RESEARCH STATEMENT

JONE LOPEZ DE GAMIZ ZEARRA

1. Introduction

My research area is geometric group theory, which is the study of infinite discrete groups through their actions on metric spaces. Historically, groups arise from geometry. In 1872, F. Klein wanted to understand geometries by studying their symmetry groups. Geometric group theory, in contrast, has the reverse philosophy. It uses the properties of spaces on which a group acts to study the group itself. I am particularly interested in certain groups called *right-angled Artin groups (RAAGs)*. The class of RAAGs extends the class of finitely generated free groups and these groups have received a lot of attention since their introduction. My main goal is to understand the algorithmic and finiteness properties that subgroups of RAAGs share with free groups. The study of subgroups of direct products of free groups is linked with many interesting problems that arise in geometric group theory, so they have been extensively studied. Therefore, I am mainly focused on working with subgroups of direct products of RAAGs to see if the results that we know for subgroups of direct products of free groups can be generalised to this setting.

I will extend the motivation for studying subgroups of direct products of RAAGs in Section 2 by giving some historical background. After that, in Section 3 and 4, I state some recent results and the research that I am carrying on currently. Finally, in Section 5 I explain some research projects that would be interesting for me to work on in the future.

2. Background and motivation

2.1. **Subgroups of direct products of limit groups.** Subgroups of free groups are themselves free. Since these subgroups are so well understood, one may wonder if subgroups of direct products of free groups can also be easily classified. It turns out that they can be fairly complicated. For example, in [29] K.A. Mihailova constructed a finitely generated subgroup of the direct product of two free groups of rank two with undecidable conjugacy and membership problems. G. Baumslag and J.E. Roseblade proved that *finitely presented* subgroups are, on the contrary, considerably better behaved (see [3]): they are virtually the direct product of two free groups.

When we consider subgroups of direct products with more than two factors, the spectrum of finiteness properties gets richer, reflecting the more complex structure of their finitely presented subgroups. M.R. Bridson, J. Howie, C.F. Miller and H. Short carried a programme to study the structure of these subgroups depending on their finiteness properties. They described not just the structure of subgroups of direct products of free groups, but also of direct products of the more general class of limit groups (see [8], [9], [10], [11], [12]).

The structure of limit groups is well understood. Sela characterised them as groups that have a faithful action on a real tree induced by a sequence of homomorphisms to a free group. In addition, one can give a hierarchical description in terms of a non-trivial cyclic JSJ-decomposition.

We now briefly explain some of their results. We say that a group G is weakly of type $FP_n(\mathbb{Q})$ (and we denote it as w$FP_n(\mathbb{Q})$) if $H_i(G_0, \mathbb{Q})$ is finite dimensional for all $G_0 < G$ of finite index. They showed that if $\Gamma_1, \ldots, \Gamma_n$ are limit groups and S is a subgroup of type w$FP_n(\mathbb{Q})$ of $\Gamma_1 \times \cdots \times \Gamma_n$, then S is virtually a direct product of limit groups. They also characterised finitely presented residually free groups. Viewing finitely generated residually free groups as subgroups of direct products of limit groups, they showed that if $S < \Gamma_1 \times \cdots \times \Gamma_n$ is finitely generated and $\Gamma_1, \ldots, \Gamma_n$ are limit groups, then S is finitely presented if and only if $p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$, where $p_{i,j}$ is the projection homomorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \Gamma_i \times \Gamma_j$, for $1 \leq i < j \leq n$. As a corollary, the conjugacy and the membership problems are decidable for finitely presented residually free groups.

2.2. **Right-angled Artin groups (RAAGs).** There is a natural class of groups that interpolates between free groups and free abelian groups, called *right-angled Artin groups*. Right-angled Artin groups (RAAGs) were first introduced by A. Baudisch ([2]) in the 1970's and further developed by C. Droms ([19], [20], [21]). The class of RAAGs generalises the class of finitely generated free groups, by allowing relations saying that some or all of the generators commute. In recent years, RAAGs have been studied extensively due to their actions on CAT(0) cube complexes and their rich subgroup structure. As a consequence of the work of I. Agol ([1]) and D. Wise ([33]), groups acting properly by combinatorial automorphisms on a CAT(0) cube complex with special quotient are subgroups of RAAGs. Therefore, many groups that are relevant in geometric group theory can be seen virtually as subgroups of RAAGs, including fundamental groups of closed, irreducible 3-manifolds, all Coxeter groups and one-relator groups with torsion.

