THE TOM DIECK SPLITTING PRINCIPLE FOR EQUIVARIANT ORTHOGONAL SPECTRA

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Abstract. We give a *zig-zag* of π_*-isomorphisms between the *genuine* fixed points of the suspension spectrum of a G-space and the wedge of other spectra. This wedge is indexed on subgroups H of G, and the H summand is the *genuine* fixed points of the suspension spectrum of a space that depends only on the H-fixed points of the original G-space. This result *lifts* the original tom Dieck Splitting in equivariant homotopy groups to a point-set level splitting in the homotopy category of orthogonal spectra which does not rely on the *Adams Isomorphism*. We emphasize that all the morphisms in the *zig-zag* are actual morphisms of orthogonal spectra. Moreover, the group G is a compact Lie group.

1. Overview

1.1. **Introduction.** For a compact Lie group G and any G-space A, in [15, Satz 2, p. 654] tom Dieck constructed a chain of explicit group homomorphisms (1.4.1) that gives rise to a splitting of equivariant stable homotopy groups as follows:

$$\pi_*^G(\Sigma^\infty A) \cong \bigoplus_{(H)\subset G} \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \tag{1.1.1}$$

where the direct sum runs over a set of representatives of all conjugacy classes of closed subgroups H of G and WH is the Weyl group of H in G, i.e., the quotient NH/H where NH is the normalizer of H in G.

Building on tom Dieck's work, Lewis, May and McCLure [6, Chapter 5] established an explicit equivalence

$$(\Sigma^\infty A)^G \simeq \bigvee_{(H)\subset G} \Sigma^\infty(EWH_+ \wedge_{WH} A^H) \tag{1.1.2}$$

in the stable homotopy category [1]. However, they do not directly lift tom Dieck's chain of group homomorphisms and their proof relies on the *Adams Isomorphism*, see [6, Theorem 11.1, p. 294] for details[2].

The purpose of this article is to present a novel approach to the tom Dieck splitting that:

- lifts tom Dieck's homomorphism to a *zig-zag* of maps of orthogonal spectra, and

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[1] This equivalence has to be understood for a finite group G.

[2] We will refer to this statement as the "*LMM Splitting*" even if it is usually called the tom Dieck Splitting.

- does not make use of the *Adams Isomorphism*.

1.2. **Motivations.** The main motivation of this paper is to provide a clean reference of the tom Dieck Splitting and its generalisations. In fact, even if the spectra level splitting (1.1.2) is well known in the community, a proof of it in the context of orthogonal spectra is not present in the literature. A second motivation is that as stated before, the LMM splitting is a generalisation of the tom Dieck Splitting which includes the Adams Isomorphism. Our approach highlights the fact that the tom Dieck Splitting and the Adams Isomorphism are two independent pieces of mathematics. Moreover, combining our result with a modern version of the Adams Isomorphism we obtain a point-set level LMM splitting as we will see in the last section (7).

1.3. **Fixed Points.** We will work in the category of orthogonal G-spectra $\mathcal{S}p^G$ as presented in Chapter 3 of [11]. Equivariant homotopy groups $\pi_*^G(X)$ of an orthogonal G-spectrum X are defined as usual and they are indexed on a complete G-universe (4.1.1). Moreover, they are functorial in G, in the sense that a continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups induces a *restriction homomorphism* $\alpha^* : \pi_*^G(X) \longrightarrow \pi_*^K(\alpha^* X)$ [11, Construction 3.1.15], where $\alpha^* X$ denote the K-spectrum with K-action induced by α . To make the goal more precise we introduce a formal definition of the *genuine* fixed points functor.

Definition 1.3.1. Let G be a compact Lie group. A genuine fixed points functor is a functor $F^G : \mathcal{S}p^G \longrightarrow \mathcal{S}p$ together with a natural isomorphism

$$\pi_*^G \overset{\cong}{\Longrightarrow} \pi_* \circ F^G \tag{1.3.1}$$

of functors $\mathcal{S}p^G \longrightarrow \mathcal{A}b$.

Given a morphism of orthogonal G-spectra $f : X \longrightarrow Y$ we have a commutative square

$$\begin{tikzcd} \pi_*^G(X) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \pi_*^G(Y) \arrow[d, "\cong"] \\ \pi_*(F^G X) \arrow[r, "(F^G f)_*"'] & \pi_*(F^G Y) \end{tikzcd}$$

where the unlabelled isomorphisms are given by the natural isomorphism of the definition 1.3.1. Given a group homomorphism $m : \pi_*^{G_1}(X_1) \longrightarrow \pi_*^{G_2}(X_2)$ for two compact Lie groups G_1, G_2 and orthogonal G_i-spectra X_i one can ask if there exists a morphism of non equivariant spectra $g : F^{G_1} X_1 \longrightarrow F^{G_2} X_2$ such that the following square

$$\begin{tikzcd} \pi_*^{G_1}(X_1) \arrow[r, "m"] \arrow[d, "\cong"'] & \pi_*^{G_2}(X_2) \arrow[d, "\cong"] \\ \pi_*(F^{G_1} X_1)) \arrow[r, "g_*"'] & \pi_*(F^{G_2} X_2) \end{tikzcd} \tag{1.3.2}$$

commutes.

If that is the case we will say that g is a lifting of m. Of course, if $G_1 = G_2 = G$ and m is induced by a morphism of G-spectra f then $F^G f$ is a lifting of f_*.

We summarize what we have said in the following definition.

Definition 1.3.2. In the previous notation we say that

- g is a *lift* of m;
- m is *induced* by g.

We now have all the ingredients to make more precise our strategy.

1.4. **Tom Dieck's Proof.** We start by investigating the original proof of the splitting 1.1.1, which is given by constructing a homomorphism

$$\zeta_H : \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \longrightarrow \pi_*^G(\Sigma^\infty A) \tag{1.4.1}$$

for each conjugacy class of closed subgroup H of G, such that

$$\zeta = \bigoplus_{(H)\subset G} \zeta_H : \bigoplus_{(H)\subset G} \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \longrightarrow \pi_*^G(\Sigma^\infty A) \tag{1.4.2}$$

is an isomorphism natural in A.

The construction of ζ_H is given explicitly in [15] and goes as follows:

Construction 1.4.1.

- The projection $p : NH \longrightarrow WH$ induces a restriction homomorphism in equivariant homotopy groups $p^* : \pi_*^{WH}(\_) \longrightarrow \pi_*^{NH}(p^*(\_))$ [11, Construction 3.1.15] and it can be applied to the domain of the homomorphism ζ_H;
- We can now apply the morphism of orthogonal NH-spectra induced by the inclusion $A^H \longrightarrow A$ landing in $\pi_*^{NH}(\Sigma^\infty(EWH_+ \wedge A))$;
- G/NH is a smooth manifold and we denote by $L = T_{eNH}(G/NH)$ the tangent space at the coset eNH of G/NH. Note that L has a natural NH-action induced by the left translation of G/NH and hence L is a NH-representation. The next homomorphism is the effect of the suspension of the NH-composite
$$EWH_+ \wedge A \cong EWH_+ \wedge A \wedge S^0 \longrightarrow EWH_+ \wedge A \wedge S^L$$
on NH-equivariant homotopy groups, where the last inclusion is given by the inclusion of 0 in L;
- The *external transfer* is a natural group isomorphism
$$Tr_{NH}^G : \pi_*^{NH}(\Sigma^\infty(EWH_+ \wedge A) \wedge S^L) \longrightarrow \pi_*^G(G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)))$$
where $G \ltimes_{NH} (\_) = G_+ \wedge_{NH} (\_)$ is the left adjoint to the restriction functor induced by the inclusion $NH \longrightarrow G$. The construction of the external transfer will be discuss in section 5.
- Finally, the NH-projection $EWH_+ \wedge A \longrightarrow A$ induces by adjunction a morphism of orthogonal G-spectra
$$G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)) \longrightarrow \Sigma^\infty A$$
the effect of which in G-homotopy groups gives a homomorphism
$$\pi_*^G(G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A))) \longrightarrow \pi_*^G(\Sigma^\infty A).$$

The map ζ_H is defined as the composite of these five homomorphisms and the direct sum over a set of representatives of all conjugacy classes of closed subgroups H of G of ζ_H was proven by tom Dieck in [15, Satz 2, p. 654] to be an isomorphism for every based G-space A.

Since homotopy groups commutes with wedges, in order to *lift* the isomorphism ζ it is enough to *lift* ζ_H for all closed subgroup H of G.

1.5. **The Main Result.** The goal of this paper is to prove the following theorem.

