20/10/23

Elliptic Curves over general base schemes and Rigidification

Today: ① define elliptic curves over general schemes

② study $[N]: E \to E/S$ & $E[N] := \mathrm{Ker}([N])$

↳ and the special case: N invertible in S

③ Rigidification in moduli problems

④ $\Gamma(N)$ & $\Gamma_1(N)$ are rigidifications

Elliptic Curves

S: scheme

smooth curve $/S$ = $\pi: C \to S$ morphism

smooth, separated, finite presentation

and of relative dimension 1

N.B: smooth $\Rightarrow$ flat

elliptic curve E/S = smooth curve with geometrically connected fibers of genus 1

with a chosen: $e: S \to E$ (identity section)

Ex: 0) E/k: elliptic curves over fields

1) $E: y^2 = x^3 + ax + b \subseteq \mathbb{P}^2_{\mathbb{Z}}$ $a, b \in \mathbb{Z}$

$\Delta = 4a^3 + 27b^2 \neq 0$

distinct prime factors: $p_1 \dots p_N$ of Δ

$$\begin{array}{c} E \\ \downarrow \\ S = \mathrm{Spec}\, \mathbb{Z} \setminus \{p_1 \dots p_N\} \end{array}$$

2) E as above, $p \nmid \Delta$ prime

$$\begin{array}{ccc} E_{\mathbb{Z}_p} & & \mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}_p) = \{0, \eta\} & & \mathbb{Q}_p \end{array}$$

3) Legendre-family: $E: y^2 = x(x-1)(x-t) \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{A}^1_t$

$$\begin{array}{c} \downarrow \\ \mathbb{A}^1_t \setminus \{0,1\} \end{array}$$

Thm: $\underset{\substack{\| \\ (E \to S, e)}}{E/S}$ elliptic curve $\Rightarrow$ E is an S-group scheme with e = identity element

Sketch:

recall: $G \to S$ is an S-group scheme if $\exists$ morphisms:

$$\left.\begin{array}{l} m: G \times G \to G \\ i: G \to G \\ e: S \to G \end{array}\right\} \text{ all satisfying group axioms}$$

$$h: \begin{array}{ccc} \mathrm{Sch}_{/S}^{opp} & \longrightarrow & [\mathrm{Sch}_{/S}^{opp}, \mathrm{Set}] \\ \rotatebox{90}{$\in$} & & \\ X & \longmapsto & \mathrm{Hom}_S(-, X) \end{array}$$

Yoneda's Lemma: h is fully-faithful

$$\begin{tikzcd} \mathrm{Sch}_{/S}^{opp} \arrow[r, "h_{G/S}"] \arrow[dr, dotted, "\exists (*)"'] & \mathrm{Set} \\ & \mathrm{Grp} \arrow[u, "\text{forgetful}"'] \end{tikzcd}$$

Lemma: X/S is an S-group scheme $\Leftrightarrow$ $\exists$ a factorisation $(*)$ as above

E/S elliptic curve

Claim: E/S group scheme – enough to show:

$h \cong [\mathrm{Sch}_{/S}^{opp} \to \mathrm{Gp}]$ some functor

$\to$ take: $\mathrm{Pic}: (T \xrightarrow{\varphi} S) \to \dfrac{\{\mathcal{L} \in \mathrm{Pic}(E_T) : \deg \mathcal{L}_t = 0 \ \forall t \in T\}}{\langle f_T^* \mathcal{L}_0 : \mathcal{L}_0 \in \mathrm{Pic}(T) \rangle}$

$(f_T: E_T \to T)$

$\square$

Def: $\mathcal{M}: \mathrm{Sch}_{/B}^{opp} \to \underline{\mathrm{Set}}$ "Moduli problem"

$(S \to B) \mapsto \left\{ \begin{array}{c} X \\ \downarrow \\ S \end{array} : \text{"objects"}/S \right\} / \text{iso.}$

fine moduli problem: $\mathcal{M} \cong h_P$ ($\mathcal{M}$ representable)

P: fine moduli space

Slogan: "non-trivial automorphisms obstruct representability"

Classical: $E/\mathbb{C}$: $\# \mathrm{Aut}_{\mathbb{C}}(E) = \begin{cases} 2 & o/w \\ 4 & j = 1728 \\ 6 & j = 0 \end{cases}$

Assume $\mathcal{M}$ representable: $\Phi: h_P \xrightarrow{\sim} \mathcal{M}$

$\mathrm{Hom}_B(P, P) \xrightarrow{\sim} \mathcal{M}(P/B)$

$\mathrm{id}_P \longmapsto u$ "universal-family"

non-trivial automorphisms $\Rightarrow$ $X \to S$ isotrivial

X_s fibers all iso.

two solutions: ① stacks

② Rigidity: $\# \mathrm{Aut}_S(E + \text{data}) = 1$

eg. data = 1) $\Gamma(n)$ structure

2) $\Gamma_1(n)$ structure

E/S elliptic curve – is a group scheme

$\Rightarrow \forall N \in \mathbb{Z}_{\geq 1}$: $[N]: E \to E$ "multiplication by N map"

- group homomorphism
- Kernel: $E[N]/S$ subscheme

↳ $\forall s \in S$: $E[N]_s$ = N-torsion of E_s

Properties: 1) $[N]: E \to E/S$ is flat, finite and locally free

$\Rightarrow$ thus its "rank" can be defined

$\mathrm{rank}([N]) := N^2$

2) If N is invertible in S: $E[N]$ is étale $/S$

Thm (Rigidity of $\Gamma(N)$) E/S elliptic curve, $N \geq 2$, S connected

$\varepsilon \in \mathrm{Aut}_S(E)$ st: $\varepsilon|_{E[N]} = \mathrm{id}$

Then: 1) $N = 2 \Rightarrow \varepsilon = \pm \mathrm{id}$

2) $N \geq 3 \Rightarrow \varepsilon = \mathrm{id}$

Thm (Rigidity of $\Gamma_1(N)$) E/S elliptic curve, $N \geq 4$, S connected

$\varepsilon \in \mathrm{Aut}_S(E)$, $G \subseteq E$ finite, locally free of rank N st:

$\varepsilon|_G = \mathrm{id}$

Then: $\varepsilon = \mathrm{id}$