

Geometric Properties of Moduli Problems

(Kat)

$\mathcal{P}: \mathcal{C}^{op} \rightarrow \text{Sets}$: representable if $\exists \mathcal{M}(\mathcal{P}) \in \text{ob}(\mathcal{C})$ st:

$$\mathcal{P}(X) \cong \text{Hom}_{\mathcal{C}}(X, \mathcal{M}(\mathcal{P})) \quad \forall X \in \text{ob}(\mathcal{C})$$

ie. $\mathcal{P} \cong \text{Hom}_{\mathcal{C}}(_, \mathcal{M}(\mathcal{P}))$

(iso. of functors)

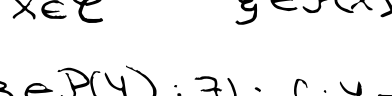
Yoneda: there is a bijective correspondence:

$$\text{Hom}(\text{Hom}(_, X), \mathcal{P}) \longleftrightarrow \mathcal{P}(X)$$

natural transformations values of \mathcal{P} at X

$\cdot \mu_X \in \text{Hom}(\text{Hom}(_, X), \mathcal{P}), \quad \xi := \mu_X(1_X)$

Def: a universal object is a pair: (X, ξ)



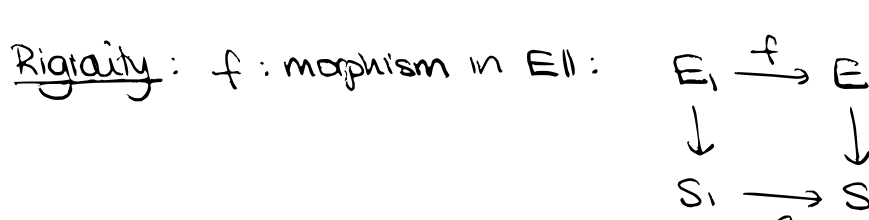
st: $\forall Y \in \text{ob}(\mathcal{C})$ and $\beta \in \mathcal{P}(Y)$: $\exists!$: $f: Y \rightarrow X$ st:

$$\beta = \mathcal{P}(f)(\xi)$$

$$(X, \xi) \longleftrightarrow \mu$$

universal object natural transformation

Prop: representable $\iff \exists$ universal object



comes with an iso. of elliptic curves:

$$E_1 \cong E \times_S S_1$$

$\mathcal{P}: (\mathcal{E}ll)^{op} \rightarrow \text{Sets}$: induces a map:

$$\mathcal{P}(E/S) \rightarrow \mathcal{P}(E_1/S_1)$$

$$\text{Aut}(E/S) := \left\{ f: E \rightarrow E : \begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow \cong & & \downarrow \cong \\ S & & S \end{array} \right\}$$

\hookrightarrow elements of $\text{Aut}(E/S)$ induce maps:

$$\mathcal{P}(E/S) \rightarrow \mathcal{P}(E/S) \text{ - by functoriality}$$

ie. $\text{Aut}(E/S)$ acts on $\mathcal{P}(E/S)$ on the right

Def: the moduli problem \mathcal{P} is rigid if $\forall E/S$ and

$\alpha \in \mathcal{P}(E/S)$: $(E/S, \alpha)$ has no non-trivial automorphisms.

equivalently:

$$f \in \text{Aut}(E/S) \text{ fixes } \alpha \iff f = \text{id}$$

ie. $\text{Aut}(E/S)$ acts freely on $\mathcal{P}(E/S)$

Prop: representable \implies rigid

Pf: assume \mathcal{P} is representable by $\mathbb{E}/\mathcal{M}(\mathcal{P})$

$$\implies \exists \text{ universal object: } \alpha \in \mathcal{P}(\mathbb{E}/\mathcal{M}(\mathcal{P}))$$

so $\forall E/S \neq \beta \in \mathcal{P}(E/S)$: $\exists!$: $f: E/S \rightarrow \mathbb{E}/\mathcal{M}(\mathcal{P})$

$$\text{s.t. } \beta = \mathcal{P}(f)(\alpha)$$

and an isomorphism: $i: E \xrightarrow{\sim} E \times_{\mathcal{M}(\mathcal{P})} S$

$\varphi \in \text{Aut}(E/S)$ and the induced map fixes β

$$\implies i \circ \varphi = i \iff \varphi = \text{id}$$

(since i is an iso. of elliptic curves) \square

Def: a moduli problem \mathcal{P} is étale/affine over $\mathcal{E}ll$ if:

- $\cdot \mathcal{P}$ is relatively representable
- \cdot the morphism: $\mathcal{P}_{E/S} \rightarrow S$ is étale/affine

Γ N.B: \mathcal{P} is étale if: $\cdot \mathcal{P}$ is relatively representable

locally of finite presentation

$\cdot \forall E/S$ and every closed subscheme $S_0 \subseteq S$

defined by a nilpotent ideal:

$$\mathcal{P}(E/S) \rightarrow \mathcal{P}(E \times_S S_0/S_0) \text{ is bijective} \quad \square$$

Thm 4.7: \mathcal{P} moduli problem, relatively representable and affine over $\mathcal{E}ll$. Then:

\mathcal{P} is representable $\iff \mathcal{P}$ is rigid

Sketch: " \implies " clear

" \impliedby " equivalent statements over $\mathbb{Z}[Y_2], \mathbb{Z}[Y_3]$

+ known properties of "naive level 3"

and "legendre"

+ recollement \square

Cor: relatively representable $\left\{ \begin{array}{l} + \\ \text{affine} + \text{étale} + \text{rigid} \end{array} \right\} \implies$ representable by a smooth affine curve/ \mathbb{Z}

Cor: The naive level N moduli problem, $N \geq 3$ is representable by a smooth affine curve: $\mathcal{Y}(N)/\mathbb{Z}[Y_N]$

Modular Families

$\cdot \mathbb{E}/\mathcal{M}$: called a modular-family if the moduli problem elliptic curve it represents \mathcal{P} is étale over $\mathcal{E}ll$

\cdot a collection of modular-families $\{\mathbb{E}_i/\mathcal{M}_i\}$ cover $\mathcal{E}ll$

if the corresponding moduli problems $\{\mathcal{P}^{(i)}\}$ satisfy:

$$\forall E/S: \bigcup_i \mathcal{P}_{E/S}^{(i)} \rightarrow S \text{ is étale and surjective}$$

Rmk: if $(N, M) = 1$: the 2 modular families corresponding to $\mathcal{Y}(N)$ and $\mathcal{Y}(M)$ cover $\mathcal{E}ll$

N.B: $\mathbb{E}/\mathcal{M} \implies \mathcal{M}$ is a smooth curve/ \mathbb{Z} .

modular-family

let \mathcal{Q} be a property of schemes which is local for the étale topology

\mathcal{P} : relatively representable moduli problem

then: \mathcal{P} has property \mathcal{Q} if for any modular-family: \mathbb{E}/\mathcal{M}

the \mathcal{M} -scheme: $\mathcal{P}_{\mathbb{E}/\mathcal{M}}$ has property \mathcal{Q} .