Durham University

Department of Mathematical Sciences

Project IV (MATH4072) Report

The Algebraic Topology of Smooth Manifolds: Differential Forms and De Rham Cohomology

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Friday 2nd May, 2025

Declaration

This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

Acknowledgements

I would like to thank my supervisor Dr Martin Kerin for his support throughout the project.

Abstract

The goal of Algebraic Topology is to construct functors from a category of topological spaces to the category of a specific algebraic structure. A particularly important class of such functors are cohomology theories which assign sequences of R-modules to topological spaces which satisfy the Eilenberg-Steenrod axioms and importantly only depend on the homotopy type of the space. By restricting to the category of smooth manifolds, we can produce a notion of cohomology: de Rham Cohomology. This report details the construction of differential forms and de Rham Cohomology with the major goal being to show that de Rham Cohomology satisfies a version of the Eilenberg-Steenrod axioms.

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I. Introduction

1.1 Motivation

Algebraic Topology can be broadly summed up as the study of algebraic invariants that can be defined on certain classes of topological spaces which allow us to distinguish them up to homeomorphism or homotopy equivalence. In the context of Category Theory, this amounts to constructing functors from the category of topological spaces and continuous maps to a category of algebraic objects with the relevant homomorphisms [BBT20].

One particularly important and interesting class of spaces are *manifolds*, which are spaces that locally look like Euclidean space. It is possible to generalise the notions of calculus on $\mathbb{R}^n$ by defining *smooth structures* on manifolds. A manifold with a smooth structure is called a *smooth manifold*. We can exploit this smooth structure to produce rich algebraic structure, the highlight of which will be *differential forms* and the *exterior derivative*. Together, they allow us to define an algebraic invariant called *de Rham cohomology* which turns out to only depend on the homotopy type of the manifold [Lee12]. The main goal of this report is to show that, in fact de Rham cohomology satisfies a version of each Eilenberg-Steenrod Axiom, not just the homotopy axiom!

We proceed first by briefly discussing some category theory since it provides us with a consistent language for discussing rigorously the association of algebraic structures to topological spaces. We then introduce some notions from homological algebra which give us the tools to actually construct examples of such algebraic structures.

1.2 Some Category Theory

Category theory is not only studies objects but also the relationships between them. In this way, we define categories to contain information about both objects and their relations. We can then talk about mapping between different categories, formalising the notion of associating algebraic structures to topological spaces [BBT20].
The following is adapted from *Topology: A Categorical Approach* [BBT20].

Definition 1.2.1. A *category*, $\mathcal{C}$ consists of the following:

1. A class of *objects* $Ob(\mathcal{C})$
2. For any two objects $X, Y \in Ob(\mathcal{C})$ a set of *morphisms*, $\mathrm{Hom}(X, Y)$
3. For all $X \in Ob(\mathcal{C})$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called the *identity morphism*
4. For all $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$, there exists a morphism $g \circ f \in \mathrm{Hom}(X, Z)$ called the *composition* of f and g

satisfying

5. $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$ for all $f \in \mathrm{Hom}(X, Y)$,
6. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $h \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$, $f \in \mathrm{Hom}(Z, W)$.

Example 1.2.2.

$\mathcal{C}$	$Ob(\mathcal{C})$	$\mathrm{Hom}(X, Y)$
Set	Sets	Set functions
Grp	Groups	Group homomorphisms
$\mathbf{Vect_k}$	Vector Spaces over the field k	k-Linear maps
Top	Topological Spaces	Continuous maps
hTop	Topological Spaces	Homotopy classes of maps
R-Mod	R-Modules where R is a ring	R-Linear maps

Looking at **Top** in more detail, we note that for a topological space X, the identity map $\mathrm{id}_X : X \to X$ is always continuous and for topological spaces X, Y and Z and continuous maps $f : X \to Y$ and $g : Y \to Z$, their composition $g \circ f : X \to Z$ is continuous. Moreover, composition of set functions is associative.

In each of the categories above, we have a notion of when the objects are "the same". For example, for topological spaces we have the notion of homeomorphisms (Definition A.1.3) and for groups we have the notion of group isomorphisms. If two spaces are homeomorphic or two groups are isomorphic, they have the same structure in the appropriate context. We can generalise this to any category.

Definition 1.2.3. Let $\mathcal{C}$ be a category, $X, Y \in Ob(\mathcal{C})$ and $f \in \mathrm{Hom}(X, Y)$.

- We say f is *left invertible* if there exists a morphism $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f = \mathrm{id}_X$. The morphism g is called a *left inverse* of f.
- We say f is *right invertible* if there exists a morphism $g \in \mathrm{Hom}(Y, X)$ such that $f \circ g = \mathrm{id}_Y$. The morphism g is called a *right inverse* of f.
- We say f is *invertible* if it is both left and right invertible. We then say f is an *isomorphism*.
- If an isomorphism $f \in \mathrm{Hom}(X, Y)$ exists, we say X and Y are *isomorphic* and we write $X \cong Y$.

Lemma 1.2.4. *Let $\mathcal{C}$ be a category, $X, Y \in Ob(\mathcal{C})$ and $f \in \mathrm{Hom}(X, Y)$ be an isomorphism. If $g, h \in \mathrm{Hom}(Y, X)$ are a left and right inverse of f respectively, then $g = h$. We call g an **inverse** of f. Moreover, the inverse of f is unique.*

Proof. First, using associativity we have that

$$g = g \circ \mathrm{id}_X = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_Y \circ h = h.$$

Next suppose that $g, \tilde{g} \in \mathrm{Hom}(Y, X)$ are both inverses of f. Then

$$g = g \circ \mathrm{id}_X = g \circ (f \circ \tilde{g}) = (g \circ f) \circ \tilde{g} = \mathrm{id}_Y \circ \tilde{g} = \tilde{g}.$$

So the inverse is indeed unique. □

In the spirit of study objects and the relationships between them, we want to study relationships between categories. The key information in a category are the identity morphisms and compositions of morphisms, so this is what we want to preserve, analogous to how group homomorphisms preserve the group operation. This motivates the next definition.

Definition 1.2.5. Let $\mathcal{C}$ and $\mathcal{D}$. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ consists of two functions:

- $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
- $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for each $X, Y \in Ob(\mathcal{C})$

which satisfy

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, for all $X \in Ob(\mathcal{C})$,
- $F(f \circ g) = F(f) \circ F(g)$, for all $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is similar but it instead maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$ and so instead satisfies $F(f \circ g) = F(g) \circ F(f)$, for all $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

The definition of contravariant functors may seem a bit arbitrary: why should we care about functors which swap the order of compositions? It turns out that many important objects in algebraic topology and beyond behave this way, including de Rham cohomology! Here is another very important example which is adapted from *Introduction to Smooth Manifolds* by Lee to fit this context [Lee12].