Theorem 1.5.1. *The homomorphsim ζ_H is induced by a zig-zag of morphisms of non equivariant spectra*

$$\begin{aligned} F^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \longrightarrow F^{NH}(\Sigma^\infty(EWH_+ \wedge A) \wedge S^L) \longleftarrow \\ \longleftarrow F^G(G \ltimes_{NH} \Sigma^\infty(EWH_+ \wedge A)) \\ \longrightarrow F^G(\Sigma^\infty A). \end{aligned}$$

The strategy will be to lift each individual homomorphism in the definition of ζ_H to a morphism of non equivariant spectra not necessarily in the same direction. We proceed by observing that the second, the third and the fifth homomorphism in the definition of ζ_H come already from morphisms of equivariant spectra and hence we can apply the genuine fixed point functor to obtain the desired non equivariant morphisms.

It remains to lift the first homomorphism and the fourth which are respectively a *restriction homomorphism* and the *external transfer*.

After introducing the general theory of equivariant orthogonal spectra following [11, Chapter 3], we will discuss in section 5 the *Wirthmüller isomorphism* which is the inverse of the *external transfer*. We will prove that for suspension spectra the *Wirthmüller isomorphism* is the composite of a restriction homomorphism and a homomorphism induced by a morphism of equivariant spectra. This is something always true when G is a finite group but not true if G is a compact Lie group and the spectrum involved is not a suspension spectrum.

After doing this we will have reduced the proof of 1.5.1 to the proof of the following lemma:

Lemma 1.5.2. *Let $\alpha : K \longrightarrow G$ be a continuous homomorphism between compact Lie groups and X an orthognal G-spectrum. Then $\alpha^* : \pi_*^G(X) \longrightarrow \pi_*^K(\alpha^* X)$ is induced by a morphism*

$$R_K^G : F^G X \longrightarrow F^K \alpha^* X$$

of non equivariant spectra.

To do so, we will choose a specific model for the genuine fixed points functor that is presented in the case where G is a finite group in [10]. We will use the same construction but for a compact Lie group G. We mention here that every time that we will refer to [10] for a proof of a statement, even if their statement is given only in the case of a finite group, their proof works in the same way in our more general contest of a compact Lie group. The reason behind the choice of restricting to finite groups is that they are main interested in the application of their results to Topological Cyclic Homology.

1.6. **Related Work.** As already mentioned the splitting was proved on the level of homotopy groups by tom Dieck in [15, Satz 2, p. 654] [3]. It was then generalised by Lewis, May and McClure to a splitting in the equivariant stable homotpy category in [6, Theorem 11.1, p. 294]. The same splitting appears also in the Alaska notes [9, Thm 1.1, Ch XIX, p. 245] with a similar proof. Both the last two last generalisation of the classical splitting rely on the *Adams Isomorphism*

[3] for the non german reader a traslation of tom Dieck's argument adapted to the case when G is a finite group can be found in [12, Ch 6, p. 60].

which appeard in [1, Thm 5.3, p. 500] for the first time and generalised in [6, Thm 7.1, p. 97] by Lewis and May.

When G is finite Guillou and May in [3, Thm 6.5, p. 43] give a categorical proof of the tom Dieck Splitting as a consequence of the equivariant Barratt-Priddy-Quillen theorem in the context of genuine G-spectra. Finally, an ∞-categorical version can be found in [2, Thm A.9, p. 46].

1.7. **Notations and Conventions.** By a *space* we mean a *compactly generated weak hausdorff space*. The reasons behind this choice can be found in [13]. We will denote by $\mathcal{T}op$ (respectively $\mathcal{T}op_*$) the category of spaces (pointed spaces) with continuous maps (based maps). The mapping space between two spaces A, B will be denoted by $map(A, B)$ and $map_*(A, B)$ for the pointed version. We refer again to [13] for a detailed exposition of the topology on $map(A, B)$. Finally, $[A, B]$ (respectively $[A, B]_*$) will denote the set of homotopy classes of maps (based homotopy classes of based maps) from A to B. For the equivariant case we fix a compact Lie group G. We refer to a G-space as a G-object in $\mathcal{T}op$, i. e., a space A together with a continuous action map

$$G \times A \longrightarrow A$$

which has to be associative and unital. A map between two G-spaces A, B is a continuous map $A \longrightarrow B$ that commutes with the action of G and we will denote by $map^G(A, B)$ the space of G-maps between A and B and $\mathcal{T}op^G$ the category of G-spaces and G- equivariant maps. If A and B are G-spaces , $map(A, B)$ will denote the mapping G-space, where the G action is given by conjugation. Finally $[A, B]^G$ will denote the set of G-homotopy classes of G-equivariant maps from A to B. For real inner product spaces V, W we denote by $\mathcal{L}(V, W)$ the space of linear isometric embedding and by S^W the one-point compactification of the finite dimensional inner product space W. In the case $W = \mathbb{R}^n$ we will write S^n instead of $S^{\mathbb{R}^n}$. In the case V and W comes with a G-action then $\mathcal{L}(V, W)$ inherits a natural G-action given by conjugation and S^W inherits a G-action that fixes the point at infinity.

1.8. **Organization.** In section 2 we will introduce the general theory of orthogonal spectra as presented in [11, Ch 3]. Section 3 is devoted to the construction of the *functor* Q as defined in [10, Definition 4.17] and in section 4 we will recollect the definitions of equivariant homotopy groups of an orthogonal G-spectrum as presented in [11, Ch 3].

These sections are needed only to fix the notation no original work is presented here.

In section 5 we will discuss the *Wirthmüller isomorphism* and the *external transfer*.

In section 6 we will introduce the *naive* and *genuine* fixed points of a G-spectrum. This section contains the proof of lemma 1.5.2 which is the main technical result of this paper.

Finally in section 7 using the modern *Adams Isomorphism* [10][Theorem 1.7, p. 4] we will recover the LMM Splitting in case when G is a finite group.

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2. Orthogonal G-Spectra

2.1. **Terminology.** In this section we recall the definition of an equivariant orthogonal spectrum and we fix the notations needed. An *equivariant orthogonal spectrum* is a continuous based functor from the *orthogonal complement category* $\mathcal{O}$ [11, Construction 3.1.1] to the category $\mathcal{T}op_*^G$ of based G-spaces [11, Definition 3.1.3], where G is a fixed compact Lie group. Similarly, an orthogonal spectrum is a continuous based functor from $\mathcal{O}$ to the category $\mathcal{T}op_*$ of based spaces. We denote by $\mathcal{S}p$ the category of orthogonal spectra and by $\mathcal{S}p^G$ the category of orthogonal G-spectra. The morphisms in these two categories are given by natural transformations of functors. In this definition are hidden all the usual data of an orthogonal spectrum. In particular, an orthogonal G-spectrum X consists of the following data:

- for each finite dimensional real inner product space V a based $G \times O(V)$-space $X(V)$;
- for every two finite dimensional real inner product spaces V, W a based *action* G-map
$$\mathcal{O}(V, W) \wedge X(V) \longrightarrow X(W);$$
- for every two finite dimensional real inner product spaces V, W a based *suspension* G-map [11, p. 230]
$$\sigma_{V,W} : X(V) \wedge S^W \longrightarrow X(V \oplus W);$$

subject to *unitality, associativity, compatibilty* conditions.

A morphism $f : X \longrightarrow Y$ between orthogonal G-spectra consists of a continuous based G-map $f(V) : X(V) \longrightarrow Y(V)$ for each finite dimensional real inner product space V which commutes with the *action* and the *suspension* map.

Another important piece of notation is the so called *untwisting homeomorphism* [11, 3.1.2]

$$\mu_{V,W} : \mathcal{O}(V, W) \wedge S^V \cong S^W \wedge \mathcal{L}(V, W)_+. \tag{2.1.1}$$

Remark 2.1.1. If V is s finite dimensional G-representation and X is an orthogonal G-spectrum then we can evaluate X on V obtaining a $(G \times G)$-space $X(V)$. We regard $X(V)$ as a G-space via the diagonal action. Moreover, the structure map $\sigma_{V,W}$ is G-equivariant.

The suspension maps of an orthogonal spectrum encode the most important structure of a spectrum hence, whenever they satisfy some extra conditions the spectrum has a special name.

Definition 2.1.2 (Good, Almost Good and **Ω-*G***-Spectra)**.** A G-spectrum X is called

- *good* if the suspension map $\sigma_{V,W}$ is a closed embedding;
- *almost good* if the adjoint of the suspension map
$$\tilde{\sigma}_{V,W} : X(V) \longrightarrow map_*(S^W, X(V \oplus W)) \tag{2.1.2}$$
is a closed embedding;
- *an* Ω-G-*spectrum* if the adjoint of the suspension map is a G-*weak equivalence*, i.e., it induces a weak equivalence in H-fixed points for all H subgroups of G

for every pair of finite dimensional G-representations V, W.

