > 0$ *or* $\liminf_{N\to\infty} \frac{|\Phi_N \cap \Psi_N|}{|\Phi_N|} > 0$, *then* $\Phi \cap \Psi = (\Phi_N \cap \Psi_N)_{N\in\mathbb{N}}$ *is a Følner sequence in G.*

(iii) If $(g_N)_{N\in\mathbb{N}}$ *is a sequence of elements of G, then the sequence of shifts* $(g_N + \Phi_N)_{N\in\mathbb{N}}$ *is a Følner sequence in G.*

(iv) $\Phi/2 = (\Phi_N/2)_{N\in\mathbb{N}}$ *is a Følner sequence in G.*

Proof. Statements (i) and (ii) follow immediately from the definitions, and while (iii) should be known to aficionados, the proof is also a simple consequence of the definitions. Let us now prove (iv). From [11, Lemma 5.4] we know that $N \mapsto \Phi_N = \Phi_N \cap 2G$, $N \in \mathbb{N}$ is a Følner sequence in $2G$. Observe that for each $N \in \mathbb{N}$, $\Phi_N/2 = \Phi_N/2$. Let $g \in G$. Then for each N we have that $\big(g + \Phi_N/2\big)\triangle\big(\Phi_N/2\big) \subset \big((2g + \Phi_N)\triangle\Phi_N\big)/2$.

Observing that $(2g+\Phi_N)\triangle\Phi_N \subset 2G$ and $\Phi_N \subset 2G$, it follows from the definition of the kernel of the doubling map D that $|\big((2g + \Phi_N)\triangle\Phi_N\big)/2| = r|(2g + \Phi_N)\triangle\Phi_N|$

and $|\Phi_N/2| = r|\Phi_N|$. Thus, we get that

$$\frac{|\big(g+\Phi_N/2\big)\triangle\big(\Phi_N/2\big)|}{|\Phi_N/2|} = \frac{|\big(g+\Phi_N/2\big)\triangle\big(\Phi_N/2\big)|}{|\Phi_N/2|} \leq \frac{|(2g+\Phi_N)\triangle\Phi_N|}{|\Phi_N|},$$

which goes to 0 as $N \to \infty$, because $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner in $2G$. This concludes the proof of (iv). □

Remark A.0.2. We remark here that statements (i), (ii) and (iii) hold in any countable abelian group, irrespectively of whether the values r, ℓ are finite or infinite.

Lemma A.0.3. *Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be a Følner sequence in G. Then*

$$\lim_{N\to\infty} \frac{|\{g \in \Psi_N : g + \ker(D) \subset \Psi_N\}|}{|\Psi_N|} = 1. \tag{A.0.1}$$

In particular, this implies that $\frac{|2\Psi_N|}{|\Psi_N|} = \frac{1}{r} + \mathrm{o}_{N\to\infty}(1)$.

Proof. Let $\ker(D) = \{h_1, \dots, h_r\}$. For each N we have that

$$\{g \in \Psi_N : g + \ker(D) \not\subset \Psi_N\} = \bigcup_{i=1}^{r} \{g \in \Psi_N : g + h_i \notin \Psi_N\} = \bigcup_{i=1}^{r} \Psi_N \setminus (\Psi_N - h_i)$$

which is a subset of $\bigcup_{i=1}^{r} \Psi_N \triangle (\Psi_N - h_i)$ and therefore

$$\frac{|\{g \in \Psi_N : g + \ker(D) \not\subset \Psi_N\}|}{|\Psi_N|} \leq \sum_{i=1}^{r} \frac{|\Psi_N \triangle (\Psi_N - h_i)|}{|\Psi_N|}. \tag{A.0.2}$$

Since Ψ is a Følner sequence in G, each summand in the right hand side of (A.0.2) goes to 0 as $N \to \infty$, so

$$\lim_{N\to\infty} \frac{|\{g \in \Psi_N : g + \ker(D) \not\subset \Psi_N\}|}{|\Psi_N|} = 0,$$

which implies (A.0.1). Now, for each N, if we let $\Phi_N = \{g \in \Psi_N : g + \ker(D) \subset \Psi_N\}$, then we have that $\Phi_N \subset \Psi_N \subset \bigcup_{g\in\Psi_N} g + \ker(D)$, which gives that

$$|2\Phi_N| \leq |2\Psi_N| \leq |2 \bigcup_{g\in\Psi_N} g + \ker(D)|. \tag{A.0.3}$$

Consider the equivalence relation in G defined by $a \equiv b \iff a - b \in \ker(D)$, and for each $g \in G$ let $[g]$ denote the equivalence class of g. Note that for $g, g' \in G$, $[g] \cap [g'] \neq \emptyset \iff [g] = [g']$. Let R_N be the number of distinct equivalence classes $[g]$ contained in Φ_N. Then $|\Phi_N| = rR_N$ and $|2\Phi_N| = R_N$, because $2(g + h_i) = 2g$

for each $g \in G$, $h_i \in \ker(D)$, combined with the fact that whenever $[g] \cap [g'] = \emptyset$ we also have that $2g \neq 2g'$. Hence, $|2\Phi_N| = \frac{1}{r}|\Phi_N|$.

