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Geometric Properties of Moduli Problems

(Kat)

$\mathcal{P} : \mathcal{C}^{op} \to$ Sets : representable if $\exists\, \mathcal{M}(\mathcal{P}) \in Ob(\mathcal{C})$ st:

$\mathcal{P}(X) \cong Hom_{\mathcal{C}}(X, \mathcal{M}(\mathcal{P})) \;\; \forall X \in Ob(\mathcal{C})$

ie. $\mathcal{P} \cong Hom_{\mathcal{C}}(\,, \mathcal{M}(\mathcal{P}))$

(iso. of functors)

Yoneda : there is a bijective correspondence :

$Hom(Hom(-, X), \mathcal{P}) \longleftrightarrow \mathcal{P}(X)$

natural transformations   values of $\mathcal{P}$ at X

- $\mu_X \in Hom(Hom(*, X), \mathcal{P})$, $\xi := \mu_X(id_X)$

Def : a universal object is a pair : (X, ξ)

$X \in \mathcal{C}$ $\xi \in \mathcal{P}(X)$

s.t : $\forall Y \in Ob(\mathcal{C})$ and $\beta \in \mathcal{P}(Y)$: $\exists!$: $f : Y \to X$ s.t :

$\beta = \mathcal{P}(f)(\xi)$

$(X, \xi) \longleftrightarrow \mu$

universal object   natural transformation

Prop : representable $\Longleftrightarrow \exists$ universal object

Rigidity : f : morphism in Ell :
$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ S_1 \arrow[r, "g"'] & S \end{tikzcd}$$

comes with an iso. of elliptic curves :

$E_1 \cong E \times_S S_1$

$\mathcal{P} : (Ell)^{opp} \to$ Sets : induces a map :

$\mathcal{P}(E/S) \longrightarrow \mathcal{P}(E_1/S_1)$

$Aut(E/S) := \{ f : E \to E : \begin{tikzcd}[column sep=small] E \arrow[rr, "f"] \arrow[dr] & \circlearrowleft & E \arrow[dl] \\ & S & \end{tikzcd} \}$

$\hookrightarrow$ elements of $Aut(E/S)$ induce maps :

$\mathcal{P}(E/S) \to \mathcal{P}(E/S)$ - by functoriality

ie. $Aut(E/S)$ acts on $\mathcal{P}(E/S)$ on the right

Def : the moduli problem $\mathcal{P}$ is rigid if $\forall$ E/S and $\alpha \in \mathcal{P}(E/S)$: $(E/S, \alpha)$ has no non-trivial automorphisms.

equivalently :

$f \in Aut(E/S)$ fixes $\alpha \Longleftrightarrow f = id$

ie. $Aut(E/S)$ acts freely on $\mathcal{P}(E/S)$

Prop : representable $\Rightarrow$ rigid

Pf : assume $\mathcal{P}$ is representable by $\mathbb{E}/\mathcal{M}(\mathcal{P})$

$\Rightarrow \exists$ universal object : $\alpha \in \mathcal{P}(\mathbb{E}/\mathcal{M}(\mathcal{P}))$

so $\forall$ E/S & $\beta \in \mathcal{P}(E/S)$: $\exists!$ $f : E/S \to \mathbb{E}/\mathcal{M}(\mathcal{P})$

s.t : $\beta = \mathcal{P}(f)(\alpha)$

and an isomorphism : $i : E \xrightarrow{\sim} \mathbb{E} \times_{\mathcal{M}(\mathcal{P})} S$

$\varphi \in Aut(E/S)$ and the induced map fixes β

$\Rightarrow i \circ \varphi = i \Longleftrightarrow \varphi = id$

(since i is an iso. of elliptic curves) $\square$

Def : a moduli problem $\mathcal{P}$ is étale/affine over Ell if :

- $\mathcal{P}$ is relatively representable
- the morphism : $\mathcal{P}_{E/S} \to S$ is étale / affine

[N.B : $\mathcal{P}$ is étale if : • $\mathcal{P}$ is relatively representable locally of finite presentation

• $\forall$ E/S and every closed subscheme $S_0 \subseteq S$ defined by a nilpotent ideal :

$\mathcal{P}(E/S) \to \mathcal{P}(E \times_S S_0 / S_0)$ is bijective]

Thm 4.7 : $\mathcal{P}$ moduli problem, relatively representable and affine over (Ell) . Then :

$\mathcal{P}$ is representable $\Longleftrightarrow \mathcal{P}$ is rigid

Sketch : "$\Rightarrow$" clear

"$\Leftarrow$" equivalent statements over $\mathbb{Z}[1/2]$, $\mathbb{Z}[1/3]$

+ known properties of "naive level 3" and "Legendre"

+ recollement $\square$

Cor : $\left.\begin{array}{c} \text{relatively representable} \\ + \\ \text{affine + étale + rigid} \end{array}\right\} \Rightarrow$ representable by a smooth affine curve / $\mathbb{Z}$

Cor : The naive level N moduli problem, $N \geq 3$ is representable by a smooth affine curve : $Y(N) / \mathbb{Z}[1/N]$

Modular Families

- $\mathbb{E}/\mathcal{M}$: called a modular family if the moduli problem it represents $\mathcal{P}$ is étale over (Ell)

elliptic curve

- a collection of modular families $\{\mathbb{E}_i/\mathcal{M}_i\}$ cover (Ell) if the corresponding moduli problems $\{\mathcal{P}^{(i)}\}$ satisfy:

$\forall$ E/S : $\coprod_i \mathcal{P}^{(i)}_{E/S} \to S$ is étale and surjective

Rmk : if $(N, M) = 1$: the 2 modular families corresponding to $Y(N)$ and $Y(M)$ cover (Ell)

N.B : $\mathbb{E}/\mathcal{M}$ modular family $\Rightarrow \mathcal{M}$ is a smooth curve / $\mathbb{Z}$.

Let Q be a property of schemes which is local for the étale topology

$\mathcal{P}$: relatively representable moduli problem

then : $\mathcal{P}$ has property Q if for any modular family : $\mathbb{E}/\mathcal{M}$ the $\mathcal{M}$-scheme : $\mathcal{P}_{\mathbb{E}/\mathcal{M}}$ has property Q .