Proposition 2.1.3. *If X is a good orthogonal G-spectrum then it is almost good.*

Proof. The proof follows from the fact that if A, B and C are based spaces and $i : A \wedge B \longrightarrow C$ is a closed embedding then so is its adjoint $\tilde{i} : A \longrightarrow map_*(B, C)$ the proof of which can be found in [14, Corollary 5.11, p. 19]. □

We point out that since the category $\mathcal{O}$ has a small skeleton given by $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ to define an orthogonal spectrum X it is enough to specify the value on the skeleton with some extra conditions, see for details [11, Remark 3.1.6]. From now on we will write X_n instead of $X(\mathbb{R}^n)$ to lighten the notation.

2.2. **Constructions.** [Mapping, Smash and Shifted Spectrum] Given a G-spectrum X and a based G-space A the *smash spectrum* $X \wedge A$ and the *mapping spectrum* $map_*(A, X)$ are defined as in [11, Construction 3.1.19].

Remark 2.2.1. The functor $\_ \wedge A$ is left adjoint to $map_*(A, \_)$ as endofunctors of $\mathcal{S}p^G$.The adjunction is given by the levelwise adjunction of the same two functors as endofunctors of $\mathcal{T}op_*^G$.

Given a finite dimensional G-representation U the *the U-th shifted spectrum $sh^U X$* is definded as in [11, Contruction 3.1.21].

Remark 2.2.2. Let X be an orthogonal G-spectrum. The assignment $U \mapsto sh^U X$ extends to a continuous functor $\mathcal{O} \longrightarrow \mathcal{S}p^G$.

Remark 2.2.3. For every finite dimensional G-representation U and G-spectrum X the suspension map of X defines a morphism of G-spectra

$$\lambda_U : X \wedge S^U \longrightarrow sh^U X, \tag{2.2.1}$$

i.e., the value of λ_U on a G-representation V is given by $\sigma_{V,U}$.

2.3. **Suspension Spectra.**

Definition 2.3.1 (Suspension Spectra)**.** Let A be a based G-space. The *suspension G-spectrum* of A $\Sigma^\infty A$ is defined as $(\Sigma^\infty A)(V) = A \wedge S^V$, with suspension map the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$.

Remark 2.3.2. Clearly, suspension spectra are *good*.

3. The functor Q

3.1. **Costruction and Properties of Q.** The functor Q is defined as in [10, Definition 4.17], i.e., given a finite dimensional G-representation U and a G-spectrum X, the value of $Q^U X$ on a finite dimensional G-representation V is

$$Q^U X(V) = sh^U(map_*(S^U, X(V)) = map_*(S^U, X(V \oplus U)).$$

Remark 3.1.1. For every finite dimensional G-representation U and G-spectrum X the adjoint of λ_U defines a morphism of G-spectra

$$\tilde{\lambda}_U : X \longrightarrow Q^U X. \tag{3.1.1}$$

Proposition 3.1.2. *The association*

$$(U, X) \mapsto Q^U X$$

extends to a continuous functor from the category $\mathcal{L}^f \times \mathcal{S}p^G$ to the category $\mathcal{S}p^G$, where $\mathcal{L}^f$ denote the topological category with objects finite dimensional G-representations and morphisms the G-space of linear isometric embeddings with G-action given by conjugation.

Proof. We refer to [10, Proposition 7.1, p. 24] for the proof of this proposition. $\square$

Remark 3.1.3. We want to extend the definition of Q to countably infinite dimensional G-representations. We denote by $\mathcal{L}$ the topological category of G-representations of possibly countably infinite dimension and morphisms the G-space of linear isometric embeddings . Being a functor category to the category $\mathcal{T}op_*^G$, the category of orthogonal G-spectra is cocomplete and $\mathcal{L}$ is skeletally small. Hence, for a countably infinite dimensional G-representation $\mathcal{U}$ and a G-spectrum X we can define

$$Q^{\mathcal{U}}X = (\mathrm{Lan}_{\mathcal{L}^f_G \subset \mathcal{L}_G} Q^{-}X)(\mathcal{U}) \tag{3.1.2}$$

as the *left Kan extension* of $Q^{(-)}X : \mathcal{L}^f \longrightarrow \mathcal{S}p^G$ along the inclusion $\mathcal{L}^f \longrightarrow \mathcal{L}$. The existence of the *Kan extension* as a continuous functor $\mathcal{L} \longrightarrow \mathcal{S}p^G$ is guaranteed by [4, Proposition 4.33, p. 63].

Notation 3.1.4. For a G-representation $\mathcal{U}$ we denote by $p(\mathcal{U})$ the poset of finite dimensional G-subrepresentations of $\mathcal{U}$ under inclusion.

Proposition 3.1.5. *For any orthogonal G-spectrum X and any G-representation $\mathcal{U}$, there is a natural isomorphism of G-spectra*

$$\mathrm{colim}_{p(\mathcal{U})} Q^{(-)}X \xrightarrow{\cong} (\mathrm{Lan}_{\mathcal{L}^f \subset \mathcal{L}} Q^{(-)}X)(\mathcal{U}) = Q^{\mathcal{U}}X. \tag{3.1.3}$$

Proof. The proof is the same in the case of finite group which can be found in [10, Proposition 8.3, p. 29]. □

Remark 3.1.6. The inclusion $0 \longrightarrow \mathcal{U}$ gives a natural morphism of G-spectra

$$r_X^{\mathcal{U}} : X \cong Q^0 X \longrightarrow Q^{\mathcal{U}}X. \tag{3.1.4}$$

Moreover, if $\iota : \mathcal{U} \longrightarrow \mathcal{V}$ is a linear isometric embedding between two G-representations the following diagram

$$\begin{tikzcd} & X \arrow[dl, "r_X^{\mathcal{U}}"'] \arrow[dr, "r_X^{\mathcal{V}}"] & \\ Q^{\mathcal{U}}X \arrow[rr, "\iota_*"'] & & Q^{\mathcal{V}}X \end{tikzcd} \tag{3.1.5}$$

commutes by functoriality of $Q^{(-)}X : \mathcal{L} \longrightarrow \mathcal{S}p^G$.

Proposition 3.1.7. *Let X be a G-spectrum then:*

- *X is almost good if and only if for all finite dimensional G-representations $U \subset V$ the morphism $Q^U X \longrightarrow Q^V X$ is a levelwise closed embedding;*
- *X is almost good then so is $Q^{\mathcal{U}}X$ for every G-representation $\mathcal{U}$ and $r_X^{\mathcal{U}} : X \longrightarrow Q^{\mathcal{U}}X$ is a levelwise closed embedding.*

Proof. This is part of the content of Lemma 5.10 of [10]. Again the proof in the case of compact Lie group is the same □

3.2. **Effect of α^* on Q.** The effect of the change of group functor on the functor Q will be crucial for the proof of lemma 1.5.2. We let X be an orthogonal G-spectrum, $\alpha : K \longrightarrow G$ be a continuous group homomorphism between compact Lie groups and U be a finite dimensional G-representation. Then we have the following lemma.

Lemma 3.2.1. *α^* defines an isomorphism of K-spectra*

$$\alpha^*(Q^U X) \longrightarrow Q^{\alpha^*(U)}\alpha^* X. \tag{3.2.1}$$

Moreover this isomorphism is compatible with the structure maps in the colimit 3.1.5 and hence, it defines a K-isomorphism

$$\alpha^* : \alpha^*(Q^{\mathcal{U}} X) \xrightarrow{\cong} Q^{\alpha^*(\mathcal{U})}\alpha^* X. \tag{3.2.2}$$

Proof. It is clear that α^* is an isomorphism levelwise, i.e., for every $n \in \mathbb{N}$

$$\alpha^*((Q^U X)_n) = \alpha^*(map_*(S^U, X(\mathbb{R}^n \oplus U)) \cong map_*(S^{\alpha^*(U)}, \alpha^*(X)(\mathbb{R}^n \oplus \alpha^*(U))$$

where the last isomorphism is exactly given by $f \mapsto \alpha^*_n f$. We now show that α^* is compatible with the suspension of $\alpha^*(Q^U X)$ and $Q^{\alpha^* U}\alpha^* X$, i.e., that the following diagram

$$\begin{tikzcd}
\alpha^*((Q^U X)_n) \wedge S^1 \arrow[r, "\alpha^*_n \wedge Id"] \arrow[d, "\sigma_n^{\alpha^*(Q^U X)}"'] & (Q^{\alpha^*(U)}\alpha^* X)_n \wedge S^1 \arrow[d, "\sigma_n^{Q^{\alpha^*(U)}\alpha^* X}"] \\
\alpha^*((Q^U X)_{n+1}) \arrow[r, "\alpha^*_{n+1}"'] & (Q^{\alpha^*(U)}\alpha^* X)_{n+1}
\end{tikzcd}$$

commutes.