Example 1.2.6. Let k be a field. We can consider $\mathrm{Hom}(-,k)$ as a functor $\mathbf{Vect_k} \to \mathbf{Vect_k}$ in the following way.
Given a vector space V over k, we define its *dual space* to be $V^* := \mathrm{Hom}(V,k)$. V^* has a natural vector space by defining addition and scalar multiplication pointwise, e.g. for $f, g \in V^*$,
$(f+g)(v) := f(v) + g(v)$ for all $v \in V$.
Given a linear map $f : V \to W$ we can define its *dual map* $f^* : W^* \to V^*$ by $f^*\varphi = \varphi \circ f$. Since the composition of linear maps is linear, $\varphi \circ f \in V^*$. Moreover, f^* is a linear map. Let $\varphi, \psi \in W^*$ and $a, b \in k$, then for all $v \in V$

$$f^*(a\varphi + b\psi)(v) = (a\varphi + b\psi)(f(v)) = a\,\varphi(f(v)) + b\,\psi(f(v)) = a\,f^*\varphi(v) + b\,f^*\psi(v).$$

So we have that $f^*(a\varphi + b\psi) = a\,f^*\varphi + b\,f^*\psi$.
We also have for $v \in V$ and $\varphi \in V^*$ that $\mathrm{id}_V^*\,\varphi(v) = \varphi(\mathrm{id}_V(v)) = \varphi(v)$. So $\mathrm{id}_V^* = \mathrm{id}_{V^*}$ and given $f \in \mathrm{Hom}(V,W)$ and $g \in \mathrm{Hom}(W,U)$ we have that

$$(g \circ f)^*\varphi(v) = \varphi(g(f(v))) = g^*\varphi(f(v)) = f^*(g^*\varphi)(v) = (f^* \circ g^*)\varphi(v).$$

So $(g \circ f)^* = f^* \circ g^*$.
Now we see that if we let $\mathrm{Hom}(-,k)(f) = f^*$, $\mathrm{Hom}(-,k)$ maps vector spaces to their duals and linear maps to their dual maps in such a way that composition is reversed. So $\mathrm{Hom}(-,k)$ is a contravariant functor.

Remark 1.2.7. It is possible to generalise this functor to **R-Mod**. See the next section for the definition of an R-module.

What makes functors so useful is that they preserve isomorphisms, as the next theorem states.

Theorem 1.2.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $X, Y \in Ob(\mathcal{C})$. Then*

$$X \cong Y \implies F(X) \cong F(Y).$$

Proof. I give the proof in the case that F is a contravariant functor. The covariant case is analogous. Since $X \cong Y$, there exists an isomorphism $f \in \mathrm{Hom}(X,Y)$ and let $g \in \mathrm{Hom}(Y,X)$ be its inverse. Then $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Applying F to both of these equations we get that

$$\begin{aligned} &F(f \circ g) = F(\mathrm{id}_Y) \text{ and } F(g \circ f) = F(\mathrm{id}_X) \\ \implies &F(g) \circ F(f) = \mathrm{id}_{F(Y)} \text{ and } F(f) \circ F(g) = \mathrm{id}_{F(X)}\,. \end{aligned}$$

So $F(f)$ and $F(g)$ are inverse of each other and are hence isomorphisms. So $F(X) \cong F(Y)$. □

Remark 1.2.9. In general, the converse is not true. For example, we can think of the fundamental group (See A.2.6) as a functor from the category of path connected spaces with morphisms as homotopy classes of loops to the category of groups. We have that $\pi_1(\mathbb{R}^2) \cong \pi_1(S^2)$ but $\mathbb{R}^2$ and S^2 are not homotopy equivalent (a proof that these are indeed not homotopy equivalent is given in chapter 5).

This result is particularly powerful because given a suitable functor, it allows us to determine quite quickly if two objects are not isomorphic!

Corollary 1.2.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $X, Y \in Ob(\mathcal{C})$. Then*

$$F(X) \ncong F(Y) \implies X \ncong Y.$$

We can actually go one step further in this realm of studying the relationships between objects by considering how functors may relate to each other. We do this by studying natural transformations.

Definition 1.2.11. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* η from F to G consists of a morphism $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ for all $X \in Ob(\mathcal{C})$ such that for any $f \in \mathrm{Hom}(X,Y)$ the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

That is $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

1.3 Homological Algebra and Cohomology Theories

As we have discussed briefly, algebraic topology is broadly the study of functors from a category of topological spaces to a category of some algebraic structures. Category theory gives us the general language to describe this but how do we actually go about constructing such functors? Homological algebra gives us one way to do this by providing a general framework for associating a sequence of groups/R-modules to topological spaces. The overall method is to first produce a functor which takes topological spaces to either chain or cochain complexes and then, as outlined below, we get other functors from those complexes to groups which we call the homology or cohomology groups of the complex respectively [Fri]. For our purposes, it is sufficient to just study cochain complexes and cohomology.

Before we define cochain complexes, we recall the definition of an R-module.

Definition 1.3.1. [Mun84] Let R be a commutative ring. An *R-module* is an Abelian group M with an operation $\cdot : R \times M \to M$, called scalar multiplication, which satisfied the following for all $a, b \in M$ and $\alpha, \beta \in R$

1. $\alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b$.
2. $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a$.
3. $\alpha \cdot (\beta \cdot a) = (\alpha\beta) \cdot a$.
4. $1 \cdot a = a$.

Let M be an R-module and let $S \subseteq M$. We say S is a *submodule of M* if it is a subgroup of M and is also an R-module with the same scalar multiplication.

Given two R-modules M and N, we call a function $\varphi : M \to N$ an *R-linear map* if φ is a group homomorphism and for all $\alpha \in R$ and $a \in M$ we have $\varphi(\alpha \cdot a) = \alpha \cdot \varphi(a)$. If $\varphi : M \to N$ is an R-linear map, we define its *kernel* to be the set $\ker(\varphi) := \{r \in M : \varphi(r) = 0\}$ and its *image* to be the set $\text{im}(\varphi) := \varphi(s) : s \in M$. $\ker(\varphi)$ is a submodule of M and $\text{im}(\varphi)$ is a submodule of N.

Examples 1.3.2. [Mun84]

1. If k is a field, then it is a ring. So vector spaces over k are also k-modules.
2. If G is an Abelian group, we can define a scalar multiplication $\cdot : \mathbb{Z} \times G \to G$ to be the n-fold sum $n \cdot g = g + \cdots + g$. In this way, we see that any Abelian group is a $\mathbb{Z}$-module.

