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Moduli Problems of Elliptic Curves: Introduction

Goal: define and study the arithmetic of moduli problems of elliptic curves

Two classical approaches:

① Deligne-Rapoport (1972): Les schémas de modules de courbes elliptiques

$\rightsquigarrow$ via "stacks"

② Katz-Mazur (1985): Arithmetic Moduli of Elliptic Curves

$\rightsquigarrow$ via "relatively representable" moduli problems

We'll follow ②.

1. Category of Elliptic Curves

E/k : (E, O) with: • E : "nice" curve $/k$ of genus 1
elliptic curve over a field $k = \bar{k}$ • $O \in E(k)$ a marked point

Facts: i) $E \hookrightarrow \mathbb{P}^2_k$ via a Weierstrass eqn

ii) E has a natural group scheme structure with $O = \mathrm{id}$.

S : a scheme

elliptic curve over S: $E/S = (\pi : E \to S, O)$

where: • π is a morphism of schemes: proper, flat and its fibers are smooth curves of genus 1

• $O \in E(S)$: a marked section

Rmk: we can view E/S as a family of elliptic curves:

$x \in S \rightsquigarrow E_x/k(x)$: elliptic curve over the residue field of $x \in S$

Facts: i) In general, E/S cannot be embedded in $\mathbb{P}^2_S$ – but a local version of this occurs

ii) E/S has a natural group scheme structure

Def: Category of Elliptic Curves: Ell

$\mathrm{Ob}(\mathrm{Ell})$: E/S – elliptic curves over schemes

$\mathrm{Hom}(E/S, E'/S')$: pairs $(f : S' \to S, g : E' \to E)$ s.t.:

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

is Cartesian and $O_{S'} : S' \to E'$ induced by O is equal to O'.

2. Moduli Problems

Def: a "moduli problem of elliptic curves" is a contravariant functor: $P : \mathrm{Ell} \to \underline{\mathrm{Set}}$

Def: $E/S \in \mathrm{Ob}(\mathrm{Ell})$, P a moduli problem of elliptic curves

$P(E/S)$ is a set and we call any element of $P(E/S)$ a "P-structure" on E/S

We define: Ell_P : category of elliptic curves with P-structures

Objects: $(E/S, \alpha)$ $\alpha \in P(E/S)$

morphisms: $\phi \in \mathrm{Hom}((E/S, \alpha), (E'/S', \alpha'))$: $\phi \in \mathrm{Hom}(E'/S', E/S)$ s.t.: $P(\phi)$ takes α to α'

Forgetful functor: $\mathcal{F}_P : \mathrm{Ell}_P \to \mathrm{Ell}$
$(E/S, \alpha) \mapsto E/S$

Def: we call a functor $\mathcal{F} : \mathcal{C} \to \underline{\mathrm{Set}}$ representable if there exists an object $X \in \mathrm{Ob}(\mathcal{C})$ st: $\mathcal{F} \cong h^X$ where:

$h^X : \mathcal{C} \to \underline{\mathrm{Set}}$: $Y \mapsto \mathrm{Hom}(Y, X)$

Our main question will be the following:

Given a moduli problem of elliptic curves, is it representable?

Rmk: $E/S \in \mathrm{Ob}(\mathrm{Ell})$: $h^{E/S} : \mathrm{Ell} \to \underline{\mathrm{Set}}$: trivially representable
$E'/S' \mapsto \mathrm{Hom}(E'/S', E/S)$

Notation: $E/S \in \mathrm{Ob}(\mathrm{Ell})$: $\mathrm{Ell}_{E/S} := \mathrm{Ell}_{h^{E/S}}$
category of elliptic curves with $h^{E/S}$-structure

$\mathcal{F}_{E/S}$: corresponding forgetful functor.

Def: P moduli problem of elliptic curves

P is "relatively representable" if: $\forall E/S \in \mathrm{Ob}(\mathrm{Ell})$:

$P \circ \mathcal{F}_{E/S} : \mathrm{Ell}_{E/S} \to \underline{\mathrm{Set}}$ is representable.

Thm: every representable moduli problem is relatively representable

Notation: P representable

$X_P := E_P/S_P \in \mathrm{Ob}(\mathrm{Ell})$: representing object of P

S_P : called "fine moduli scheme associated to P"

E_P : called "universal elliptic curve over S_P"

P : moduli problem

$E/S \in \mathrm{Ob}(\mathrm{Ell}) \to \mathrm{Aut}(E/S) = \{g : E \xrightarrow{\sim} E : \pi g = g\}$
$= (\pi : E \to S, O)$

$\mathrm{Aut}(E/S) \circlearrowright P(E/S)$: since P is a functor

Def: with notation as above, P is "rigid" if $\mathrm{Aut}(E/S)$ acts freely on $P(E/S)$ $\forall E/S \in \mathrm{Ob}(\mathrm{Ell})$

Rmk: P relatively representable $\Rightarrow$ $P \times h^{E/S}$ relatively representable $\forall E/S \in \mathrm{Ob}(\mathrm{Ell})$

$X_{E/S} := \hat{E}_{E/S}/\hat{S}_{E/S}$: representing object of $P \times h^{E/S}$

Def: P relatively representable moduli problem

P has a scheme theoretic property (eg. affine/étale/...) if $\forall E/S$: $\hat{S}_{E/S}$ has that scheme theoretic property

Thm (Katz-Mazur) Let P be relatively representable & affine.

Then: P is representable $\Leftrightarrow$ P is rigid.

If P is representable, then S_P is affine.

If P is étale & representable then: S_P is a smooth curve $/\mathbb{Z}$

3. Basic Moduli Problems

$[\Gamma(n)]$: $E/S \longmapsto \{\text{iso. } (\mathbb{Z}/n\mathbb{Z})^2_S \xrightarrow{\sim} E[n]\}$

$[\Gamma_1(n)]$: $E/S \longmapsto \{(\mathbb{Z}/n\mathbb{Z})_S \hookrightarrow E[n]\}$

$[\Gamma_0(n)]$: $E/S \longmapsto \{\text{subschemes of } E[n], \text{ locally iso. to } (\mathbb{Z}/n\mathbb{Z})\}$

Thm (Katz-Mazur)

- $\forall n \geq 1$: $[\Gamma(n)], [\Gamma_1(n)], [\Gamma_0(n)]$ are relatively representable, finite and étale
- $\forall n \geq 4$: $[\Gamma(n)], [\Gamma_1(n)]$ are representable and their moduli schemes are smooth affine curves over $\mathrm{Spec}\, \mathbb{Z}[1/n]$
- $\forall n \geq 1$: $[\Gamma_0(n)]$ is not rigid (and hence not representable)

Rmk: moduli schemes as above are what we usually call "affine" or "open" modular curves – they are not proper $/\mathbb{Z}$

We'll discuss how they can be compactified via Drinfeld level structures.