Moreover, subgroups of RAAGs play an important role as examples of groups with interesting finiteness properties. If F_2^n is the direct product of n free groups (which is itself a RAAG) of rank two and H_n is the kernel of the homomorphism $F_2^n \mapsto \mathbb{Z}$ sending all the standard generators to 1, then H_n is of type F_{n-1} but not of type FP_n (see [31] and [5]). M. Bestvina and N. Brady in [4] gave a general construction which provides examples of subgroups of direct products of RAAGs having one type of finiteness property but not the other, extending the previous result. In particular, they produced groups of type FP but not finitely presented.

For a given group G, we can define a limit group over G by extending the definition for limit groups. Limit groups arise naturally from several points of view and so they can be defined in several ways. From an algebraic point of view, a *limit group over G* is just a finitely generated fully residually G group. An important fact about these groups is that finitely generated residually G groups are finitely generated subgroups of direct products of limit groups over G.

In [14], M. Casals-Ruiz and I. Kazachkov studied limit groups over RAAGs. They gave a dynamical characterisation of limit groups over RAAGs generalising the work of Sela. They showed that a finitely generated group is a limit group over a RAAG if and only if it acts *nicely* on a real cubing, which is a higher-dimensional generalisation of real trees.

3. Recent results

The richness of subgroups of RAAGs and the previous work on finitely presented subgroups of direct products of free groups motivates the study of subgroups of direct products of RAAGs. Since finitely presented subgroups of direct products of free groups have good algorithmic behaviour, one can wonder if this is the case for all finitely presented subgroups of direct products of RAAGs. In his work [7], M.R. Bridson shows that this is not the case as he shows that there is a RAAG A and a finitely presented subgroup S of $A \times A$ for which the conjugacy problem and the membership problem in $A \times A$ are undecidable. Hence, on the one hand, there is a subclass of the class of

RAAGs containing free groups where finitely presented subgroups of the direct product have a nice structure, and on the other hand, by the result of Bridson, we know that this is not true in the entire class of RAAGs. As a consequence, one may wonder which is the class of RAAGs where finitely presented subgroups of the direct product of groups in this class have good algorithmic behaviour. My research has been focused on answering this question, i.e. studying the structure of subgroups of direct products of certain RAAGs, and more generally, of direct products of limit groups over these RAAGs.

3.1. **Subgroups of direct products of limit groups over Droms RAAGs.** In general, not all subgroups of RAAGs are again RAAGs, and Droms RAAGs are precisely those with the property that all of their finitely generated subgroups are again RAAGs. In particular, finitely generated free groups are Droms RAAGs. They were characterised by Droms as the RAAGs where the defining graph does not contain full subgraphs that are squares or lines of length 3 (see [21]).

In this class of RAAGs, the aforementioned results for limit groups and finitely presented residually free groups can be generalised to limit groups over Droms RAAGs and finitely presented residually Droms RAAGs, respectively.

Theorem 3.1. [28, Theorem 3.1] *If $\Gamma_1, \dots \Gamma_n$ are limit groups over Droms RAAGs and S is a subgroup of $\Gamma_1 \times \cdots \times \Gamma_n$ of type $wFP_n(\mathbb{Q})$, then S is virtually a direct product of limit groups over Droms RAAGs.*

Bridson, Howie, Miller and Short's characterisation of finitely presented residually free groups can also be generalised to finitely presented residually Droms RAAGs. In order to state the result, we introduce the following definition: an embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ of a finitely generated group that is residually a Droms RAAG as a full subdirect product of limit groups over Droms RAAGs is *neat* if Γ_0 is abelian (possibly trivial), $S \cap \Gamma_0$ is of finite index in Γ_0 and Γ_i has trivial center for $i \in \{1, \dots, n\}$.