Definition 1.3.3. [Fri] Let R be a commutative ring. A *cochain complex over R*, $(C^\bullet, \delta_\bullet)$, is a sequence of R-modules, C^n, and R-linear maps $\delta_n : C^n \to C^{n+1}$ which we call *coboundary maps*, such that $\delta_{n+1} \circ \delta_n = 0$ for all $n \geq 0$:

$$0 \longrightarrow C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} \cdots \longrightarrow C^n \xrightarrow{\delta_n} C^{n+1} \longrightarrow \cdots$$

We call C^n the *nth cochain module* and elements of C^n are called *n-cochains* or simply *cochains*.

From the fact that $\delta_{n+1} \circ \delta_n = 0$, we immediately get this useful result.

Corollary 1.3.4. *Let $(C^\bullet, \delta_\bullet)$ be a cochain complex. Then* $\text{im}(\delta_n) \subseteq \ker(\delta_{n+1})$.

Proof. Let $b \in \text{im}(\delta_n)$. Then there exists $c \in C^n$ such that $b = \delta_n(c)$. Then $\delta_{n+1}(b) = \delta_{n+1} \circ \delta_n(c) = 0$. So $b \in \ker(\delta_{n+1})$. □

This fact allows us to define a sequence of R-modules associated to a cochain complex by considering the following quotient modules.

Definition 1.3.5. [Fri] Let $(C^\bullet, \delta_\bullet)$ be a cochain complex. We call elements of $\ker(\delta_n)$ *n-cocycles* and elements of $\mathrm{im}(\delta_{n-1})$ *n-coboundaries*. We define the *nth cohomology group* of $(C^\bullet, \delta_\bullet)$ to be the quotient

$$H^n(C^\bullet) := \frac{\ker(\delta_n)}{\mathrm{im}(\delta_{n-1})}.$$

If $z, w \in C^n$ are n-cocycles such that $[z] = [w]$, we say they are *cohomologous*. That is, z and w are cohomologous if $\exists\, b \in \mathrm{im}(\delta_{n-1})$ such that $z - w = \delta_{n-1}(b)$.

Remark 1.3.6. Corollary 1.3.4 implies that all n-coboundaries are n-cocycles.

Continuing with the theme of studying objects and the relations between them, we want to consider maps between cochain complexes: cochain maps. The definition may seem arbitrary at first but it will then use it to show that we get induced maps between cohomology groups.

Definition 1.3.7. [Fri] Let R be a ring and $(C^\bullet, \delta_\bullet)$ and $(D^\bullet, \varepsilon_\bullet)$ be two cochain complexes over R. A *cochain map* $f_\bullet : C^\bullet \to D^\bullet$ is a sequence of R-linear maps $f_n : C^n \to D^n$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{\delta_{n-1}} & C^n & \xrightarrow{\delta_n} & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow{\varepsilon_{n-1}} & D^n & \xrightarrow{\varepsilon_n} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

That is, $f_{n+1} \circ \delta_n = \varepsilon_n \circ f_n$ for all $n \geq 0$.

Lemma 1.3.8. *[Fri] Let R be a commutative ring. Then cochain complexes over R with cochain maps forms a category which we denote* **Ch**.

Proof. We have two things to check: existence of identity cochain maps and the composition of a cochain map is a cochain map. First let $(C^\bullet, \delta_\bullet), (D^\bullet, \varepsilon_\bullet)$ and $(E^\bullet, \eta_\bullet)$ be cochain complexes over R and let $f_\bullet : C^\bullet \to D^\bullet$ and $g_\bullet : D^\bullet \to E^\bullet$ be cochain maps. Define $h_\bullet : C^\bullet \to E^\bullet$ by $h_n = g_n \circ f_n$ for all $n \geq 0$. Then since $f^\bullet$ and $g^\bullet$ are cochain maps, we have

$$\begin{aligned}
& h_{n+1} \circ \delta_n = (g_{n+1} \circ f_{n+1}) \circ \delta_n = g_{n+1} \circ (f_{n+1} \circ \delta_n) = g_{n+1} \circ (\varepsilon_n \circ f_n) \\
\implies\ & h_{n+1} \circ \delta_n = (g_{n+1} \circ \varepsilon_n) \circ f_n = (\eta_n \circ g_n) \circ f_n = \eta_n \circ h_n.
\end{aligned}$$

Then the associativity of R-linear maps gives us that the composition of cochain maps is also associative. Now consider $\mathrm{id}^\bullet_C : C^\bullet \to C^\bullet$ defined by $\mathrm{id}^n_C := \mathrm{id}_{C^n}$. Then

$$\mathrm{id}^{n+1}_C \circ \delta_n = \mathrm{id}_{C^{n+1}} \circ \delta_n = \delta_n = \delta_n \circ \mathrm{id}_{C^n} = \delta_n \circ \mathrm{id}^n_C\,.$$

So $\mathrm{id}^\bullet_C$ is a cochain map. Moreover, since for each $n \geq 0$, id_{C^n} is the identity map, we have for any cochain map $f_\bullet : C^\bullet \to D^\bullet$ that

$$\mathrm{id}^\bullet_D \circ f_\bullet = f_\bullet \circ \mathrm{id}^\bullet_C\,.$$

□

Now given a cochain map, we can define well defined homomorphisms between the respective cohomology groups.

Lemma 1.3.9. *[Fri] Let $(C^\bullet, \delta_\bullet)$ and $(D^\bullet, \varepsilon_\bullet)$ be two cochain complexes over a commutative ring R and let $f_\bullet : C^\bullet \to D^\bullet$ be a cochain map. Then the map $f^* : H^n(C^\bullet) \to H^n(D^\bullet)$ given by $f^*[z] = [f_n(z)]$ is a well-defined R-linear map.*

Proof. The fact that f^* is R-linear follows from the definition of quotient modules. It remains to show that if $z, w \in C^n$ are cohomologous n-cocycles, i.e. $[z] = [w]$, we have that $f^*[z] = f^*[w]$. Let $b \in \mathrm{im}(\delta_{n-1})$ such that $z - w = \delta_{n-1}(b)$. Then $f_n(z - w) = f_n(\delta_{n-1}(b)) = \varepsilon_n(f_n(b))$. So $f_n(z) - f_n(w) = \varepsilon_n(f_n(b))$. Hence, $f_n(z)$ and $f_n(w)$ are cohomologous, as required. □

Corollary 1.3.10. *Let $n \geq 0$. The assignment of a cochain complex to its nth cohomology group and cochain maps to their induced homomorphisms is a covariant functor $H^n : \mathbf{Ch} \to \mathbf{R\text{-}Mod}$.*

