Quotients of Moduli Problems

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G : finite group

$\mathcal{P}$: moduli problem for Ell/R (some ring)

G acts on $\mathcal{P}$ if : $\forall E/S$ G acts on $\mathcal{P}(E/S)$ and the following commutes:

$$E_1/S_1 \to E/S \quad \rightsquigarrow \quad \begin{array}{ccc} G \times \mathcal{P}(E/S) & \longrightarrow & \mathcal{P}(E/S) \\ \downarrow & & \downarrow \\ G \times \mathcal{P}(E_1/S_1) & \longrightarrow & \mathcal{P}(E_1/S_1) \end{array}$$

__Rmk__ : $\mathcal{P}$ relatively representable & $G \curvearrowright \mathcal{P}$ $\Rightarrow$ G acts on $\mathcal{P}_{E/S}$ $\forall E/S$

__EX__ : $\mathcal{P} = [\Gamma_1(N)]$, $G = (\mathbb{Z}/N\mathbb{Z})^\times$

G acts on $\mathcal{P}$: $a.P := aP \quad P \in \mathcal{P}(E/S)$

it follows : $\mathcal{P}(E/S)/G = [\Gamma_0(N)](E/S)$

- $\mathcal{P}, \mathcal{P}'$ relatively representable, both with a G-action and suppose $\exists$ a G-equivariant : $\mathcal{P} \to \mathcal{P}'$

Def : $\mathcal{P}'$ is __the__ quotient of $\mathcal{P}$ by G if :

1) G acts trivially on $\mathcal{P}'$
2) for any representable moduli problem $\mathcal{S}$, étale: $\mathcal{M}(\mathcal{S},\mathcal{P})/G$ and is isomorphic to $\mathcal{M}(\mathcal{S},\mathcal{P}')$

   ↳ equivalently : $\forall$ modular family E/S $(\mathcal{P}_{E/S})/G$ exists and is isomorphic to : $\mathcal{P}'_{E/S}$

__Thm__ : $\mathcal{P}$: relatively representable and affine
G : some finite group acting on $\mathcal{P}$

1) $\mathcal{P}/G$ exists, is relatively representable and affine

any relatively rep^n $\mathcal{P}'$ with : trivial G action, then any G-equivariant map $\mathcal{P} \to \mathcal{P}'$ factors uniquely through : $\mathcal{P} \to \mathcal{P}/G$

2) $\mathcal{P}$ as a torsor ...
3) $\forall E/S$: $\mathcal{P}_{E/S}/G$ exists and there exists a map:

$$\mathcal{P}_{E/S}/G \to (\mathcal{P}/G)_{E/S}$$: bijective on geometric points

Moreover : If any of the following hold, the above map is an isomorphism.

a) E/S is flat over Ell/R ← as a representable moduli problem
b) $\#G$ is invertible in S
c) G acts freely

4) $\mathcal{P} \to \mathcal{P}/G$ is finite
5) $\mathcal{P}$ normal $\Rightarrow$ $(\mathcal{P}/G)$ normal
6) "finiteness"

Pf idea : $\mathcal{Q}$: rel. rep^n moduli problem

$E/S \mapsto \mathcal{Q}_{E/S}$: rel. rep^n moduli problems $\xrightarrow{\sim}$ $(Sch/S \to Sch/R)$

- good if G acts free
- G not acting freely : add extra ℓ-structure for some prime ℓ □

__Rmk__ : $R \to R'$ ring hom., $\mathcal{P}$, G as above

$\exists$ morphism : $(\mathcal{P} \otimes_R R')/G \to (\mathcal{P}/G) \otimes_R R'$ ↓ an isomorphism if:

(n.b : always surjective)

- $R \to R'$ is flat

__or__ • $\#G$ is invertible in R'

__or__ • G acts free

__Prop__ : (Descent) $\mathcal{P}, \mathcal{P}'$: relatively representable & affine

Suppose $\exists \mathcal{P} \to \mathcal{P}'$ G-equivariant

If further G acts freely on the RHS then we get a Cartesian diagram :

$$\begin{array}{ccc} \mathcal{P} & \to & \mathcal{P}/G \\ \downarrow & & \downarrow \\ \mathcal{P}' & \to & \mathcal{P}'/G \end{array}$$

equivalently : $\forall E/S$

$$\begin{array}{ccc} \mathcal{P}_{E/S} & \to & (\mathcal{P}/G)_{E/S} \\ \downarrow & & \downarrow \\ \mathcal{P}'_{E/S} & \to & (\mathcal{P}'/G)_{E/S} \end{array}$$