We let $a : (Q^U X)_n \wedge S^1 \longrightarrow (Q^U(X \wedge S^1))_n$ be the assembly map that sends $f \wedge t$ to the map $(v \mapsto f(v) \wedge t)$. Since S^1 comes with the trivial action, the suspension map for $\alpha^*(Q^U X)$ is

$$\alpha^*(\sigma_n^{Q^U X}) = \alpha^*((\tau_{U,\mathbb{R}} \circ \sigma^X_{\mathbb{R}^n \oplus U, \mathbb{R}})_* \circ a) \tag{3.2.3}$$

where $\tau_{U,\mathbb{R}} : X((\mathbb{R}^n \oplus U) \oplus \mathbb{R}) \cong X(\mathbb{R}^n \oplus \mathbb{R} \oplus U)$ flips the coordinates. And similarly for the suspension map of $Q^{\alpha^*(U)}\alpha^* X$. So we let $f \wedge t$ be an element in $\alpha^*((Q^U X)_n) \wedge S^1$ and $u \in U$. Then

$$\begin{aligned}
&(\alpha^*_{n+1} \circ \alpha^*((\tau_{U,\mathbb{R}} \circ \sigma^X_{\mathbb{R}^n \oplus U,\mathbb{R}})_* \circ a))(f \wedge t)(u) = \alpha^*_{n+1} \circ \alpha^*((\tau_{U,\mathbb{R}} \circ \sigma^X_{\mathbb{R}^n \oplus U,\mathbb{R}})_*(f(u) \wedge t) = \\
&= \alpha^*_{n+1}(\tau_{U,\mathbb{R}}(\sigma^X_{\mathbb{R}^n \oplus U,\mathbb{R}}(f(u) \wedge t))) = \alpha^*(\tau_{U,\mathbb{R}})(\alpha^*(\sigma^X_{\mathbb{R}^n \oplus U,\mathbb{R}}(f(u) \wedge t))) = \\
&= (\tau_{\alpha^*(U),\mathbb{R}} \circ \sigma^{\alpha^* X}_{\mathbb{R}^n \oplus \alpha^* U,\mathbb{R}})_*(\alpha^*(f(u) \wedge t)) = ((\tau_{\alpha^*(U),\mathbb{R}} \circ \sigma^{\alpha^* X}_{\mathbb{R}^n \oplus \alpha^* U,\mathbb{R}})_* \circ a \circ (\alpha^*_n \wedge Id))(f \wedge t)(u)
\end{aligned}$$

as we wanted. A similar computation proves that for an inclusion of G-representations $U \subset V$ the following diagram

$$\begin{tikzcd}
\alpha^*(Q^U X) \arrow[r, "\alpha^*"] \arrow[d] & Q^{\alpha^*(U)}\alpha^* X \arrow[d] \\
\alpha^*(Q^V X) \arrow[r, "\alpha^*"'] & Q^{\alpha^*(V)}\alpha^* X
\end{tikzcd}$$

commutes, where the vertical arrow comes from the functoriality of Q with respect representations. Using that

$$(\alpha^*(Q^{\mathcal{U}} X))_n \cong \alpha^*(\mathrm{colim}_{U \in p(\mathcal{U})}(Q^U X)_n) = \mathrm{colim}_{U \in p(\mathcal{U})} \alpha^*((Q^U X)_n)$$

we arrives to the desired K-isomorphism

$$\alpha^*(Q^{\mathcal{U}} X) \xrightarrow{\cong} Q^{\alpha^*(\mathcal{U})}\alpha^* X. \tag{3.2.4}$$

□

4. Homotopy theory of Orthogonal G-spectra

4.1. **Equivariant Homotopy Groups.** In this section we present the definitions of *equivariant homotopy groups* of an orthogonal G-spectrum. We fix a compact Lie group G, an orthogonal G-spectrum X and a G-universe $\mathcal{U}$ as defined in [6, Chapter 1, p. 11]. Then $\pi_*^{G,\mathcal{U}}(X)$ is defined as in [11, 3.1.11] (note that we are not requiring that $\mathcal{U}$ is a *complete* G-universe).

Notation 4.1.1. By definition equivariant homotopy groups of an orthogonal G-spectrum X depend on a G-universe $\mathcal{U}$. We give special names depending on the property of $\mathcal{U}$.

- when $\mathcal{U}$ is a complete G-universe we set

$$\pi_*^{G,\mathcal{U}}(X) = \pi_*^G(X). \tag{4.1.1}$$

  and we refer to it as *the equivariant homotopy groups* of X. In the literature $\pi_*^G(X)$ are usually called the *genuine* equivariant homotopy group of X.
- when $\mathcal{U}$ is a trivial G-universe, i.e., G-isomorphic to $\mathbb{R}^\infty$ we set

$$\tilde{\pi}_*^G(X) = \pi_*^{G,\mathbb{R}^\infty}(X) \tag{4.1.2}$$

  and we refer to it as the *trivial universe homotopy groups.*

Remark 4.1.2. Let G be a compact Lie group. We recall some basic facts about universes:

- complete G-universes always exist;
- if H is a closed subgroup of G then:
  - the underlying H-representation of a complete G-universe is a complete H-universe [11, Rem 1.1.13, p.21];
    the H-fixed points of a complete G-universe is a complete WH-universe;
- if V is a G-representation and $\mathcal{U}$ a G-universe such that V embeds into $\mathcal{U}$ then the G-space $\mathcal{L}(V,\mathcal{U})$ is G-equivariantly contractible [11, Proposition 1.1.21, p. 24].
- if $\mathcal{V}$ is a G-universe and $\mathcal{U}$ is a complete G-universe then any two equivariant isometric embeddings $\iota, \iota' : \mathcal{V} \longrightarrow \mathcal{U}$ are G-homotopic through G-linear isometric embeddings.

Proposition 4.1.3. *Let X be an orthogonal G-spectrum. Choose a G-linear isometric embedding $\iota\colon \mathbb{R}^\infty \longrightarrow \mathcal{U}$ where $\mathcal{U}$ is a complete G-universe. Then we have a natural group homomorphism*

$$\tilde{\pi}_*^G(X) \longrightarrow \pi_*^G(X) \tag{4.1.3}$$

induced by ι which is an isomorphism if X is an Ω-G-spectrum.

Proof. Recall that for an Ω-G-spectrum X the adjoint suspension maps are G-weak equivalences. In particular if V is a finite dimensional G-representation then $\tilde{\sigma}_{0,V}$ realizes a G-weak equivalence between $X(0)$ and $map_*(S^V, X(V))$. Moreover we have the following natural isomorphisms

$$[S^{\mathbb{R}^k\oplus V}, X(V)]_*^G \cong [S^{\mathbb{R}^k}, map_*(S^V, X(V))]_*^G \xleftarrow[\cong]{(\tilde{\sigma}_{0,V})_*} [S^{\mathbb{R}^k}, X(0)]_*^G$$
$$\xrightarrow[\cong]{(\tilde{\sigma}_{0,\mathbb{R}^n})_*} [S^{\mathbb{R}^k}, map_*(S^{\mathbb{R}^n}, X(\mathbb{R}^n))]_*^G \cong [S^{\mathbb{R}^k\oplus\mathbb{R}^n}, X(\mathbb{R}^n)]_*^G$$

where the unlabelled isomorphisms are given by adjunction. These isomorphisms are compatible with the suspension maps defining the homotopy groups of X hence they define a homomorphism $\pi_*^G(X) \longrightarrow \tilde{\pi}_*^G(X)$ which is the identity if precomposed with 4.1.3. This implies that 4.1.3 has a retraction which is an isomorphism and so is 4.1.3. □

4.2. **π_*-isomorphisms and $\mathcal{H}o(\mathcal{S}p^G)$.** A morphism of orthogonal G-spectra $f: X \longrightarrow Y$ induces a homomorphism on homotopy groups $f_*: \pi_*^G(X) \longrightarrow \pi_*^G(Y)$.

Definition 4.2.1. A morphism $f: X \longrightarrow Y$ between orthogonal G-spectra is

- a π_**–isomorphism* if the induced map $f_*: \pi_k^H(i^*X) \longrightarrow \pi_k^H(i^*Y)$ is an isomorphism for all closed subgroups H of G and $i: H \longrightarrow G$ is the natural inclusion;
- a *level strong equivalence* if $f(V): X(V) \longrightarrow Y(V)$ is a G-weak equivalence for all finite dimensional G-representations.
- a *homotopy equivalence* if there exists a morphism of G-spectra $g: Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are homotopic to the respective identities as morphism of G-spectra.