Proof. Let $(C^\bullet, \delta_\bullet), (D^\bullet, \varepsilon_\bullet)$ and $(E^\bullet, \eta_\bullet)$ be cochain complexes over R and let $f_\bullet : C^\bullet \to D^\bullet$ and $g_\bullet : D^\bullet \to E^\bullet$ be cochain maps. Then if $z \in C^n$ is an n-cocycle, we have

$$(g \circ f)^*[z] = [(g_n \circ f_n)(z)] = [g_n(f_n(z))] = g^*[f_n(z)] = g^*(f^*[z]) = (g^* \circ f^*)[z],$$

and

$$(\mathrm{id}_C)^*[z] = [\mathrm{id}_{C^n}(z)] = [z] = \mathrm{id}_{H^n(C^\bullet)}[z].$$

So $(g \circ f)^* = g^* \circ f^*$ and $(\mathrm{id}_C)^* = \mathrm{id}_{H^n(C^\bullet)}$. □

A natural question to ask is "Do two different cochain maps ever induce the same homomorphisms in cohomology?" This motivates the next definition.

Definition 1.3.11. [Fri] Let $(C^\bullet, \delta_\bullet)$ and $(D^\bullet, \varepsilon_\bullet)$ be two cochain complexes over a commutative ring R and let $f_\bullet, g_\bullet : C^\bullet \to D^\bullet$ be two cochain maps. A *cochain homotopy* between $f_\bullet$ and $g_\bullet$ is a family $h = \{h_n\}_{n \geq 0}$ of R-linear maps $h_n : C^n \to D^{n-1}$ such that for all $n \geq 0$

$$\varepsilon_n \circ h_n + h_{n+1} \circ \delta_n = f_n - g_n.$$

If such a cochain homotopy exists, we say $f_\bullet$ and $g_\bullet$ are *cochain homotopic*.

Lemma 1.3.12. *[Fri] Let $f_\bullet, g_\bullet : C^\bullet \to D^\bullet$ be two cochain homotopic maps. Then $f^* = g^* : H^n(C^\bullet) \to H^n(D^\bullet)$. That is, cochain homotopic maps induce the same homomorphisms in cohomology.*

Proof. Let h be a cochain homotopy between $f_\bullet$ and $g_\bullet$ and let $z \in C^n$ be an n-cocycle. To show that $f^*[z] = g^*[z]$ we show that $f_n(z)$ and $g_n(z)$ are cohomologous. Since z is a cocycle, $\delta_n(z) = 0$. So we have

$$f_n(z) - g_n(z) = \varepsilon_n \circ h_n(z) + h_{n+1} \circ \delta_n(z) = \varepsilon_n(h_n(z)).$$

So $f_n(z)$ and $g_n(z)$ are indeed cohomologous and hence $[f_n(z)] = [g_n(z)]$, as required. □

We now introduce some tools which relate the cohomology groups of certain cochain complexes which will be invaluable for calculations later on.

Definition 1.3.13. [Mun84] We say the following (potentially infinite) sequence of R-modules

$$0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots \longrightarrow M_n \xrightarrow{\varphi_n} M_{n+1} \longrightarrow \cdots$$

is *exact at* M_n if $\mathrm{im}(\varphi_{n-1}) = \ker(\varphi_n)$. We say the sequence is *exact* if it is exact at M_n for all $n \geq 0$. If the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact, we call it a *short exact sequence*.

We also have a notion of short exact sequences for cochain complexes which play an important role in the calculation of cohomology groups.

Definition 1.3.14. [Lee12] A *short exact sequence of cochain complexes* consists of three cochain complexes $A^\bullet, B^\bullet, C^\bullet$

$$0 \longrightarrow A^\bullet \xrightarrow{f_\bullet} B^\bullet \xrightarrow{g_\bullet} C^\bullet \longrightarrow 0 \tag{1.1}$$

such that for all $n \geq 0$ the sequence

$$0 \longrightarrow A^n \xrightarrow{f_n} B^n \xrightarrow{g_n} C^n \longrightarrow 0$$

is exact.

We can now state the Snake Lemma for cohomology, which is the result that allows us to compare the cohomology groups of different complexes. The proof is analogous to the proof of the Snake Lemma for homology which is proven in Algebraic Topology IV and is therefore omitted.

Lemma 1.3.15 (Snake Lemma)**.** *[Lee12] Given a short exact sequence of cochain complexes as in (1.1), there exists an R-linear map $\Delta^n : H^n(C^\bullet) \to H^{n+1}(A^\bullet)$ for all $n \geq 0$, called the* ***connecting homomorphism****, such that the following sequence is exact:*

$$\cdots \xrightarrow{\Delta^{n-1}} H^n(A^\bullet) \xrightarrow{f^*} H^n(B^\bullet) \xrightarrow{g^*} H^n(C^\bullet) \xrightarrow{\Delta^n} H^{n+1}(A^\bullet) \xrightarrow{f^*} \cdots$$

To get a notion of cohomology for topological spaces, what remains is to construct a functor which associates cochain complexes to topological spaces and continuous maps to cochain maps. The standard example of cohomology is *singular cohomology*, a discussion of which can be found in *Algebraic Topology* by Munkres [Mun84]. Many examples of cohomology for topological spaces have some useful properties which were axiomatised by Eilenberg and Steenrod to form what we call a cohomology theory [ES52].

Definition 1.3.16. [Jos95] Let G be a module over a commutative ring R. A *cohomology theory* with coefficients in G is a sequence of contravariant functors $\{H^k(-;G)\}_{k\geq 0}$ from the category of pairs of topological spaces to the category of R-modules and a natural transformation
$\delta^* : H^*(A;G) \to H^*(X,A;G)$ which satisfy the following axioms:

(A1) Homotopy Axiom: If $f, g : (X,A) \to (Y,B)$ are homotopic maps, then $f^* = g^* : H^k(Y,B;G) \to H^k(X,A;G)$ for all $k \geq 0$, where $f^* := H^k(-;G)(f)$

(A2) Exactness Axiom: Let $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ be inclusions. Then there is an exact sequence

$$\cdots \xrightarrow{\delta^*} H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta^*} H^{n+1}(X,A;G) \xrightarrow{j^*} \cdots \tag{1.2}$$