Theorem 3.2. [28, Theorem 10.1] *Let S be a finitely generated group that is residually a Droms RAAG. The following are equivalent:*

(1) S is finitely presentable;

(2) S is of type $FP_2(\mathbb{Q})$;

(3) S is of type $wFP_2(\mathbb{Q})$;

(4) There exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs such that the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \le i < j \le n$;

(5) For every neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \le i < j \le n$.

Corollary 3.3. *For all $n \in \mathbb{N}$, a residually Droms RAAG is of type F_n if and only if it is of type $FP_n(\mathbb{Q})$.*

Corollary 3.4. [28, Theorem 10.6] *The multiple conjugacy problem is decidable in every finitely presented group that is residually a Droms RAAG.*

Thanks to the understanding of limit groups over Droms RAAGs and finitely presented residually Droms RAAGs, we know other interesting properties. In joint work with D.H. Kochloukova (see [24]), we studied their growth of homology groups and their volume gradients.

Theorem 3.5. [24, Theorem C] *Let Γ be a limit group over a coherent RAAG and let (B_n) be an exhausting normal chain in Γ. Then,*

(1) the rank gradient $RG(\Gamma, (B_n)) = \lim_{n\to\infty} \frac{d(B_n)}{[\Gamma : B_n]} = -\chi(\Gamma)$.

(2) the deficiency gradient $DG(\Gamma,(B_n)) = \lim_{n\to\infty} \frac{def(B_n)}{[\Gamma\colon B_n]} = \chi(\Gamma)$.

(3) the k-dimensional volume gradient $\lim_{n\to\infty} \frac{vol_k(B_n)}{[\Gamma\colon B_n]} = 0$ *for* $k \geq 2$.

Theorem 3.6. [24, Theorem D] *Let K be a field, Γ a limit group over a coherent RAAG and (B_n) an exhausting normal chain in Γ. Then,*

(1) $\lim_{n\to\infty} \frac{\dim_K H_1(B_n,K)}{[\Gamma\colon B_n]} = -\chi(\Gamma)$.

(2) $\lim_{n\to\infty} \frac{\dim_K H_j(B_n,K)}{[\Gamma\colon B_n]} = 0$ *for all* $j \geq 2$.

In [25], we computed the Bieri–Neumann–Strebel–Renz invariants for limit groups over Droms RAAGs and the first Σ-invariant for finitely presented residually Droms RAAGs.

Theorem 3.7. [25, Proposition A] *Let Γ be a limit group over Droms RAAGs such that Γ has trivial center. Then $\Sigma^n(\Gamma) = \Sigma^n(\Gamma,\mathbb{Z}) = \Sigma^n(\Gamma,\mathbb{Q}) = \emptyset$ for every $n \geq 1$.*

Theorem 3.8. [25, Corollary C] *a) Let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1,\ldots,\Gamma_m$ where each Γ_i has trivial center. Then*

$$\Sigma^1(H) = \Sigma^1(H,\mathbb{Q}) = \{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = L_i \text{ for every } 1 \leq i \leq m\},$$

and thus,

$$S(H) \setminus \Sigma^1(H) = \bigcup_{1\leq i\leq m} S(H, \ker(p_i)),$$

where $p_i\colon H \mapsto \Gamma_i$ is the canonical projection.

b) If H is a finitely presented residually Droms RAAG, then there exist finitely many subgroups $H_1,\ldots,H_m$ of H such that

$$S(H) \setminus \Sigma^1(H) = \bigcup_{1\leq i\leq m} S(H, H_i).$$

3.2. **Subgroups of the direct product of two 2-dimensional coherent RAAGs.** A group is *coherent* if all its finitely generated subgroups are also finitely presented. In particular, Droms RAAGs are coherent RAAGs. Droms characterised coherent RAAGs as the RAAGs where the associated graphs do not contain induced cycles of length greater than 3. The RAAG associated to a tree, which is called a *tree group*, is therefore coherent. Tree groups with trivial center are no longer Droms RAAGs and the fact that they have finitely generated subgroups which are not RAAGs make them more difficult to work with.