__Prop__ : Quotients of Products

$\mathcal{P}_1, \mathcal{P}_2$ relatively representable, affine

$G_i \curvearrowright \mathcal{P}_i$, with $\mathcal{P}_1$ flat, $\mathcal{P}_2/G_2$ flat

$\Rightarrow (\mathcal{P}_1, \mathcal{P}_2)/(G_1 \times G_2) \simeq (\mathcal{P}_1/G_1, \mathcal{P}_2/G_2)$

Applications

- $GL_2(\mathbb{Z}/N\mathbb{Z}) \curvearrowright [\Gamma(N)]$: acts on Drinfeld bases in the usual way
- $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) \curvearrowright [\text{bal}.\Gamma_1(N)]$:

$$(a,b) \cdot (P, E \rightleftarrows E', P') := (a \cdot P, E \rightleftarrows E', bP')$$

- $(\mathbb{Z}/N\mathbb{Z})^\times \curvearrowright [\Gamma_1(N)]$: as previously defined

__Thm__ :

	map of moduli problems	quotient
1)	$d \mid N$: $[\Gamma(N)] \longrightarrow [\Gamma(d)]$ $(P,Q) \longmapsto (N/d\, P, N/d\, Q)$	$[\Gamma(d)] \cong [\Gamma(N)]/*$ $* = \{A \in GL_2(\mathbb{Z}/N\mathbb{Z}) : A \equiv I \bmod d\}$
2)	$[\Gamma(N)] \longrightarrow [\text{bal}.\Gamma_1(N)]$ $(P,Q) \longmapsto (P, Q \bmod P)$	$[\text{bal}.\Gamma_1(N)] \cong [\Gamma(N)]/\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$
3)	$[\Gamma(N)] \longrightarrow [\Gamma_1(N)]$ $(P,Q) \longmapsto P$	$[\Gamma_1(N)] \cong [\Gamma(N)]/\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$
4)	$[\Gamma(N)] \longrightarrow [\Gamma_0(N)]$ $(P,Q) \longmapsto \langle P \rangle$	$[\Gamma_0(N)] \cong [\Gamma(N)]/\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
5)	$[\text{bal}.\Gamma_1(N)] \longrightarrow [\Gamma_1(N)]$ $(P, E \rightleftarrows E', P') \mapsto (E, P)$	$[\Gamma_1(N)] \cong [\text{bal}.\Gamma_1(N)]/1 \times (\mathbb{Z}/N\mathbb{Z})^\times$
6)	$[\text{bal}.\Gamma_1(N)] \longrightarrow [\Gamma_0(N)]$ $(P, E \rightleftarrows E', P') \mapsto (E, \ker(E \to E'))$	$[\Gamma_0(N)] \cong [\text{bal}.\Gamma_1(N)]/(\mathbb{Z}/N\mathbb{Z})^{*2}$
7)	$[\Gamma_1(N)] \longrightarrow [\Gamma_0(N)]$ $P \longmapsto \langle P \rangle$	$[\Gamma_0(N)] \cong [\Gamma_1(N)]/(\mathbb{Z}/N\mathbb{Z})^*$

Pf : (1) clear : $\mathcal{P} \to \mathcal{P}'$ G-equivariant and $G \curvearrowright \mathcal{P}'$ trivial

$\mathcal{M}(\mathcal{S}, \Gamma(N))$ nice & N invertible in S

– follows from previous □

__Regularity__ :

$\mathcal{P}$: relatively rep^n moduli problem and satisfies regularity axioms : (R1) – (R4) for some p

$G \curvearrowright \mathcal{P}$ (G : some finite group)

assume : (G1) G acts freely on $\mathcal{P} \otimes \mathbb{Z}[1/p]$

(G2) G action only depends on p-divisible gps

(G3) some deformation assumptions

__Thm__ Assume (G1) – (G3). Then :

1) $\mathcal{P}/G$ is finite, flat over Ell/R, constant rank ≥ 1 and regular

+ deformation statements

2) $\mathcal{P} \to \mathcal{P}/G$ finite, flat, of degree : $\#G$

⋮

+ deformation statements

Applications

__Thm__ : 1) $\forall p$ prime, $n \geq 1$

$G \subseteq \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subseteq GL_2(\mathbb{Z}/p^n\mathbb{Z})$ or $G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$

then : $[\Gamma(p^n)/G]$ is regular, finite and flat

2) $\forall G, H \subseteq (\mathbb{Z}/p^n\mathbb{Z})^\times$: $[\text{bal}.\Gamma_1(p^n)]/G \times G$

$[\Gamma_1(p^n)]/G$

– regular, finite, flat