(A3) Excision Axiom: Let (X,A) be a pair of topological spaces and let $U \subset A$ such that $\overline{U} \subset \mathring{A}$, where $\overline{U}$ is the closure of U and $\mathring{A}$ is the interior of A. Then the inclusion $j : (X\backslash U, A\backslash U) \hookrightarrow (X,A)$ induces an isomorphism for all $k \geq 0$:

$$j^* : H^k(X,A;G) \to H^k(X\backslash U, A\backslash U; G)$$

(A4) Dimension Axiom: Let $\mathcal{P}$ be the one point space. Then

$$H^k(\mathcal{P};G) = \begin{cases} G, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

II. Constructions on Smooth Manifolds

2.1 Smooth Manifolds and Smooth Maps

The remainder of the report will be dedicated to studying a particular class of topological spaces: smooth manifolds. We first generalise notions from calculus that then allow us to associate a cochain complex to these manifolds which then naturally leads to a notion of cohomology: de Rham cohomology [Lee12]. This chapter is dedicated to building up the foundations of the theory of smooth manifolds before we begin to study the algebraic structures in chapters 3 to 5. What follows is adapted from *Introduction to Smooth Manifolds* by John M. Lee [Lee12] unless indicated otherwise. This section is largely a review from Riemannian Geometry IV and as such, many details are omitted and can be found in [Lee12]. We begin by giving the definition of a topological manifold.

Definition 2.1.1. Let $n \in \mathbb{Z}_{\geq 0}$. We say a second countable (A.1.5) Hausdorff (A.1.6) topological space M is an *n-dimensional (topological) manifold* if it is *locally Euclidean*. That is, for all $p \in M$ there exists an open neighbourhood $U \subseteq M$ of p and a homeomorphism $\varphi : U \to V$ for some open subset V of either $\mathbb{R}^n$ or $\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x^n \geq 0\}$ with the subspace topology. We define the *dimension* of M to be $\dim(M) = n$. We call the pair (U, φ) a *chart centred at p*. We write $\mathrm{Int}(\mathbb{H}^n) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x^n > 0\}$ and $\partial\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$. We say that $p \in M$ is an *interior point* if there exists a chart (U, φ) containing p such that $\varphi(U) \subseteq \mathrm{Int}(\mathbb{H}^n)$. We say that $p \in M$ is a *boundary point* if there exists a chart (U, φ) containing p such that $p \in \varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. We define the *interior* of M, $\mathrm{Int}(M)$, to be the set of interior points of M and the *boundary* of M, ∂M, to be the set of boundary points of M. We say M is a *manifold without boundary* if $\partial M = \varnothing$ and we say M is a *manifold with boundary* otherwise. Lastly, we call a collection of charts $\mathcal{A} := \{(U_i, \varphi_i)\}_{i \in J}$ an *atlas* for M if $\{U_i\}_{i \in J}$ is a cover of M.

Remark 2.1.2. A point $p \in M$ is either an interior point or a boundary point, but not both. That is $M = \mathrm{Int}(M) \sqcup \partial M$. For the details, see Theorems 1.37 and 1.46 in [Lee12].

To define smooth manifolds, we want a notion of our atlas being "smooth". Since up until this point, we only have the definition of smooth maps between Euclidean spaces, we have to associate our charts to smooth maps between Euclidean spaces. We do this by considering when charts overlap.

Definition 2.1.3. Let (U, φ) and (V, ψ) be two charts of a manifold M such that $U \cap V \neq \varnothing$. We call the maps $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ and $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ *transition maps*. We say two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \varnothing$ or the transition maps are smooth as maps between open sets of $\mathbb{R}^n$ or $\mathbb{H}^n$. We say that an atlas $\mathcal{A}$ for M is a *smooth atlas* for M if all charts in $\mathcal{A}$ are smoothly compatible. We call charts of a smooth atlas *smooth charts*.

Remark 2.1.4. A map $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{H}^n$ is smooth if for all $x \in U$ there exist $\tilde{U} \subseteq \mathbb{R}^n$ open neighbourhood of x and a smooth map $\tilde{f} : \tilde{U} \to \mathbb{R}^n$ such that $\tilde{f}|_{\tilde{U} \cap \mathbb{H}^n} = f|_{\tilde{U} \cap \mathbb{H}^n}$.

It is possible that two different smooth atlases for a manifold produce the "same" structure in that they determine all the same smooth functions on the manifold (see Definition 2.1.10). So trying to use smooth atlases to define smooth structures does not appear to be particularly useful. However, this next definition and proposition will fix this problem.

Definition 2.1.5. A smooth atlas $\mathcal{A}$ for a manifold M is said to be *maximal* if it is not properly contained in a larger smooth atlas. That is, if $\mathcal{A}'$ is another smooth atlas for M with $\mathcal{A} \subseteq \mathcal{A}'$, then $\mathcal{A} = \mathcal{A}'$.

Proposition 2.1.6. *Let M be a manifold. Then*

*(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the **maximal smooth atlas determined by** $\mathcal{A}$.*

(b) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is also a smooth atlas for M.

Proof. See Proposition 1.17 in [Lee12]. □

This now allows us to define smooth structures on manifolds.

Definition 2.1.7. A *smooth structure* on a manifold M is a maximal smooth atlas $\mathcal{A}$ for M. We call the pair $(M, \mathcal{A})$ a *smooth manifold* if M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. We often omit the smooth structure and refer to M as a smooth manifold.

Remark 2.1.8. Proposition 2.1.6 (a) implies that to show a manifold M has a smooth structure, we only need to find a smooth atlas on M. It need not be maximal since the proposition guarantees the existence of a maximal smooth atlas that contains it and hence M has a smooth structure.

Examples 2.1.9.

1. $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ is a smooth atlas for $\mathbb{R}^n$. So $\mathbb{R}^n$ is a smooth manifold with $\dim(\mathbb{R}^n) = n$
2. Let $S^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ be the n-sphere. The standard smooth structure on M is given by stereographic projection. So S^n is a smooth manifold with $\dim(S^n) = n$. See Example 1.31 in [Lee12].
3. $U \subseteq M$ be an open subset of a smooth manifold without boundary M. Let $\mathcal{A}$ be the smooth structure on M, then $\mathcal{A}_U := \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}$ is a smooth structure on U. We call U an *open submanifold* of M and $\dim(M) = \dim(U)$.
4. Let $\mathbb{RP}^n$ be the quotient space of S^n obtained by identifying antipodal points, called the *real projective space*. Then $\mathbb{RP}^n$ is an n-dimensional smooth manifold. See Example 1.33 in [Lee12].
5. Let $M_1, \ldots, M_k$ be smooth manifolds without boundary. Then $M_1 \times \cdots \times M_k$ is a smooth manifold without boundary. If M_k is a smooth manifold with boundary, then $M_1 \times \cdots \times M_k$ is a smooth manifold with boundary. See Example 1.34 and Proposition 1.45 in [Lee12].