Tree groups are of special interest, as they are essentially the only examples of 3-manifold groups that are RAAGs. Indeed, Droms proved in [19] that the RAAG GX is the fundamental group of a 3-manifold if and only if each connected component of X is either a tree or a triangle. Hence, GX is the free product of tree groups and free abelian groups of rank three. For instance, the RAAG associated to the path with four vertices, P_4, is the figure 8 knot group. Moreover, 2-dimensional coherent RAAGs (coherent RAAGs where the Salvetti complex has dimension 2) are defined by graphs which are forests, that is, they are free products of tree groups. Thus, the class of tree groups is an appropriate starting point to study the structure of finitely presented subgroups of direct products of coherent RAAGs.

In [16], together with M. Casals-Ruiz, we generalised Baumslag and Roseblade's result for free groups and we described the structure of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs. A new phenomenon appears. Finitely presented subgroups of the direct product of two Droms RAAGs are virtually direct products themselves. In the case of tree groups, the finitely presented subgroups have a more complicated structure: they are virtually abelian extensions of direct products.

Theorem 3.9. [16] *Let S be a finitely presented subgroup of the direct product of two 2-dimensional coherent RAAGs. Then, S is virtually H-by-(free abelian), where H is the direct product of two subgroups of 2-dimensional coherent RAAGs.*

The main reason for this new behaviour comes from the fact that tree groups fiber, while free groups and Droms RAAGs with trivial center do not. Thus, on the one hand, these groups have normal subgroups which are not of finite index, and on the other hand, the intersections of a subgroup with each of the factors need not be finitely generated. However, we exploit other properties of tree groups to obtain the Structure Theorem 3.9.

Proposition 3.10. [16, Proposition 4.4] *Let G be a tree group and let G' be any group. Suppose that $S < G \times G'$ is a finitely presented subdirect product. Define L_1 and L_2 to be $S \cap G$ and $S \cap G'$, respectively, and assume that L_1 is non-trivial. Then, there is $y \in G'$ such that $\langle L_2, y\rangle$ is finitely generated.*

Proposition 3.11. [16, Proposition 4.5] *Let G be a tree group. Suppose that N is a non-trivial finitely generated normal subgroup of G. Then, either N has finite index in G or G/N is virtually $\mathbb{Z}$.*

This structure result in fact applies to a wider class of groups. Since subgroups of coherent RAAGs do not need to be RAAGs themselves, we define a wider class of groups, $\mathcal{G}$, which is the class of finitely generated cyclic subgroup separable graphs of groups with free abelian vertex groups and cyclic edge groups. This class contains 2-dimensional coherent RAAGs and residually finite tubular groups among others.

One of the consequences of the subgroup structure is that finitely presented subgroups have a good algorithmic behaviour:

Corollary 3.12. *Finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs have decidable multiple conjugacy problem and membership problem.*

These finitely presented subgroups also have good finiteness properties. In joint work with D.H. Kochloukova, we showed that finitely presented subgroups of the direct product of two groups in $\mathcal{G}$ are of type F_∞ (see [25, Corollary 5.10]).

Bridson's example of a right-angled Artin group A and an algorithmically bad finitely presented subgroup of $A \times A$ is neither a Droms RAAG nor a 2-dimensional coherent RAAG. Indeed, what we have proved is that in these two classes, finitely presented subgroups of the direct product of two groups do have good algorithmical and finiteness behaviour.

4. Current projects

Recently, I have been studying subgroups of direct products of more than two 2-dimensional coherent RAAGs. In their paper [3], Baumslag and Roseblade mentioned that they could not extend their structure theorem for finitely presented subgroups of the direct product of two free groups to finitely presented subgroups of direct products of finitely many free groups. The generalisation of M.R. Bridson, J. Howie, C.F. Miller and H. Short was published almost 30 years later and the proof is trickier. In the case of 2-dimensional coherent RAAGs also, the methods I used to prove Theorem 3.9 cannot be applied to the direct product of finitely many 2-dimensional coherent RAAGs.

The path that I have taken so far is to follow the steps in [12]. One of the key ingredients in the study of finitely presented subgroups of the direct product of free groups is the fact that the subgroups virtually contain a term of the lower central series, that is, they are a nilpotent extension of a direct product of free groups. This result brings the problem to the world of nilpotent groups. In nilpotent groups, a subgroup has either finite index or it is virtually a subgroup of a kernel of the nilpotent group to $\mathbb{Z}$. This property together with the fact that kernels from the direct product of n free groups to $\mathbb{Z}$ are not of type FP_n are the main ingredients to conclude that if S is a subgroup of the direct product of n free groups of type FP_n, then the nilpotent part actually needs to be finite and so the subgroup is virtually a direct product of free groups.