Let also $F_N = \bigcup_{g \in \Psi_N} g + \ker(D)$, and let L_N be the number of different equivalent classes appearing in Ψ_N. As before we see that $|F_N| = rL_N$ and $|2F_N| = L_N$, so that $|2F_N| = \frac{1}{r}|F_N|$. Moreover,

$$F_N \setminus \Psi_N \subset \bigcup_{\substack{g \in \Psi_N \\ g + \ker(D) \not\subset \Psi_N}} g + \ker(D),$$

so $|F_N \setminus \Psi_N| \leq r\,|\{g \in \Psi_N : g + \ker(D) \not\subset \Psi_N\}| = r|\Psi_N \setminus \Phi_N| = r(|\Psi_N| - |\Phi_N|)$. Dividing by $|\Psi_N|$, taking $N \to \infty$ and using that $\frac{|\Phi_N|}{|\Psi_N|} \to 1$, we get that $\frac{|F_N \setminus \Psi_N|}{|\Psi_N|} \to 0$, so also $\frac{|F_N \setminus \Psi_N|}{|F_N|} \to 0$, and since $\Psi_N \subset F_N$, we get that $\frac{|\Psi_N|}{|F_N|} \to 1$. Now, dividing by $|\Psi_N|$ in (A.0.3), we get that $\frac{|2\Phi_N|}{|\Psi_N|} \leq \frac{|2\Psi_N|}{|\Psi_N|} \leq \frac{|2F_N|}{|\Psi_N|}$. Then $\frac{|2\Phi_N|}{|\Psi_N|} = \frac{1}{r}\frac{|\Phi_N|}{|\Psi_N|} \to \frac{1}{r}$ and $\frac{|2F_N|}{|\Psi_N|} = \frac{1}{r}\frac{|F_N|}{|\Psi_N|} \to \frac{1}{r}$, which implies that $\frac{|2\Psi_N|}{|\Psi_N|} \to \frac{1}{r}$ as $N \to \infty$ and concludes the proof of the lemma. □

As a special case of the above result we also obtain the following.

Lemma A.0.4. *For any Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ in G we have that $\frac{|\Phi_N/2|}{|\Phi_N \cap 2G|} = r + \mathrm{o}_{N\to\infty}(1)$ and $\frac{|\Phi_N/2|}{|\Phi_N|} = \frac{r}{\ell} + \mathrm{o}_{N\to\infty}(1)$.*

Proof. Applying Lemma A.0.3 with $\Psi_N = \Phi_N/2$ and observing that $2(\Phi_N/2) = \Phi_N \cap 2G$, we get that $\frac{|\Phi_N/2|}{|\Phi_N \cap 2G|} = r + \mathrm{o}_{N\to\infty}(1)$. Combining the previous with [11, Lemma 5.4] we get that $\frac{|\Phi_N/2|}{|\Phi_N|} = \frac{r}{\ell} + \mathrm{o}_{N\to\infty}(1)$. □

Lemma A.0.5. *If a Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ in G is quasi-invariant with respect to doubling, then $(\Phi_N \cap \Phi_N/2)_{N \in \mathbb{N}}$ is also a Følner sequence in G.*

Proof. From Lemma A.0.1 (iv) we know that $\Phi/2 = (\Phi_N/2)_{N \in \mathbb{N}}$ is also a Følner in G. Now, from Lemma A.0.4 and since Φ is q.i.d. we see that $\Phi, \Phi/2$ satisfy the assumptions of Lemma A.0.1 (ii), and the lemma follows. □

Appendix B

Progressive measures, averages and seminorms

The purpose of this appendix is to justify our choice of the way to define the progressive measures used in Chapter 7 and to explain why it is necessary to consider double averages in Theorem 7.7.6.

