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3/11/23 (Talk 5)

Representability and Moduli Problems for Elliptic Curves

E/S : elliptic curve

$n \in \mathbb{Z}_{\geq 1}$

Recall : 4 basic n-structures on E/S

- $\Gamma(n)$-structures (or full level n structures)

  group hom. : $\phi : (\mathbb{Z}/n\mathbb{Z})^2 \to E[n]$ s.t. :

  $$\sum_{a \in (\mathbb{Z}/n\mathbb{Z})^2} [\phi(a)] = E[n] \quad : \text{equality of Cartier divisors}$$

- $\Gamma_1(n)$-structures : group hom. : $\phi : (\mathbb{Z}/n\mathbb{Z}) \to E[n]$ s.t. :

  $$\sum_{a \in (\mathbb{Z}/n\mathbb{Z})} [\phi(a)] \subseteq E[n] \quad \text{subgroup-scheme}$$

- Balanced $\Gamma_1(n)$-structures : $\begin{cases} \phi : E \to E' & n\text{-isogeny} \\ \hat{\phi} : E' \to E & \text{: dual isogeny} \\ P \in \ker(\phi), \hat{P} \in \ker(\hat{\phi}) & \text{: generators} \end{cases}$

- $\Gamma_0(N)$-structure : cyclic n-isogeny $\phi : E \to E'$

  N.B: cyclic = fppf locally on S and $\ker(\phi)$ admits a generator

Goal : understand the moduli problems of elliptic curves defined by the above

Today : prove "representability" and formally define moduli problems of elliptic curves

1. Some Category Theory

Def : $\mathcal{C}$ a category

a moduli problem for $\mathcal{C}$ is a contravariant functor:

$$P : \mathcal{C} \to \underline{\text{Sets}}$$

- for any $X \in \text{Ob}(\mathcal{C})$: $P(X)$ is the set of level P structures on X

General problem : understand the functor:

$$\mathcal{M}_{g,n} : \underline{\text{Sch}} \to \text{Sets}$$
$$S \longmapsto \{[C/S, P_1 .. P_n \in C(S)] : C : \text{curve of genus } g\}$$

N.B : $g = 1, n = 1$: elliptic curves

Def : $P : \mathcal{C} \to$ sets moduli problem

- P is representable if $\exists\, \mathcal{M}(P) \in \text{Ob}(\mathcal{C})$ s.t. $\forall X \in \text{Ob}(\mathcal{C})$:

  there exists a bijection of sets : $P(X) \cong \text{Hom}_{\mathcal{C}}(X, \mathcal{M}(P))$

  – which is functorial in X

  * such $\mathcal{M}(P)$ is called a fine moduli space for P

- If $\exists\, \mathcal{M}(P) \in \text{Ob}(\mathcal{C})$ and a natural transformation of functors : $\tau : P \to \text{Hom}_{\mathcal{C}}(*, \mathcal{M}(P))$ and τ is "universal"

  $\Rightarrow \mathcal{M}(P)$ is called a coarse moduli space for P

Rmk : fine is coarse

Objects + morphisms ↗ ↖ objects only

2. Representability

- fix E/S : elliptic curve & $n \in \mathbb{Z}_{\geq 1}$

define : $P_A : \underline{\text{Sch}}_{/S} \to \underline{\text{Sets}}$

$$T \longmapsto \{A\text{-structures on } E_T\}$$

where $A \in \{\Gamma(n), \Gamma_1(n), \text{Balanced } \Gamma_1(n), \Gamma_0(n)\}$

Thm (Katz-Mazur 3.6) With notation as above:

P_A is representable for $A \in \{\Gamma(n), \Gamma_1(n), \text{Balanced } \Gamma_1(n)\}$ and the representing object is finite (as an S-scheme).

Sketch : C/S : smooth, commutative gp scheme$/S$ of relative dimension 1.

T : S-scheme : $C[N](T) := \ker([N] : C(T) \to C(T))$

$A \cong (\mathbb{Z}/N_1\mathbb{Z}) \times .. \times (\mathbb{Z}/N_t\mathbb{Z})$ finite abelian group

Prop 1.6.2 : $\text{Sch}_{/S} \to \text{Sets}$

$T \longmapsto \{A\text{-structures on } C_{/T}\}$

is represented by a closed subscheme of :

$$\text{Hom}_{S\text{-gp}}(A, C) \cong C[N_1] \times .. \times C[N_t]$$

– defined locally by : $1 + |A| + |A|^2$ eqns.

for the above result : take $A = (\mathbb{Z}/n\mathbb{Z})^2$ & $(\mathbb{Z}/n\mathbb{Z})$ for $\Gamma(n)$ and $\Gamma_1(n)$ resp.