In the case of tree groups, the situation is more complex, the subgroup is virtually an extension of a direct product by a solvable group. More precisely, we can prove that the solvable group is of the form (finitely generated nilpotent)-by-(finitely generated free abelian). We also understand the finiteness properties of kernels from direct products of tree groups to $\mathbb{Z}$.

As a consequence, we know that if S is a finitely presented subdirect product of $G_1 \times \cdots \times G_n$ where each G_i is a tree group, then $p_{i,j}(S)$ is virtually a kernel of $H_i \times H_j \mapsto \mathbb{Z}^n$ for some $n \in \mathbb{N} \cup \{0\}$, H_i of finite index in G_i. However, this does not give a complete characterisation of finitely presented subgroups in the same way that we have a characterisation of finitely presented residually free groups. The key property for nilpotent groups that allows to fit the subgroup inside a kernel does not hold in the whole generality for solvable groups. Therefore, the problem requires a more subtle study.

Problem 4.1. Give a complete characterisation of finitely presented subgroups of direct products of 2-dimensional coherent RAAGs.

Problem 4.2. For $n \geq 3$, understand the structure of w$FP_n(\mathbb{Q})$ subgroups of the direct product of n 2-dimensional coherent RAAGs.

5. Further directions

There are many questions that arise from the above work. For example, which other properties of subgroups of direct products of limit groups can be generalised to direct products of limit groups over Droms RAAGs or to direct products of 2-dimensional coherent RAAGs?

Recall that we have seen that the multiple conjugacy problem is decidable in every finitely presented group that is residually a Droms RAAG. For finitely presented residually free groups, it is also known the decidability of the membership problem for finitely presented subgroups of finitely presented residually free groups ([10]). A key point in the proof of this result is that limit groups are subgroup separable (see [32]). This is unknown for limit groups over Droms RAAGs.

Question 5.1. Are limit groups over Droms RAAGs subgroup separable?

Question 5.2. Is the membership problem decidable for finitely presented subgroups of finitely presented residually Droms RAAGs?

Recall that one of the consequences of the Structure Theorem 3.9 of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs is that the multiple conjugacy and membership problems are decidable. The isomorphism problem, however, is still open.

Question 5.3. Is the isomorphism problem for finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs decidable?

In our structure theorem, we require the groups to be cyclic subgroup separable. There are many groups that are not cyclic subgroup separable, for instance, non-unimodular (generalised) Baumslag-Solitar groups. I would like to understand subgroups of direct products of these groups.

Problem 5.4. Understand the structure of finitely presented subgroups of the direct product of two non-unimodular (Generalised) Baumslag-Solitar groups.

A next step in this research line would be to study subgroups of direct products of general coherent RAAGs, not just the 2-dimensional coherent RAAGs. General coherent RAAGs split as amalgamated free products where the vertex groups are free abelian, but instead of dealing with infinite cyclic edge groups, in this case we would have to work with free abelian edge groups of rank (possibly) greater than one. For 2-dimensional coherent RAAGs, we first show that the finitely presented subgroup is an extension of a direct product by a $\mathbb{Z}$-by-$\mathbb{Z}$ group. We then prove that this $\mathbb{Z}$-by-$\mathbb{Z}$ group is a quotient of a Baumslag-Solitar group, and using the structural theory of Baumslag-Solitar groups, we conclude that the $\mathbb{Z}$-by-$\mathbb{Z}$ group is in fact free abelian. In the general case of coherent RAAGs, the current proof would need to study groups of the form $\mathbb{Z}^m$-by-$\mathbb{Z}^n$. One can probably reduce the problem to the study of the Leary-Minasyan groups, but still we would need to develop some structural results for these groups.

Problem 5.5. Understand the structure of finitely presented subgroups of direct products of coherent RAAGs of arbitrary dimension.

This research project fits into other more general and long-term goals, which I discuss separately in the next sections.