B.0.1 Progressive measures coming from finite-step nilsystems are distinct

For an ergodic system (X, μ, T) with continuous pronilfactors, the progressive measures σ^{Z_k} for $k \geq 1$ are defined in [45] as follows: Consider $a \in \mathtt{gen}(\mu, \Phi)$ and Z_k the k-step pronilfactor with continuous factor map $\pi_k \colon X \to Z_k$. Define $\xi^{Z_k} \in (Z_k^k)$ to be the Haar measure of the orbit closure of $(\pi_k(a), \dots, \pi_k(a))$ under the transformation $T \times \cdots \times T^k$ and for $f_0, f_1, \dots, f_k \in C(X)$ the measure $\sigma^{Z_k} \in (X^{k+1})$ is defined via

$$\int_{X^{k+1}} f_0 \otimes f_1 \otimes \cdots \otimes f_k \, \mathrm{d}\sigma^{Z_k} = f_0(a) \int_{Z_k^k} \mathbb{E}(f_1 \mid Z_k) \otimes \cdots \otimes \mathbb{E}(f_k \mid Z_k) \, \mathrm{d}\xi^{Z_k}. \quad \text{(B.0.1)}$$

Herein, we want to illustrate that there are cases where if $P_{k+1} \colon X^{k+2} \to X^{k+1}$ is the projection to the first $k+1$ coordinates, then $P_{k+1}\sigma^{Z_{k+1}}$ and σ^{Z_k} are distinct (see also [3, Example 3.19]).

Fix $s \geq 2$ and let $X = \mathbb{T}^s$ and $T \colon X \to X$ be the affine transformation given by $T(x_1, \dots, x_s) = (x_1 + \alpha, x_2 + 2x_1 + \alpha, \dots, x_s + sx_{s-1} + \binom{s}{2}x_{s-2} + \cdots + sx_1 + \alpha)$. This transformation has been repeatedly used in the literature; see, for example, [19, Proposition 3.11]. In particular if $a = (0, \dots, 0)$ then $T^n a = (n\alpha, n^2\alpha, \dots, n^s\alpha)$

Notice that the factors $Z_k(X)$ for $k \leq s$ in this case are the systems $Z_k = \mathbb{T}^k$ with

factor maps $\pi_k \colon X \to Z_k$ given by projecting onto the first k coordinates. Moreover, for every $k \leq s$, $\sigma^{Z_{k+1}}$ is the Haar measure on the manifold

$$\Omega_k = \left\{ \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k+1} \end{pmatrix} \in (\mathbb{T}^s)^{k+2} \;\middle|\; \begin{array}{c} v_0 = a, \\ v_r = (rt_1, r^2t_2, \dots, (r+1)^r t_k, u_{r,1}, \dots, u_{r,s+1-k}), \\ t_1, \dots, t_k \in \mathbb{T} \quad \text{and} \\ u_{r,1}, \dots, u_{r,s+1-k} \in \mathbb{T} \quad \text{for } r = 1, \dots k+1 \end{array} \right\}$$

where we recall that $a = (0, \dots, 0) \in \mathbb{T}^s = X$. For instance, when $s = 3$,

$$\Omega_1 = \{(0,0,0,\ t,u,v,\ 2t,w,w') \mid t,u,v,w,w' \in \mathbb{T}\} \subset X^{1+2}$$

$$\Omega_2 = \{(0,0,0,\ t,r,u,\ 2t,4r,v,\ 3t,9r,w) \mid t,r,u,v,w \in \mathbb{T}\} \subset X^{2+2}$$

$$\Omega_3 = \{(0,0,0,\ t,r,s,\ 2t,4r,8s,\ 3t,9r,27s,\ 4t,16r,64s) \mid t,r,s \in \mathbb{T}\} \subset X^{3+2}$$

Notice that $P_k(\Omega_k)$ is strictly contained in Ω_{k-1}, so in particular they are distinct. Since Haar measures of nilmanifolds are fully supported, it follows that the measures $P_{k+1}\sigma^{Z_k}$ and $\sigma^{Z_{k-1}}$ are distinct for all $k \leq s$.

For the same reason, the measures σ^{Z_k} are not ergodic under the action of $Id \times T \times \cdots \times T^k$ for $k < s$. In fact, the nilmanifold $P_{s,k}(\Omega_s)$, where $P_{s,k}$ denotes the projection map $P_{s,k} \colon X^{s+2} \to X^{k+2}$ to the first $k+2$ coordinates, is a minimal component of Ω_k for the transformation $Id \times T \times \cdots \times T^k$. This shows that Ω_k itself is not minimal for $Id \times T \times \cdots \times T^k$ and hence its Haar measure σ^{Z_k} is not ergodic.

B.0.2 Double average for seminorm control

Here we use the same family of examples to prove that the seminorms control is only guaranteed in the double averaging version of Lemma 7.7.6. More precisely, we build an example where if one consider a single average, the L^2 norm is no longer controlled by the seminorms of f_1, f_2.