Now that we have described the objects of interest, we want to describe relationships between them: smooth maps. The intuition to keep in mind here is that smooth maps between smooth manifolds locally look like smooth maps between Euclidean spaces.

Definition 2.1.10. Let M and N be smooth manifolds. Then a function $F : M \to N$ is said to be a *smooth map* if for all $p \in M$ there exist smooth charts (U, φ) of M containing p and (V, ψ) of N containing $F(p)$ with $F(U) \subseteq V$ such that the map of Euclidean spaces $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

Proposition 2.1.11. *Let $F : M \to N$ be a smooth map. Then F is continuous.*

Proof. Let $p \in M$ and let (U, φ) and (V, ψ) be smooth charts of M and N as above. Then we get the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \varphi(U) & \xrightarrow[\psi \circ F \circ \varphi^{-1}]{} & \psi(V) \end{array}$$

Both φ and ψ are homeomorphisms by assumption, so they and their inverses are continuous and $\psi \circ F \circ \varphi^{-1}$ is also continuous since smooth maps of Euclidean spaces are also continuous. From the diagram, we see that $F = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$. So on a neighbourhood of p, F is the composition of continuous maps and is hence continuous on a neighbourhood of p. Since $p \in M$ was arbitrary, F is continuous on all of M. □

One might wonder whether smoothness depends on the choice of smooth charts, however the next result guarantees this is not the case.

Proposition 2.1.12. *Let $F : M \to N$ be a smooth map and let (U, φ) and (V, ψ) be any smooth charts for M and N respectively. Then the map $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(U \cap F^{-1}(V))$ is a smooth map of Euclidean spaces.*

Proposition 2.1.13.

(a) Let $F : M \to N$ be a function between smooth manifolds. Then if for every $p \in M$ there exists an open neighbourhood U of p such that $F|_U$ is smooth, then F is smooth.

(b) Gluing Lemma: Let $\{U, V\}$ be an open cover for a smooth manifold M and let $G : U \to N$ and $H : V \to N$ be smooth maps such that $G|_{U \cap V} = H|_{U \cap V}$. Then there exists a smooth map $F : M \to N$ with $F|_U = G$ and $F|_V = H$.

Proof. See Proposition 2.6 and corollary 2.8 in [Lee12] for the proofs of (a) and (b) respectively. □

The next result implies that smooth manifolds and smooth maps form a category, which we denote **Diff**.

Proposition 2.1.14. *Let M, N and K be smooth manifolds. Then the following maps are smooth:*

(a) a constant map $c : M \to N$,

(b) the identity map $\mathrm{id}_M : M \to M$,

(c) the inclusion of an open submanifold $U \subseteq M$: $i : U \hookrightarrow M$,

(d) the composition $G \circ F : M \to K$ of smooth maps $G : N \to K$ and $F : M \to N$.

Proof. This is Proposition 2.10 in [Lee12] and the proofs of (a)-(b) are exercises so the proof of those given is my own.

(a): Let $p \in M$ and let (U, φ) be a smooth chart containing p and (V, ψ) be a smooth chart containing $c(p)$. Then $\psi \circ c \circ \varphi^{-1}$ is a constant map between subsets of Euclidean space and is hence smooth.

(b): Let $p \in M$ and let (U, φ) be a smooth chart containing p. Then (U, φ) also contains $\mathrm{id}_M(p) = p$. Then $\varphi \circ \mathrm{id}_M \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$ and so is smooth.

(c): Let $p \in U$ and let (V, φ) be a smooth chart for U containing p. Then by definition of the smooth structure on U (see Examples 2.1.9 3.), (V, φ) is also a smooth chart for M containing $i(p) = p$. Then since $i|_V = \mathrm{id}_V$, we again have $\varphi \circ i \circ \varphi^{-1} = \mathrm{id}_{\varphi(V)}$ and so is smooth.

(d): Let $p \in M$. By smoothness of G there are smooth charts (V, θ) containing $F(p)$ and (W, ψ) containing $G(F(p))$ with $G(V) \subseteq W$ such that $\psi \circ G \circ \theta^{-1} : \theta(V) \to \psi(W)$ is smooth. By Proposition 2.1.11 F is continuous and so $F^{-1}(V)$ is an open neighbourhood of p. So there is a smooth chart (U, φ) of M with $p \in U \subseteq F^{-1}(V)$. By Proposition 2.1.12 $\theta \circ F \circ \varphi^{-1}$ is smooth. Then $(G \circ F)(U) \subseteq G(V) \subseteq W$ and $\psi \circ (G \circ F) \circ \varphi^{-1} = (\psi \circ G \circ \theta^{-1}) \circ (\theta \circ F \circ \varphi^{-1})$, which is the composition of smooth maps of Euclidean spaces and is hence smooth, as required.

□

Remark 2.1.15. Combining this result with Proposition 2.1.11 we have that **Diff** can be thought of as a category of a specific class of topological spaces and continuous maps that have extra structure!

An important class of smooth maps are those from a smooth manifold to $\mathbb{R}$.

Definition 2.1.16. Let M be a smooth manifold. We define $C^\infty(M)$ to be the set of smooth maps $M \to \mathbb{R}$:

$$C^\infty(M) := \{f : M \to \mathbb{R} : f \text{ is smooth}\}.$$

We call elements of $C^\infty(M)$ *smooth real maps.*

Remark 2.1.17. Since $\mathbb{R}$ has a global chart $(\mathbb{R}, \mathrm{id}_\mathbb{R})$, to check smoothness of a map $f : M \to \mathbb{R}$ we only need check that there exists a smooth chart (U, φ) for each point in M such that $f \circ \varphi^{-1}$ is smooth.

Lemma 2.1.18. *Let M be a smooth manifold. Then $C^\infty(M)$ with pointwise scalar multiplication and addition is a real vector space and with pointwise addition and multiplication is a commutative ring.*

Proof. This is Exercise 2.1 in [Lee12] and so the proof given is my own. Firstly, $0_M : M \to \mathbb{R}$ and $1_M : M \to \mathbb{R}$ given by $0_M(p) = 0$ and $1_M(p) = 1$ are both constant maps so are smooth by Proposition 2.1.14. These will act as the additive and multiplicative identities respectively. Now, let $\lambda \in \mathbb{R}$ and $f, g \in C^\infty(M)$. We define the following operations pointwise:

- scalar multiplication: $(\lambda f)(p) = \lambda \cdot f(p)$,

- addition: $(f+g)(p) = f(p) + g(p)$,
- multiplication: $(f \cdot g)(p) = f(p) \cdot g(p)$.

