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Katz-Mazur Chapter 5

Thm 5.1.1: Let $N \geq 2$.
The moduli problems $[\Gamma(N)]$, $[\Gamma_1(N)]$, [balanced $\Gamma_1(N)$] and $[\Gamma_0(N)]$ is relatively representable over (Ell).
Each is finite, flat over (Ell) and regular.
Each tensored with $\mathbb{Z}[1/N]$ is finite & étale over $(Ell_{\mathbb{Z}[1/N]})$.

Rmk: sufficient to consider $N = p^n$, $n \geq 1$, p: prime — due to factorisation of moduli problems

$\mathcal{P}$: moduli problem

Reg1. $\mathcal{P}$: relatively representable and flat over Ell

Reg2. $\mathcal{P} \otimes \mathbb{Z}[1/p]$ finite étale over $Ell_{\mathbb{Z}[1/p]}$

Reg3. $\mathcal{P}$ depends only on its p-division group

ie. E/S, E'/S s.t. $E[p^\infty] \cong E'[p^\infty]$

$\Rightarrow \mathcal{P}_{E'/S} \simeq \mathcal{P}_{E/S}$

Reg4. $k = \bar{k}$, $char(k) = p$, E_0/k supersingular

$E/W(k)[[T]]$: universal deformation of E_0/k

(4A) $\mathcal{P}(E_0/k) = \{$single object$\}$

(4B) $\mathcal{P}_{E/W(k)[[T]]}$ is the spectrum of a regular 2-dimensional ring.

Thm 5.21: any moduli problem $\mathcal{P}$ satisfying (Reg1) - (Reg4) is finite, flat over Ell, regular and constant of rank ≥ 1

Applications of (5.1.1)

Cor 5.5.4: E/S elliptic curve, $N \geq 2$ integer, then:

1. Locally fppf on S: any $\Gamma_1(N)$-structure on E/S can be completed to a $\Gamma(N)$-structu

2. locally fppf on S: E/S admits $\Gamma_1(N)$, balanced $\Gamma_1(N)$ and $\Gamma(N)$ structures.

3. K cyclic, $K \subseteq E[N]$ + other conditions $\Rightarrow E[N]/K = K'$ is cyclic and has "good" properties

4. $P \in E(S)$ has exact order N $\iff$ P can be completed to a $\Gamma(N)$-structure: (P,Q) on E/S

Cor 5.5.5: $P \in E[N](S)$ has exact order N on E/S,
$Q \in E[N](S)$, $NQ = 0$
$\langle P \rangle = K \subseteq E[N]$, $E' = E/K$
$K' = \pi(E[N])$ $\quad \pi: E \to E/K$

Then:
(P,Q) is a $\Gamma(N)$-structure $\iff$ (P, K, Q') is a balanced $\Gamma_1(N)$ structure on E/S
ie. $\mathbb{Z}P + \mathbb{Z}Q = E[N]$ $\quad Q' := \pi(Q)$

In this case, it follows: $K' = \langle Q' \rangle$

Thm 5.5.7: E/S elliptic curve
$P, Q \in E[N](S)$. Then:

1. If P,Q is a Drinfeld basis for $E[N]$, then: $\forall D | N$: DP, DQ is a Drinfeld basis for $E[N/D]$

2. P has exact order N $\Rightarrow \forall D | N$: DP has exact order N/D on E/S

3. $N = p^n$, $p \geq 2$, $n \geq 2$: (P,Q) is a Drinfeld basis $\iff$ (pP, pQ) is a Drinfeld basis for $E[p^{n-1}]$