5.1. **Finiteness properties of groups.** The study of higher finiteness properties of groups began with the work of Serre and Wall in the 1960's and it has remained a rich and active area of research.

The homotopical finiteness property F_k implies the homological finiteness property FP_k, but the converse is not true in general. However, in the case of residually Droms RAAGs, Corollary 3.3 tells us that these two properties are equivalent. It would be fascinating to study the class of groups with this property. That is,

Problem 5.6. Define the largest possible class of groups $\mathcal{G}$ such that for $G \in \mathcal{G}$, G is of type F_k if and only if it is of type FP_k.

A good starting point to attack this problem, based on the previous work, would be to understand if direct products of coherent RAAGs lie in $\mathcal{G}$. I am confident that the finite presentability condition in Theorem 3.9 may be exchanged by FP_2. As a consequence, we would get that if S is a subgroup of type FP_2 of the direct product of two 2-dimensional coherent RAAGs, then S is of type F_∞. In particular, it is finitely presented. Assuming that, as in the case of limit groups, a similar structure theorem may be showed for direct products of 2-dimensional coherent RAAGs, I conjecture the following:

Conjecture 5.7. Subgroups of direct products of finitely many 2-dimensional coherent RAAGs lie in $\mathcal{G}$.

5.2. **Dehn functions of finitely presented subgroups of RAAGs.** Dehn functions capture the isoperimetric behavior of Cayley complexes of groups. RAAGs have Dehn functions that are linear or quadratic, but Dehn functions of finitely presented subgroups of RAAGs vary considerably. For instance, Dehn functions of Bestvina-Brady kernels of RAAGs (kernels of homomorphisms from RAAGs to $\mathbb{Z}$) are at most quartic (see [18]), but M.R. Bridson's example of a RAAG A and an algorithmically bad behaved finitely presented subgroup S of $A \times A$ has exponential Dehn function. Moreover, N. Brady and I. Soroko showed that for each positive integer k there is a 3-dimensional RAAG which contains a finitely presented subgroup with Dehn function $\simeq n^k$ (see [6]). The goal would be to produce examples with other type of Dehn functions.

If our intuition for subgroups with good finiteness properties of direct products of 2-dimensional coherent RAAGs is correct, these should be of Bestvina-Brady type, so in particular, they should have Dehn function at most quartic. As a consequence, another possible approach for understanding these subgroups may pass from computing their Dehn functions.

5.3. $n - (n+1) - (n+2)$ **Conjecture.** In [10], when the authors characterised finitely presented residually free groups, they actually proved a very general result, known as The Virtually Surjective on Pairs Criterion. This criterion states that if S is a subgroup of a direct product $G_1 \times \cdots \times G_n$ of finitely presented groups $G_1, \ldots, G_n$ and if S is virtually surjective on pairs, then S is finitely presented.

After having seen the significance of projections onto pairs, B. Kuckuck suggested in [26] the following generalisation, named The Virtual Surjection Conjecture. Let $n \leq m$ be positive integers and S be a subgroup of a direct product $G_1 \times \cdots \times G_m$, where G_i is of type F_n for $1 \leq i \leq m$. If S is virtually surjective on n-tuples, then S is of type F_n.

The Virtual Surjection Conjecture is a consequence of the $n - (n+1) + (n+2)$ Conjecture. This last conjecture states that if $1 \to N_1 \to \Gamma_1 \xrightarrow{\pi_1} Q$ and $1 \to N_2 \to \Gamma_2 \xrightarrow{\pi_2} Q$ are two short exact sequences of groups, N_1 is of type F_n, both Γ_1 and Γ_2 are of type F_{n+1} and Q is of type F_{n+2}, then the fibre product,

$$P = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \pi_1(\gamma_1) = \pi_2(\gamma_2)\},$$

is of type F_{n+1}.

This conjecture is at its core a topological question, as the finiteness properties F_n are concerned with the existence of classifying spaces of a certain type. To show The Virtually Surjective on Pairs Criterion, M.R. Bridson, J. Howie, C.F. Miller and H. Short actually proved the $1-2-3$ Conjecture. In this case, the topological point of view is translated to a combinatorial characterisation of the second homotopy group of a 2-complex in terms of identity sequences and crossed modules. Unfortunately, such a characterisation is not available for higher homotopy groups.