Then provided each of these is a smooth map, the relevant properties to be a vector space or a ring follow from the properties of addition and multiplication of real numbers.
First we show the smoothness of λf. Let $p \in M$. The smoothness of f implies there exists a smooth chart (U, φ) containing p such that $f \circ \varphi^{-1}$ is smooth. Then for all $q \in \varphi(U)$ we have $((\lambda f) \circ \varphi^{-1})(q) = \lambda \cdot (f \circ \varphi^{-1})(q)$. Then since scalar multiplication of smooth maps between Euclidean spaces is again smooth, $\lambda f \circ \varphi^{-1}$ is smooth, as required.
Next, we check the smoothness of $f + g$ and $f \cdot g$ simultaneously. Let $p \in M$, then the smoothness of f and g imply there exist smooth charts (U, φ) and (V, ψ) containing p such that $f \circ \varphi^{-1}$ and $g \circ \psi^{-1}$ are smooth. Then $p \in U \cap V$ and $U \cap V$ is open since the intersection of open sets is open. Then since the restriction of a homeomorphism is a homeomorphism, the map $\tilde{\varphi} = \varphi|_{U \cap V} : U \cap V \to \varphi(U \cap V)$ is a homeomorphism. In particular, this implies that $(U \cap V, \tilde{\varphi})$ is a chart containing p. Then by Proposition 2.1.12, both $f \circ \tilde{\varphi}^{-1}$ and $g \circ \tilde{\varphi}^{-1}$ are smooth maps. Then since the addition and multiplication of smooth maps between Euclidean spaces produces smooth maps, we have that the maps $(f+g) \circ \tilde{\varphi} = (f \circ \tilde{\varphi}) + (g \circ \tilde{\varphi})$ and $(f \cdot g) \circ \tilde{\varphi} = (f \circ \tilde{\varphi}) \cdot (g \circ \tilde{\varphi})$ are smooth, as required. □

Another important class of smooth maps are diffeomorphisms. These are the isomorphisms in **Diff**.

Definition 2.1.19. We say a bijective smooth map $F : M \to N$ is a *diffeomorphism* if its inverse $F^{-1} : N \to M$ is also smooth. If a diffeomorphism from M to N exists, we say M and N are *diffeomorphic* and write $M \approx N$.

Remark 2.1.20. If $F : M \to N$ is a diffeomorphism, Proposition 2.1.11 implies that since both F and F^{-1} are smooth they are both continuous. Hence F is also a homeomorphism and a homotopy equivalence!

Example 2.1.21. Let (U, φ) be a smooth chart of a smooth manifold M. Then $\varphi : U \to \varphi(U)$ is a diffeomorphism. We already have that φ is bijective by assumption, so it remains to show that φ and φ^{-1} are smooth. Let $p \in U$ and note that $\big(\varphi(U), \mathrm{id}_{\varphi(U)}\big)$ is a global chart containing $\varphi(p)$. The maps $\mathrm{id}_{\varphi(U)} \circ \varphi \circ \varphi^{-1}$ and $\varphi \circ \varphi^{-1} \circ \mathrm{id}^{-1}_{\varphi(U)}$ are both equal to $\mathrm{id}_{\varphi(U)}$ and hence both are smooth, as required.

Throughout our study of smooth manifolds it will be useful to be able to "glue" smooth maps defined on parts of a manifold together. To do this, we make use of *partitions of unity*. First, recall the definition of the support of a function.

Definition 2.1.22. Let X be a topological space. We define the *support* of a function $f : X \to \mathbb{R}$ to be the set

$$\mathrm{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

If $\mathrm{supp}(f) \subseteq U \subseteq X$, we say f is *supported in* U. If $\mathrm{supp}(f)$ is compact, we say f is *compactly supported*.

Theorem 2.1.23. *Let M be a smooth manifold and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of M. Then there exists a family of smooth maps $(\psi_a : M \to \mathbb{R})_{a \in A}$, called a* ***smooth partition of unity subordinate to*** *$\mathcal{X}$, which satisfies the following:*

(a) $0 \leq \psi_\alpha(p) \leq 1$ *for all* $\alpha \in A$ *and* $p \in M$,

(b) ψ_α *is supported in* X_a, *i.e.* $\mathrm{supp}(\psi_\alpha) \subseteq X_\alpha$,

(c) The family $(\mathrm{supp}(\psi_\alpha))_{\alpha \in A}$ *is locally finite, i.e. for all* $p \in M$ *there exists an open neighbourhood* $U \subseteq M$ *of* p *such that* $U \cap \mathrm{supp}(\psi_\alpha) = \varnothing$ *for all but finitely map* $\alpha \in A$.

(d) $\sum_{\alpha \in A} \psi_\alpha(p) = 1$ *for all* $p \in M$.

Proof. The proof heavily relies on the fact that M is second countable. For the details, see Theorem 2.23 in [Lee12]. □

The existence of smooth partitions of unity allows us to immediately prove a few useful results.

Lemma 2.1.24. *Let M be a smooth manifold.*

*(a) For any open set $U \subseteq M$ and closed set $A \subseteq M$ contained in U, there exists a smooth map $\psi : M \to \mathbb{R}$, called a **smooth bump function for** A **supported in** U such that $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A and $\operatorname{supp}(\psi) \subseteq U$.*

(b) Let $A \subseteq M$ be closed and let $f : A \to \mathbb{R}^k$ be smooth. Then for any open set $U \subseteq M$ containing A, there exists a smooth map $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\operatorname{supp}(\tilde{f}) \subseteq U$.

Proof. See Proposition 2.25 and Lemma 2.26 in [Lee12]. □

2.2 Tangent Spaces and Vector Fields

One of our motivations for studying smooth manifolds is to generalise calculus to spaces which are not $\mathbb{R}^n$. So a natural question to ask is "can we find linear approximations of smooth maps?". The answer is yes! Recall from multivariable calculus that the Jacobian, DF, of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix of all the first order partial derivatives of the components of F and hence is a linear map $DF : \mathbb{R}^n \to \mathbb{R}^m$. The goal, then, is to find a sensible vector space associated to each manifold and, given a smooth map, a linear map between them associated to that smooth map. Everything that follows, unless stated otherwise, is taken from [Lee12].

Recall from multivariable calculus that we have a notion of directional derivative of smooth maps $f : \mathbb{R}^n \to \mathbb{R}$. Given a point $p \in \mathbb{R}^n$, this is the first order approximation of f in a given direction $\mathbf{v} \in \mathbb{R}^n$ at the point p. We can think of the directional derivative as a linear map $\nabla_{\mathbf{v}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ that satisfies the Leibniz rule: $\nabla_{\mathbf{v}}(fg) = (\nabla_{\mathbf{v}} f)g(p) + f(p)(\nabla_{\mathbf{v}} g)$. We use this as the template for the following definition.