We define the *G-equivariant stable category* $\mathcal{H}o(\mathcal{S}p^G)$ as the category obtained from $\mathcal{S}p^G$ by formally inverting π_*-isomorphisms.

Proposition 4.2.2. *If a morphism of G-spectra $f: X \longrightarrow Y$ is a level strong equivalence then it is a π_*-isomorphism and the converse is true if X and Y are both Ω-G-spectra.*

Proof. This is Theorem 3.4 in [8, p. 41] the proof of which is presented in[8, Section 8, p. 56]. □

Remark 4.2.3. Note that being isomorphic in $\mathcal{H}o(\mathcal{S}p^G)$ means that there exists a *zig-zag* of π_*-isomorphisms of orthogonal G-spectra.

The next theorem is a recollection of all the important properties satisfied by the functor Q (3.1.5). This theorem is stated and proved in [10, Theorem 1.1, p. 30] in the context of a finite group G, but the proof presented there works also in the more general context of a compact Lie group.

Theorem 4.2.4. *Let G be a compact Lie group then for any orthogonal G-spectrum X the functor Q defined in 3.1.5 together with the natural transformation 3.1.4 $r: X \longrightarrow Q^{\mathcal{U}}X$ satisfy the following properties:*

- *if X is almost good (2.1.2) then $r: X \longrightarrow Q^{\mathcal{U}}X$ is a π_*-isomorphism;*
- *if X is almost good and $\mathcal{U}$ is a complete G-universe then $Q^{\mathcal{U}}X$ is an Ω-G-spectrum.*

5. The Wirthmüller Isomorphism

The *Wirthmüller isomorphism* is the key construction in order to define the external transfer. We will not give the formal definition of the morphism since it is too technical and one can find all the details in [11, Chapter 3, p. 262].

We let H be a closed subgroup of a compact Lie group G. Then the inclusion $i: H \longrightarrow G$ induces a change of group functor $i^*: \mathcal{S}p^G \longrightarrow \mathcal{S}p^H$ given by restriction of the G-action along the homomorphism i. This functor has a left adjoint given by the *induced spectrum*. We give the explicit construction of coordinatized spectra as defined in [11, Remark 3.1.6]. Let Y be an orthogonal H-spectrum.

Construction 5.0.1 (Induced Spectrum)**.** The left adjoint of $i^*: \mathcal{S}p^G \longrightarrow \mathcal{S}p^H$ is the *induced-G-spectrum*

$$G \ltimes_H Y$$

given in level n by $(G \ltimes_H Y)_n = G_+ \wedge_H Y_n$ with induced suspension map and $O(n)$-action. Using the formula [12, 2.2, p. 7] the value on a G-representation V can be identified via the following G-equivariant homeomorphism

$$(G \ltimes_H Y)(V) \cong G \ltimes_H Y(i^*(V)), \quad [l_g \circ \phi, g \ltimes y] \mapsto g \ltimes [\phi, y] \tag{5.0.1}$$

where $l_g : V \longrightarrow V$ is the left-translation by $g \in G$.

Definition 5.0.2. Let B be a based G-space and A a based H-space. Then the *shearing isomorphism* is the G-homeomorphism

$$(G \ltimes_H A) \wedge B \cong G \ltimes_H (A \wedge i^*(B)), \qquad [g, a] \wedge a \mapsto [g, a \wedge g^{-1}b]. \tag{5.0.2}$$

Let now $L = T_{eH}(G/H)$ be the tangent H-representation, i.e., the tangent space at the coset eH of the smooth manifold G/H and A and B as in the previous definition. In [11, Chapter 3, p. 262-264] a natural (in A) H-map

$$l_A : G \ltimes_H A \longrightarrow A \wedge S^L \tag{5.0.3}$$

is constructed.

The main property of this map is given by the following proposition.

Proposition 5.0.3. *The following square*

$$\begin{tikzcd} (G \ltimes_H A) \wedge B \arrow[r, "l_A \wedge Id"] \arrow[d, "\cong"'] & A \wedge S^L \wedge B \arrow[d, "\cong"] \\ G \ltimes_H (A \wedge i^*(B)) \arrow[r, "l_{A \wedge i^*(B)}"'] & A \wedge B \wedge S^L \end{tikzcd} \tag{5.0.4}$$

where the vertical left arrow is the shearing isomorphism 5.0.2 and the vertical right arrow just flips S^L and B, commutes up to H-equivariant based homotopy.

Proof. This is the content of Proposition 3.2.3 in [11, p 264]. ☐

Remark 5.0.4. Proposition 5.0.3 tells us that if Y is an orthogonal H-spectrum then the collection of H maps l_{Y_n}

$$l_{Y_n} : G \ltimes_H Y_n \longrightarrow Y_n \wedge S^L$$

do not assemble in general into a morphism of H-spectra

$$G \ltimes_H Y \longrightarrow Y \wedge S^L$$

since they only commute with the structure maps of Y up to based H-equivariant homotopy (see last diagram in [11, p. 265]) . Nevertheless, this is enough to induce a homomorphism in homotopy groups

$$(l_Y)_* : \pi_*^H(G \ltimes_H Y) \longrightarrow \pi_*^H(Y \wedge S^L). \tag{5.0.5}$$

Definition 5.0.5. Let H be a closed subgroup of a compact Lie group G, Y an orthogonal H-spectrum and denote by L the tangent H-representation of G/H at the coset eH. Then we define the *Wirthmüller* map Ψ_H^G by the composite

$$\pi_*^G(G \ltimes_H Y) \xrightarrow{i^*} \pi_*^H(G \ltimes_H Y) \xrightarrow{(l_Y)_*} \pi_*^H(Y \wedge S^L). \tag{5.0.6}$$

Theorem 5.0.6. *Under the same notation of definition 5.0.5 the homomorphism Ψ_H^G is an isomorphism $\pi_*^G(G \ltimes_H Y) \cong \pi_*^H(Y \wedge S^L)$.*

Proof. This can be found in [11, Theorem 3.2.15, p. 271]. □

Definition 5.0.7. We define the *external transfer* $Tr_H^G : \pi_*^H(Y \wedge S^L) \longrightarrow \pi_*^G(G \ltimes_H Y)$ as the inverse of the isomorphism Ψ_H^G.

Remark 5.0.8. We emphasize that even if the notation suggests that $(l_Y)_*$ comes from a morphism of orthogonal H-spectra this is not the case in general. But, if Y is a suspension spectrum then we can do something better. Let A be a based H-space then the shearing isomorphism 5.0.2 gives the isomorphism

$$(G \ltimes_H \Sigma^\infty A)_n = G \ltimes_H (A \wedge S^n) \cong (G \ltimes_H A) \wedge S^n = (\Sigma^\infty(G \ltimes_H A))_n.$$

In particular as G-spectra (and also as H-spectra) $G \ltimes_H \Sigma^\infty A$ and $\Sigma^\infty(G \ltimes_H A)$ are isomorphic. Moreover, also $\Sigma^\infty(A \wedge S^L)$ is isomorphic to $\Sigma^\infty(A) \wedge S^L$ via the isomorphism which switches the spheres coordinates levelwise.

Hence, we obtain a morphism of H-spectra

$$\Sigma^\infty(l_A) : \Sigma^\infty(G \ltimes_H A) \longrightarrow \Sigma^\infty(A \wedge S^L)$$

such that the following diagram

$$\begin{tikzcd}
\pi_*^H(G \ltimes_H \Sigma^\infty A) \arrow[r, "(l_{\Sigma^\infty A})_*"] \arrow[d, "\cong"'] & \pi_*^H(\Sigma^\infty(A) \wedge S^L) \arrow[d, "\cong"] \\
\pi_*^H(\Sigma^\infty(G \ltimes_H A)) \arrow[r, "(\Sigma^\infty(l_A))_*"'] & \pi_*^H(\Sigma^\infty(A \wedge S^L))
\end{tikzcd} \tag{5.0.7}$$

commutes. Indeed, let V be a G-representation and $f : S^{i^*V} \longrightarrow G \ltimes_H (A \wedge S^{i^*V})$ be an H-equivariant map representing an element in $\pi_*^H(G \ltimes \Sigma^\infty A)$. Then the commutativity of the diagram is implied by the fact that the following diagram

$$\begin{tikzcd}
S^{i^*V} \arrow[r, "f"] & G \ltimes_H (A \wedge S^{i^*V}) \arrow[r, "l_{A \wedge S^{i_*V}}"] \arrow[d, "\cong"'] & A \wedge S^{i_*V} \wedge S^L \arrow[d, "\cong"] \\
& (G \ltimes_H A) \wedge S^{i^*V} \arrow[r, "l_A \wedge S^{i^*V}"'] & A \wedge S^L \wedge S^{i_*V}
\end{tikzcd}$$

commutes up to H-equivariant homotopy by proposition 5.0.3.