B. Kuckuck showed that the homological version of the $n-(n+1)-(n+2)$ Conjecture holds in full generality. That is, if we assume that N_1 is of type wFP_n, Γ_1 is wFP_{n+1}, Γ_2 FP_{n+1} and Q F_{n+2}, then P is wFP_{n+1}. Nonetheless, being of type wFP_n does not imply being FP_n (see [27]). B. Kuckuck also showed that the conjecture holds if Q is abelian, or if the second short exact sequence splits. In the case of the abelian quotient, milder conditions are necessary: it is enough if N_1 is F_l and N_2 F_k such that $l + k \geq n$.

I believe that this is a beautiful conjecture, and proving any result related to it would be a great achievement. For instance, a natural question to ask is whether the condition of N_1 being F_l and N_2 being F_k with $l + k \geq n$ are also enough for the general statement.

In addition, recall that finitely presented subgroups of the direct product of two tree groups do not satisfy the Virtually Surjective on Pairs Criterion. So another interesting direction of research is two understand finiteness properties of fibre products of tree groups (or, more generally, coherent RAAGs).

5.4. **Finite index subgroups of RAAGs.** There are several reasons to care about finite index subgroups of RAAGs. Grunewald and Lubotzky presented in [22] a rich collection of linear representations of $Aut(F_n)$ arising through the action of finite index subgroups of it on relation modules of finite quotient groups of F_n. One way of trying to generalise this result to the automorphism group of a RAAG passes through understanding when is it that finite index normal subgroups of RAAGs are again RAAGs. Moreover, understanding finite index subgroups of RAAGs would be helpful to classify RAAGs up to commensurability. M. Casals-Ruiz, I. Kazachkov and A. Zakharov characterised commensurability classes of RAAGs defined by paths (see [15]), but the general classification is not solved.

In [17], P. Dani and I. Levcovitz, with the aim of understanding which RACGs are commensurable to RAAGs, looked to finite index subgroups of RACGs that are isomorphic to RAAGs. They gave graph-theoretic conditions to obtain a complete characterisation, so I wonder if the same can be carried out for RAAGs.

5.5. **Kernels of RAAGs to polycyclic groups.** My motivation for understanding these kernels comes from the work in progress. Recall that in the case of the direct product of finitely many 2-dimensional coherent RAAGs, finitely presented subgroups are an extension of a direct product by a polycyclic group. There are some places in the literature where it seems that this study would be also helpful. For instance, in [30], A. Minasyan proves that RAAGs are hereditary conjugacy separable. He also showes that if G is a RAAG and N is a finitely generated normal subgroup such that G/N is polycyclic, then N is hereditary conjugacy separable and has solvable conjugacy problem. He points out that the case when G/N is abelian is of particular interest, because in this case one can tell whether or not the given normal subgroup N is finitely generated using Σ-invariants.

Recently, D. Kielak has constructed a polytope (see [23]), based on Thurston's polytope for controlling which $M \mapsto S^1$ fibres (here M is a 3-manifold), to understand which epimorphism $G \mapsto \mathbb{Z}$ fibres (here G is free-by-cyclic). It would be interesting to study if a similar polytope can be constructed for epimorphisms onto polycyclic groups by means of an inductive argument. In that case, a similar theory to the theory of the Bieri–Neumann–Strebel–Renz invariants could be developed.

5.6. **Category of RAAGs.** Even though this project diverges from the point of view of geometric group theory, I believe that it is interesting to understand the categorical behaviour of RAAGs. If we denote by GrpR the full subcategory of the category of groups consisting of RAAGs, there are many questions which could be considered to begin with.

The inclusion functor $\iota\colon \mathrm{GprR} \mapsto \mathrm{Grp}$ preserves all finite products and coproducts. Extending this, one can study the behaviour with respect to other (co)limits, and in general, the finite (co)completeness of this category. It is quite easy to show that the category GrpR does not have equalisers, so it is not finitely complete. However, the existence of coequalisers is more difficult to decide.

There is another obvious functor FinGraph $\mapsto$ GrpR to consider. One may investigate the kind of structure preserved/reflected by the functor, if it is a left/right adjoint, a topological functor, etc.

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