- similar argument for Balanced $\Gamma_1(n)$ □

Thm (Katz-Mazur 3.7) $n \geq 1$ and S a $\mathbb{Z}[1/n]$-scheme

E/S : elliptic curve

For all 4 as above : P_A is representable by a finite, étale S-scheme

Sketch :

① n-invertible $\Rightarrow E[n]$ is finite étale$/S$ and locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2_S$

② results of (1.5) – (1.8) similar to above

① + ② $\Rightarrow$ representability.

A	representing object
$\Gamma(n)$	$S \times \{\text{basis of } (\mathbb{Z}/n\mathbb{Z})^2\}$
$\Gamma_1(n)$	$S \times \{P \in (\mathbb{Z}/n\mathbb{Z})^2 : \#P = n\}$
Balanced $\Gamma_1(n)$	$S \times \{(K, P, P') : K \subseteq (\mathbb{Z}/n\mathbb{Z})^2 \text{ cyclic}, \#K = n, \langle P \rangle = K, \langle P' \rangle = (\mathbb{Z}/n\mathbb{Z})^2/K\}$
$\Gamma_0(n)$	$S \times \{K \subseteq (\mathbb{Z}/n\mathbb{Z})^2, K : \text{cyclic}, \#K = n\}$

□

3. Formal Moduli Problems of Elliptic Curves

Def : Category of Elliptic Curves : Ell

Objects : E/S

Morphisms : $\text{Hom}_{\text{Ell}}(E'/S', E/S)$.

$(f : E' \to E, g : S' \to S)$ s.t. :
$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
is Cartesian

i.e. • commutative

• $E' \xrightarrow{\sim} E \times_S S'$ is an iso. of elliptic curves $/S'$

Rmk : the category Ell is the "modular stack" defined by Deligne-Rapoport

Def : a moduli problem for elliptic curves is a contravariant functor : $P : \text{Ell} \to \underline{\text{Sets}}$

- E/S : $P(E/S)$ = set of level P-structures on E/S

fix a moduli problem for elliptic curves : P

Def : P is relatively representable if : $\forall E/S$:

the functor : $\text{Sch}_{/S} \to \text{Sets}$

$$T \longmapsto P(E_T/T)$$

is representable by an S-scheme : $P_{E/S}$

Def : P is representable if it's representable as a functor on Ell

i.e. $\exists\, \mathbb{E}/\mathcal{M}(P)$ and a functorial isomorphism:

$$P(E/S) \cong \text{Hom}_{\text{Ell}}(E/S, \mathbb{E}/\mathcal{M}(P))$$

Rmk : associated to P we have a functor:

$$\mathcal{F} : \underline{\text{Sch}} \to \underline{\text{Sets}} : S \longmapsto \{[E/S, \alpha] : E/S \text{ elliptic curve}, \alpha \in P(E/S)\}$$

- If P is represented by $\mathbb{E}/\mathcal{M}(P) \Rightarrow \mathcal{M}(P)$ represents $\mathcal{F}$
- Converse also holds : if we assume P is also "rigid"

Observation : E/S any elliptic curve $\Rightarrow \text{Hom}_{\text{Ell}}(*, E/S)$ is trivially a representable moduli problem

Prop : P representable $\Rightarrow$ P is relatively representable

Pf : $\forall E/S$: the corresponding functor is represented by :

$$P_{E/S} = \text{Isom}_{S \times \mathcal{M}(P)}(\pi_1^* E, \pi_2^* \mathbb{E})$$

$$\left.\begin{array}{l} \pi_1 : E \times \mathbb{E} \to E \\ \pi_2 : E \times \mathbb{E} \to \mathbb{E} \end{array}\right\} \text{projections}$$

□

Prop : P representable, P' relatively representable

"product functor" : $P \times P' : E/S \longmapsto P(E/S) \times P'(E/S)$

– is representable by $P'_{\mathbb{E}/\mathcal{M}(P)}$