The upshot of the previous remark is that for suspension spectra the homomorphism $(l_{\Sigma^\infty A})_*$ is induced by the morphism of orthogonal H-spectra

$$G \ltimes_H \Sigma^\infty A \cong \Sigma^\infty(G \ltimes_H A) \xrightarrow{\Sigma^\infty l_A} \Sigma^\infty(A \wedge S^L) \cong \Sigma^\infty(A) \wedge S^L \tag{5.0.8}$$

Hence, the inverse of the external transfer for a suspension spectrum is given by the composite of a restriction homomorphism and a morphism of H-spectra as mentioned in the introduction.

It is now clear that lemma 1.5.2 implies the main theorem 1.5.1.

6. Fixed points

We now start the discussion of fixed points of an orthogonal G-spectrum. There are several notions of it and we start with the *naive* approach.

6.1. **Naive fixed points.**

Definition 6.1.1 (Naive Fixed Point)**.** Let X be an orthogonal G-spectrum. The *naive* fixed points spectrum of X is the orthogonal spectrum X^G given by the levelwise fixed points. Using the definition of a orthogonal G-spectrum as a continuous functor $X : \mathcal{O} \longrightarrow \mathcal{T}op_*^G$ then the spectrum X^G is given by the following composition of functors

$$\mathcal{O} \xrightarrow{X} \mathcal{T}op_*^G \xrightarrow{(-)^G} \mathcal{T}op_*.$$

The naive fixed points construction is a functor $(-)^G : \mathcal{S}p^G \longrightarrow \mathcal{S}p$ from orthogonal G-spectra to orthogonal spectra. This is the most natural construction that can be made, but it is not *homotopical* in the sense that it does not preserve π_*-isomorphisms.

Remark 6.1.2. By adjunction we have a natural isomorphism

$$\tilde{\pi_k}^G(X) \cong \mathrm{colim}_n[S^{k+n}, X(\mathbb{R}^n)]_*^G \cong \mathrm{colim}_n[S^{k+n}, (X(\mathbb{R}^n))^G]_* = \pi_k(X^G) \tag{6.1.1}$$

where the first isomorphism uses that $\{\mathbb{R}^n\}_n$ is a cofinal sequence in $\mathbb{R}^\infty$, and the second one uses that S^{k+n} comes with the trivial G-action. Hence, *naive fixed points* detect *trivial universe* π_*-isomorphisms.

The naive fixed points functor preserves strong level equivalences and since π_*-isomorphisms between Ω-G-spectra are strong level equivalences by proposition 4.2.2, precomposing $(\_)^G$ with a functorial Ω-G-approximation yields a fixed points functor which preserves π_*-isomorphisms. We restrict the following discussion to the case of *almost good* G-spectra for which we already have an Ω-G-replacement functor.

6.2. **Genuine fixed points.**

Definition 6.2.1 (Genuine Fixed Points)**.** Let $\mathcal{U}$ be a complete G-universe and X an almost good G-spectrum (2.1.2). The *genuine* fixed points spectrum of X is the non equivariant orthogonal spectrum $(Q^{\mathcal{U}}X)^G$.

The most important feature of the genuine fixed points is that the homotopy groups of $(Q^{\mathcal{U}}X)^G$ calculate the G-homotopy groups of X. More precisely we have the following proposition.

Proposition 6.2.2. *Let X be an almost good spectrum and $i : (Q^{\mathcal{U}}X)^G \longrightarrow Q^{\mathcal{U}}X$ be the levelwise inclusion of G-fixed points. Then the homomorphism*

$$\pi_*((Q^{\mathcal{U}}X)^G) \longrightarrow \pi_*^G(X), \qquad [f] \mapsto ((r_X^{\mathcal{U}})_*)^{-1}([i \circ f]) \tag{6.2.1}$$

is a natural isomorphism.

Proof. Note first of all that since X is almost good and $\mathcal{U}$ is a complete G-universe $r_X^{\mathcal{U}} : X \longrightarrow Q^{\mathcal{U}}X$ is a π_*-isomorphism. Now, the homomorphism 6.2.2 is given by the composite

$$\pi_*((Q^{\mathcal{U}}X)^G) \longrightarrow \tilde{\pi}_*^G(Q^{\mathcal{U}}X) \longrightarrow \pi_*^G(Q^{\mathcal{U}}X) \xrightarrow{((r_X^{\mathcal{U}})_*)^{-1}} \pi_*^G(Q^{\mathcal{U}}X)$$

where the first arrow is post composition with i, i.e., the isomorphism 6.1.2 and the second one is induced by a chosen inclusion $\mathbb{R}^\infty \longrightarrow \mathcal{U}$, and it is an isomorphism by proposition 4.1.3 since $Q^{\mathcal{U}}X$ is an Ω-G-spectrum. Hence, the composite is an isomorphism . $\square$

Remark 6.2.3. Proposition 6.2.2 tells us that $(Q^{\mathcal{U}}\_)^G$ is a model for a genuine fixed points functor as in definition 1.3.1. Moreover, taking naive fixed points of the morphism 3.1.4 yields a comparison morphism of non-equivariant orthogonal spectra

$$(r_X^{\mathcal{U}})^G : X^G \to (Q^{\mathcal{U}}X)^G. \tag{6.2.2}$$

$(Q^{\mathcal{U}}(\_))^G$ is not an easy functor to study and the tom Dieck splitting witness this complexity. Nevertheless, it permits us to compare without losing homotopical information, equivariant spectra with action of different groups as in definition 1.3.2.

6.3. **Restriction Homomorphism as a Morphism of Spectra.** Any continuous homomorphism $\alpha : K \longrightarrow G$ between two compact Lie groups induces a group homomorphism in equivariant homotopy groups

$$\alpha^* : \pi_*^G(X) \longrightarrow \pi_*^K(\alpha^*X) \tag{6.3.1}$$

In this section we will construct the morphism $R_K^G : (Q^{\mathcal{U}_G}X)^G \longrightarrow (Q^{\mathcal{U}_K}\alpha^*X)^K$ mentioned in lemma 1.5.2 of the introduction, where $\mathcal{U}_G$ and $\mathcal{U}_K$ are respectively complete G and K universes. This is the key lemma of the paper and it is what we need in order to prove theorem 1.5.1, the goal of this paper.

Notation 6.3.1. We will identify $\alpha^*(Q^{\mathcal{U}}X)$ with $Q^{\alpha^*(\mathcal{U})}\alpha^*X$ as explained in 3.2.4. In particular we will include this identification in the change of group homomorphism $\alpha^* : \pi_*^G(Q^{\mathcal{U}}X) \longrightarrow \pi_*^K(Q^{\alpha^*(\mathcal{U})}\alpha^*X)$. We also observe that, under this identification $\alpha^*(r_X^{\mathcal{U}})$ (3.1.4) corresponds to $r_{\alpha^*X}^{\alpha^*(\mathcal{U})}$ in the sense that the following diagram

$$\begin{tikzcd} & \alpha^*(Q^{\mathcal{U}}X) \arrow[dd, "\cong"] \\ \alpha^*X \arrow[ur, "\alpha^*(r_X^{\mathcal{U}})"] \arrow[dr, "r_{\alpha^*X}^{\alpha^*(\mathcal{U})}"'] & \\ & Q^{\alpha^*(\mathcal{U})}\alpha^*X \end{tikzcd} \tag{6.3.2}$$

commutes where the vertical isomorphism is the identification 3.2.4 and it is given by α^*. We will also refer to α^* as the restriction functor (resp. homomorphism).

Remark 6.3.2. If $\alpha : K \longrightarrow G$ is a continuous group homomorphism between compact Lie groups and $\mathcal{U}_G$ and $\mathcal{U}_K$ are complete G and K-universes respectively then $\alpha^*(\mathcal{U}_G)$ is a non necessarily complete K-universe. Nevertheless, we can choose a K-linear isometric embedding $\iota : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$.

Lemma 6.3.3. *Under the notations of the previous remark we let X be an orthogonal G-spectrum. Denote by $\iota_* : Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X \longrightarrow Q^{\mathcal{U}_K}\alpha^*X$ the morphism of orthogonal K-spectra induced by functoriality of $Q^{(-)}\alpha^*X$. Let α^* be the restriction homomorphism. Then the following diagram*

$$\begin{tikzcd} \pi_*^G(X) \arrow[rr, "\alpha^*"] \arrow[d, "(r_X^{\mathcal{U}_G})_*"] & & \pi_*^K(\alpha^*X) \arrow[dl, "(r_{\alpha^*X}^{\alpha^*(\mathcal{U}_G)})_*"'] \arrow[d, "(r_{\alpha^*X}^{\mathcal{U}_K})_*"] \\ \pi_*^G(Q^{\mathcal{U}_G}X) \arrow[r, "\alpha^*"] & \pi_*^K(Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X) \arrow[r, "(\iota_*)_*"] & \pi_*^K(Q^{\mathcal{U}_K}\alpha^*X) \end{tikzcd} \tag{6.3.3}$$

commutes.