Consider $X = \mathbb{T}^3$ with the transformation $T \colon \mathbb{T}^3 \to \mathbb{T}^3$ given by $(x, y, z) \mapsto (x + \alpha, y + 2x + \alpha, z + 3y + 3x + \alpha)$ for α irrational. Let also $\mathbf{0} = (0, 0, 0)$ and

$$\Omega = \overline{Orb_{T \times T^2}(\mathbf{0}, \mathbf{0})} = \{(t, s, r, 2t, 4s, 9r) \mid t, s, r \in \mathbb{T}\} \subset X^2$$

with its Haar measure ξ. Then, $(\Omega, \xi, T \times T^2)$ is a minimal and ergodic system. Notice that, for $F \in C(\Omega)$ and $(t_0, s_0, r_0, 2t_0, 4s_0, 9r_0) \in \Omega$, the conditional expectation onto $W = Z_2(\Omega, \xi, T \times T^2)$ is given by

$$\mathbb{E}(F \mid W)(t_0, s_0, r_0, 2t_0, 4s_0, 9r_0) = \int_{\mathbb{T}} F(t_0, s_0, r, 2t_0, 4s_0, 9r)\, \mathrm{d}m(r).$$

Then, if $f_1, f_2 \in C(\mathbb{T}^3)$ are given by $f_1 = \mathbb{1}_{\mathbb{T}^2 \times (0, 1/9^2)} + 1$ and $f_2 = \mathbb{1}_{\mathbb{T}^2 \times (1/9, 2/9)} + 1$, we have that

$$\begin{aligned}
\mathbb{E}(f_1 \mid Z_2) =& \mathbb{E}(\mathbb{1}_{\mathbb{T}^2 \times (0, 1/9^2)} \mid Z_2) + 1 = 1 + 1/9^2 \\
\mathbb{E}(f_2 \mid Z_2) =& \mathbb{E}(\mathbb{1}_{\mathbb{T}^2 \times (1/9, 2/9)} \mid Z_2) + 1 = 1 + 1/9 \\
\mathbb{E}(f_1 \otimes f_2 \mid W) =& \mathbb{E}(\mathbb{1}_{\mathbb{T}^2 \times (0, 1/9^2)} \otimes \mathbb{1}_{\mathbb{T}^2 \times (1/9, 2/9)} \mid W) + \mathbb{E}(\mathbb{1}_{\mathbb{T}^2 \times (0, 1/9^2)} + \mathbb{1}_{\mathbb{T}^2 \times (1/9, 2/9)} \mid Z_2) + 1 \\
=& 0 + 1/9^2 + 1/9 + 1
\end{aligned}$$

In particular, $\mathbb{E}(f_1 \otimes f_2 \mid W) - \mathbb{E}(f_1 \mid Z_2) \otimes \mathbb{E}(f_2 \mid Z_2) \neq 0$. Moreover, we have that

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n \leq N} (T^n f_1 \cdot T^{2n} f_1) \otimes (T^n f_2 \cdot T^{2n} f_2) \right\|_{L^2(\xi)}$$

is controlled by

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n \leq N} \tilde{T}^{4n} (f_1 \otimes \mathbb{1}) \cdot \tilde{T}^{2n} (f_1 \otimes f_2) \cdot \tilde{T}^n (\mathbb{1} \otimes f_2) \right\|_{L^2(\xi)},$$

by similar arguments as in the proof of Lemma 7.7.6, where here $\tilde{T} = T \times T^2$. By [32] we can replace it by

$$\begin{aligned}
&\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n \leq N} \tilde{T}^{4n} \mathbb{E}(f_1 \otimes \mathbb{1} \mid W) \cdot \tilde{T}^{2n} \mathbb{E}(f_1 \otimes f_2 \mid W) \cdot \tilde{T}^n \mathbb{E}(\mathbb{1} \otimes f_2 \mid W) \right\|_{L^2(\xi)} \\
=& (1/9^3 + 1)(1/9^3 + 1/9 + 1)(1/9 + 1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n \leq N} \tilde{T}^{4n} (\mathbb{E}(f_1 \mid Z_2) \otimes \mathbb{1}) \cdot \tilde{T}^{2n} (\mathbb{E}(f_1 \mid Z_2) \otimes \mathbb{E}(f_2 \mid Z_2)) \cdot \tilde{T}^n (\mathbb{1} \otimes \mathbb{E}(f_2 \mid Z_2)) \right\|_{L^2(\xi)} \\
=& (1/9^3 + 1)^2 (1/9 + 1)^2.
\end{aligned}$$

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