Proof. The right triangle is induced by the triangle

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ \alpha^*(\mathcal{U}_G) \arrow[rr, "\iota"'] & & \mathcal{U}_K \end{tikzcd}$$

of K-representations which obviously commutes hence by functoriality of $Q^{(-)}\alpha^* X$ it commutes. The left trapezium commutes by 6.3.2. □

Remark 6.3.4. Note that in the diagram 6.3.3 the homomorphism $(\iota_*)_*$ is independent of the choice of the K-linear isometric embedding $\alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ by remark 4.1.2.

We now want to construct a morphism of spectra $R^G_K : (Q^{\mathcal{U}_G} X)^G \longrightarrow (Q^{\mathcal{U}_K}\alpha^* X)^K$ which realizes the group homomorphism α^* when X is an almost good-G-spectrum.

Notation 6.3.5. Let A_λ be a diagram of based G-spaces indexed by a category Λ. Then $\operatorname{colim}_\Lambda A_\lambda$ inherits a G-action and we denote by Θ_G the canonical map

$$\operatorname{colim}_\Lambda A^G_\lambda \xrightarrow{\Theta_G} (\operatorname{colim}_\Lambda A_\lambda)^G. \tag{6.3.4}$$

Lemma 6.3.6. *Θ_G is a homeomorphism if the colimit is a filtered colimit taken along closed embeddings and G is a compact Lie group.*

Proof. This can be found in [7, Proposition 1.2, p.1]. □

Recall that if U is a finite dimensional G-representation then α^* defines a isomorphism of K-spectra $\alpha^*(Q^U X) \longrightarrow Q^{\alpha^*(U)}\alpha^* X$ (3.2.1). Taking naive K-fixed points of this isomorphism yields a morphism $(\alpha^*(Q^U X))^K \longrightarrow (Q^{\alpha^*(U)}\alpha^* X)^K$. We can now precomposing with the levelwise inclusion $(Q^U X)^G \longrightarrow (\alpha^*(Q^U X))^K$ obtaining a morphism of non equivariant spectra

$$C^G_K : (Q^U X)^G \longrightarrow (Q^{\alpha^* U}\alpha^* X)^K. \tag{6.3.5}$$

Lemma 6.3.7. *Let U be a finite dimensional G-representation then C^G_K defines a morphism of spectra. Moreover, if $U \subset V$ is an inclusion of G-subrepresentations inside the chosen complete G-universe $\mathcal{U}_G$ then the following diagram*

$$\begin{tikzcd} (Q^U X)^G \arrow[r, "C^G_K"] \arrow[d] & (Q^{\alpha^*(U)}\alpha^* X)^K \arrow[d] \\ (Q^V X)^G \arrow[r, "C^G_K"'] & (Q^{\alpha^*(V)}\alpha^* X)^K \end{tikzcd}$$

commutes where the vertical morphisms are the respective fixed points of the morphism induced by functoriality of Q.

Proof. C^G_K is a morphism of spectra since is the composition of the inclusion $(Q^U X)^G \longrightarrow (\alpha^*(Q^U X))^K$ and $(\alpha^*)^K$ which is a morphism of spectra since α^* is by lemma 3.2.1. Denote by $i : U \longrightarrow V$ the given inclusion of G-representations. Using that $Q^i X = i_*$ as is defined in [11, 3.1.9, p. 232] it is easy to show that the previous diagram commutes by explicit computation. □

Remark 6.3.8. Lemma 6.3.7 tells us that C_K^G assemble into a morphism of spectra

$$\operatorname{colim}_{U\in p(\mathcal{U}_G)}(Q^U X)^G \longrightarrow \operatorname{colim}_{U\in p(\mathcal{U}_G)}(Q^{\alpha^*(U)}\alpha^* X)^K \tag{6.3.6}$$

which by abuse of notation we denote again by C_K^G.

Remark 6.3.9. Using lemma 6.3.6 the canonical map

$$\operatorname{colim}_{U\in p(\mathcal{U}_G)}(Q^U X)^G \xrightarrow{\Theta_G} (\operatorname{colim}_{U\in p(\mathcal{U}_G)} Q^U X)^G \cong (Q^{\mathcal{U}_G} X)^G$$

is an isomorphism of orthogonal spectra whenever X is almost good.Moreover, since

$$\operatorname{colim}_{U\in p(\mathcal{U}_G)}(Q^{\alpha^* U}\alpha^* X) \cong Q^{\alpha^*(\mathcal{U}_G)}\alpha^* X$$

and if X is almost good then so is $\alpha^* X$ we have that C_K^G as in 6.3.7 defines a morphism of spectra

$$(Q^{\mathcal{U}_G} X)^G \longrightarrow (Q^{\alpha^*(\mathcal{U}_G)}\alpha^* X)^K$$

which we denote by abuse of notation by C_K^G. Postcomposing with $(\iota_*)^K : (Q^{\alpha^*(\mathcal{U}_G)}\alpha^*(X))^K \longrightarrow (Q^{\mathcal{U}_K}\alpha^* X)^K$ where $\iota : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ is the chosen K-linear isometric embedding we obtain a morphism

$$R_K^G : (Q^{\mathcal{U}_G} X)^G \longrightarrow (Q^{\mathcal{U}_K}\alpha^* X)^K. \tag{6.3.7}$$

In the next proposition we will have to prove that certain diagrams of homotopy groups commutes. After choosing representatives and chase the diagrams we will be reduced to prove that two diagrams of K-spaces commute. Both will be of the form

$$\begin{tikzcd} A^K \arrow[r, "i"] \arrow[d, "g^K"'] & A \arrow[d, "g"] \\ B^K \arrow[r, "i"'] & B \end{tikzcd} \tag{6.3.8}$$

where A and B are K-spaces and the horizontal arrows are inclusions of the fixed points and the diagram clearly commutes.

Proposition 6.3.10. *Let $\alpha : K \longrightarrow G$ be a continuous group homomorphism between compact Lie groups. Let X be an almost good G-spectrum, $\mathcal{U}_G$ and $\mathcal{U}_K$ be a complete G and K-universes respectively and $\iota : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ be the chosen K-linear isometric embedding. Then the following diagram*

$$\begin{tikzcd} \pi_*^G(Q^{\mathcal{U}_G} X) \arrow[r, "\alpha^*"] & \pi_*^K(Q^{\alpha^*(\mathcal{U}_G)}\alpha^* X) \arrow[r, "(\iota_*)_*"] & \pi_*^K(Q^{\mathcal{U}_K}\alpha^* X) \\ \pi_*((Q^{\mathcal{U}_G} X)^G) \arrow[u] \arrow[rr, "(R_K^G)_*"] & & \pi_*((Q^{\mathcal{U}_K}\alpha^* X)^K) \arrow[u] \end{tikzcd} \tag{6.3.9}$$

commutes, where the vertical arrows are the natural isomorphisms 6.2.2.

Proof. Unpacking the definition of R^G_K we have to show that the diagram

$$\begin{tikzcd}
\pi^G_*(Q^{\mathcal{U}_G}X) \arrow[r, "\alpha^*"] & \pi^K_*(Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X) \arrow[r, "(\iota_*)_*"] & \pi^K_*(Q^{\mathcal{U}_K}\alpha^*X) \\
\pi_*((Q^{\mathcal{U}_G}X)^G) \arrow[u] \arrow[r, "(C^G_K)_*"'] & \pi_*((Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)^K) \arrow[u] \arrow[r, "((\iota_*)^K)_*"'] & \pi_*((Q^{\mathcal{U}_K}\alpha^*X)^K) \arrow[u]
\end{tikzcd}$$

commutes. We now show that each square of the last diagram commutes.

- *The right square commutes:* Let $i \geq 0$ and $f : S^{i+n} \longrightarrow ((Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n)^K$ be a continuous based map representing a class in $\pi_*(Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)^K$. Denote by $i^{\mathcal{U}_K}_K : ((Q^{\mathcal{U}_K}\alpha^*X)_n)^K \longrightarrow (Q^{\mathcal{U}_K}\alpha^*X)_n$ the inclusion of the K-fixed points and similarly for $i^{\alpha^*(\mathcal{U}_G)}_K$. Then the vertical maps are given on representatives by post composition with the inclusions of the respective fixed points (6.2.2). Hence, the commutativity of the right square follows from the diagram

(6.3.10)
$$\begin{tikzcd}
 & (Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n \arrow[r, "\iota_*"] & (Q^{\mathcal{U}_K}\alpha^*X)_n \\
S^{i+n} \arrow[r, "f"] & ((Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n)^K \arrow[u, "i^{\alpha^*(\mathcal{U}_G)}_K"] \arrow[r, "(\iota_*)^K"] & ((Q^{\mathcal{U}_K}\alpha^*X)_n)^K \arrow[u, "i^{\mathcal{U}_K}_K"]
\end{tikzcd}$$

which clearly commutes because is of the form 6.3.8.
- *The left square commutes:* Let $i \geq 0$ and $f : S^{i+n} \longrightarrow ((Q^{\mathcal{U}_G}X)_n)^G$ be a continuous based map representing a class in $\pi_*(Q^{\mathcal{U}_G}X)^G$. Recall that by definition $(C^G_K)_n$ (6.3.5) is the composite of the inclusion of the fixed points $((Q^{\mathcal{U}_G}X)_n)^G \longrightarrow (\alpha^*((Q^{\mathcal{U}_G}X)_n))^K$ followed by
$$(\alpha^*_n)^K : (\alpha^*((Q^{\mathcal{U}_G}X)_n))^K \longrightarrow ((Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n)^K.$$
Let now $i^{\mathcal{U}_G}_G : ((Q^{\mathcal{U}_G}X)_n)^G \longrightarrow (Q^{\mathcal{U}_G}X)_n$ be the inclusion of the G-fixed points and and similarly for $i^{\alpha^*(\mathcal{U}_G)}_K$. Then the commutativity of the left square is implied by the commutativity of the following diagram

(6.3.11)
$$\begin{tikzcd}
S^{i+n} \arrow[r, "f"] & ((Q^{\mathcal{U}_G}X)_n)^G \arrow[r, "\alpha^*(i^{\mathcal{U}_G}_G)"] \arrow[d] \arrow[dd, bend right, "(C^G_K)_n"'] & \alpha^*((Q^{\mathcal{U}_G}X)_n) \arrow[dd, "\alpha^*_n"] \\
 & (\alpha^*((Q^{\mathcal{U}_G}X)_n))^K \arrow[ur] \arrow[d, "(\alpha^*_n)^K"] & \\
 & ((Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n)^K \arrow[r, "i^{\alpha^*(\mathcal{U}_G)}_K"'] & (Q^{\alpha^*(\mathcal{U}_G)}\alpha^*X)_n
\end{tikzcd}$$

where the unlabelled arrows are inclusions of fixed points. The upper triangle of diagram 6.3.11 commutes since all the maps are inclusions of fixed points. The lower trapezium commutes since is induced by a diagram of the form 6.3.8.

The case $i < 0$ is similar. □

Combining the two diagrams 6.3.3 and 6.3.9 we obtain the commutative diagram

$$\begin{tikzcd}
\pi_*^G(X) \arrow[rr, "\alpha^*"] \arrow[d, "r_X^{\mathcal{U}_G}"'] & & \pi_*^K(\alpha^* X) \arrow[dl, "r_{\alpha^* X}^{\alpha^* \mathcal{U}_G}"'] \arrow[d, "r_{\alpha^* X}^{\mathcal{U}_K}"] \\
\pi_*^G(Q^{\mathcal{U}_G} X) \arrow[r, "\alpha^*"] & \pi_*^K(Q^{\alpha^*(\mathcal{U}_G)} \alpha^* X) \arrow[r, "(\iota_*)_*"] & \pi_*^K(Q^{\alpha^*(\mathcal{U}_K)} \alpha^* X) \\
\pi_*((Q^{\mathcal{U}_G} X)^G) \arrow[u] \arrow[r, "(C_K^G)_*"'] \arrow[rr, bend right, "(R_K^G)_*"'] & \pi_*((Q^{\alpha^*(\mathcal{U}_G)} \alpha^* X)^K) \arrow[u] \arrow[r, "((\iota_*)^K)_*"'] & \pi_*((Q^{\alpha^*(\mathcal{U}_K)} \alpha^* X)^K). \arrow[u]
\end{tikzcd}$$

When X is an almost good spectrum then $r_K^{\mathcal{U}_G}$, $r_{\alpha^* X}^{\mathcal{U}_K}$ and all the unlabelled vertical arrows except the central one are isomorphisms. By an easy diagram chasing we have the following corollary.

Corollary 6.3.11. *The change of group homomorphism $\alpha^* : \pi_*^G(X) \longrightarrow \pi_*^K(\alpha^* X)$ is induced by the morphism R_K^G in the sense of 1.3.2. In particular, the following diagram*

$$\begin{tikzcd}
\pi_*^G(X) \arrow[r, "\alpha^*"] \arrow[d, "(r_X^{\mathcal{U}_G})_*"'] & \pi_*^K(\alpha^* X) \arrow[d, "(r_{\alpha^* X}^{\mathcal{U}_K})_*"] \\
\pi_*^G(Q^{\mathcal{U}_G} X) & \pi_*^K(Q^{\mathcal{U}_K} \alpha^* X) \\
\pi_*((Q^{\mathcal{U}_G} X)^G) \arrow[u] \arrow[r, "(R_K^G)_*"'] & \pi_*((Q^{\mathcal{U}_K} \alpha^* X)^K) \arrow[u]
\end{tikzcd} \tag{6.3.12}$$

commutes.

This is the most important result that we needed and the splitting 1.5.1 applied for $F^G = (Q^{\mathcal{U}_G}(\_))^G$ follows formally.

7. LMM Splitting

As mentioned in the introduction the spectrum level statement of the original tom Dieck Splitting 1.1.1 presented in [6, Thm 11.1, p. 294] use the *Adams isomorphism*. We want to see how our theorem 1.5.1 together with the modern Adams isomorphism presented in [10, Thm 1.7, p. 4] gives a translation of the LMM splitting 1.1.2 to our setting of orthogonal spectra. To do so, we have to restrict to the case when G is a finite group.

Definition 7.0.1. A G-spectrum X is called *G-free* if the projection

$$EG_+ \wedge X \longrightarrow X \tag{7.0.1}$$

is a π_*- isomorphism.

Recall the definition of *good* spectra in 2.1.2.

Theorem 7.0.2 (Modern Adams Isomorphism)**.** *Let X be an orthogonal G-spectrum and $\mathcal{U}$ a complete G-universe. Then there exists a natural morphism of non equivariant spectra*

$$A_X : EG_+ \wedge_G X \longrightarrow (Q^{\mathcal{U}} X)^G \tag{7.0.2}$$

that is a π_*-isomorphism whenever X is good and G-free.

Proof. See [10, Ch 14, p. 49]. □

Now, the projection to the first factor $p : EG_+ \wedge EG_+ \longrightarrow EG_+$ is a G-homotopy equivalence. Hence, if X is good and G-free the composite

$$EG_+ \wedge_G X \longrightarrow (EG_+ \wedge EG_+ \wedge X)_G \xrightarrow{A_{EG_+ \wedge X}} (Q^{\mathcal{U}} EG_+ \wedge X)^G \tag{7.0.3}$$

is a π_*-isomorphism where the unlabelled morphism is the diagonal, which is a homotopy inverse of the projection $p : EG_+ \wedge EG_+ \longrightarrow EG_+$.

Note that if A is a G-space then $\Sigma^\infty(EG_+ \wedge A)$ is a G-free spectrum. Indeed, the projection 7.0.1 for $\Sigma^\infty(EG_+ \wedge A)$ factors as the composite

$$EG_+ \wedge \Sigma^\infty(EG_+ \wedge A) \cong EG_+ \wedge EG_+ \wedge \Sigma^\infty A \xrightarrow{p \wedge Id} EG_+ \wedge \Sigma^\infty A \cong \Sigma^\infty(EG_+ \wedge A).$$

Since p is a G-homotopy equivalence then so is the projection $EG_+ \wedge \Sigma^\infty(EG_+ \wedge A) \longrightarrow \Sigma^\infty(EG_+ \wedge A)$. Applying 7.0.3 with $G = WH$ and $X = \Sigma^\infty A^H$ and $\mathcal{U} = \mathcal{U}^H$ we obtain the following corollary.

Corollary 7.0.3. *The wedge sum over a set of representatives of all conjugacy classes of subgroup H of G of the morphisms 7.0.3 composed with the zig-zag of the splitting 1.5.1 is an equivalence*

$$(Q^{\mathcal{U}} \Sigma^\infty A)^G \simeq \bigvee_{(H) \subset G} \Sigma^\infty(EWH_+ \wedge_{WH} A^H) \tag{7.0.4}$$

in the stable homotopy cateogry of ortoghonal spectra.

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