Definition 2.2.1. Let M be a smooth manifold and let $p \in M$. A *derivation at p* is a linear map $v : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule: $v(fg) = g(p)\, vf + f(p)\, vg$ for all $f, f \in C^\infty(M)$. We denote the set of all derivations at p by T_pM and call it the *tangent space of M at p*. We call elements of T_pM *tangent vectors at p*.

Proposition 2.2.2. *Let M be a smooth manifold and let $p \in M$. Then T_pM is a real vector space with $\dim(T_pM) = \dim(M)$.*

Proof. To show that T_pM is a vector space, it is sufficient to show that it is a subspace of the vector space of all linear maps $C^\infty(M) \to \mathbb{R}$. Using the subspace criterion from linear algebra, it is sufficient to check that T_pM is non-empty and closed under scalar multiplication and addition, which in this case are defined pointwise. First, let $\lambda \in \mathbb{R}$, $f, g \in C^\infty(M)$ and $v \in T_pM$. Then

$$(\lambda v)(fg) = \lambda \cdot v(fg) = \lambda \cdot g(p)v(f) + \lambda \cdot f(p)v(g) = g(p) \cdot (\lambda v f) + f(p) \cdot (\lambda v)g.$$

So λv satisfies the Leibniz rule and is hence a derivation at p. Similarly, let $f, g \in C^\infty(M)$ and $v, w \in T_pM$. Then

$$\begin{aligned} &(v+w)(fg) = v(fg) + w(fg) = g(p)vf + f(p)wg + g(p)vf + f(p)wg \\ \implies\ &(v+w)(fg) = g(p)(vf + wf) + f(p)(vg + wg) = g(p)(v+w)f + f(p)(v+w)g. \end{aligned}$$

So $v + w$ satisfied the Leibniz rule and is hence a derivation at p.

To see that T_pM is non-empty note that the zero map $0 : f \mapsto 0 \in \mathbb{R}$ is trivially a derivation. The proof that $\dim(T_pM) = \dim(M)$ requires some more work. See Propositions 3.10 and 3.12 in [Lee12]. □

The next result further illustrates that derivations behave like derivatives.

Lemma 2.2.3. *Let M be a smooth manifold and $p \in M$. If $f \in C^\infty(M)$ is constant, then $vf = 0$ for all $v \in T_pM$.*

Proof. This is part of Lemma 3.4 in [Lee12] and was left as Exercise 3.5, so the proof given is my own. Suppose $f(p) = \lambda \in \mathbb{R}$ for all $p \in M$ and let $g \in C^\infty(M)$ be arbitrary. Then $(f \cdot g)(p) = \lambda\, g(p)$ for all $p \in M$. So on one hand, $v(fg) = v(\lambda\, g) = \lambda\, vg = f(p)\, vg$. On the other hand, by the Leibniz rule, $v(fg) = f(p)\, vg + g(p)\, vf$. Combining these we get that $f(p)\, vg + g(p)\, vf = f(p)\, vg$. So $g(p)\, vf = 0$. Since $g \in C^\infty(M)$ is arbitrary, we must have that $vf = 0$. □

Given a smooth map $F : M \to N$ and a point $p \in M$, we would like to associate tangent vectors at p to tangent vectors at $T_{F(p)}N$ in a linear way. This next lemma gives us a way of doing this.

Lemma 2.2.4. *Let $F : M \to N$ be a smooth map and let $p \in M$. Define the **differential of** F **at** p to be the map $dF_p : T_pM \to T_{F(p)}N$ by $dF_p(v)(f) = v(f \circ F)$ for all $f \in C^\infty(N)$. Then dF_p satisfies the following:*

(a) dF_p is linear

(b) If $G : N \to K$ is another smooth map, then $d(G \circ F)_p = dG_{F(p)} \circ dF_p$

(c) $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}$

(d) If F is a diffeomorphism, then dF_p is an isomorphism of vector spaces and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

(e) The inclusion of an open submanifold $i : U \hookrightarrow M$ induces an isomorphism $di_p : T_pU \to T_pM$ for all $p \in U$.

Proof. For (a)-(d) see Proposition 3.6 and for (e) see Proposition 3.9 in [Lee12]. □

Remark 2.2.5. The differential is well-defined since if $f \in C^\infty(N)$ and $F : M \to N$ is smooth, then $f \circ F : M \to \mathbb{R}$ is smooth as it is the composition of smooth maps and so $f \circ F \in C^\infty(M)$. Therefore it makes sense to apply v to $f \circ F$ and evaluating $f \circ F$ at $p \in M$ is equivalent to evaluating f at $F(p) \in N$.

It is often useful to do computations by locally defining coordinates on our manifolds. One benefit is they allow us to locally define a basis for T_pM.

Definition 2.2.6. Let (U, φ) be a smooth chart for a smooth manifold M. We call U the *coordinate domain* and φ the *(local) coordinate map*. We define the *local coordinates on U* to be the component functions of φ: $(x^1, \ldots, x^n)$, where $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

Lemma 2.2.7. *Let $x^1, ..., x^n$ be the standard coordinated of $\mathbb{R}^n$, $a \in \mathbb{R}^n$ and for all $1 \le i \le n$ define the following derivation at a*

$$\left.\frac{\partial}{\partial x^i}\right|_a f := \frac{\partial f}{\partial x^i}(a).$$

*Then $\left\{\left.\frac{\partial}{\partial x^i}\right|_a\right\}_{i=1}^n$ is a basis of $T_a\mathbb{R}^n$, called the **coordinate basis** for $T_a\mathbb{R}^n$.*

Proof. See Corollary 3.3 in [Lee12]. □

Now, given a smooth chart (U, φ), $\varphi(U) \subseteq \mathbb{R}^n$ so by Lemma 2.2.4 (e) we get the isomorphisms $T_pU \cong T_pM$ and $T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n$ for all $p \in U$. Moreover, since $\varphi : U \to \varphi(U)$ is an diffeomorphism, we also have by Lemma 2.2.4 (d) that $d\varphi_p : T_pU \to T_{\varphi(p)}\varphi(U)$ is an isomorphism. Putting this all together we get that $T_\varphi(p)\mathbb{R}^n \cong T_pM$. This allows us to associate the coordinate basis for $T_{\varphi(p)}\mathbb{R}^n$ with a basis for T_pM, as the next result states.

Proposition 2.2.8. *Let M be a smooth manifold with $\dim(M) = n$ and let (U, φ) be a smooth chart containing $p \in M$, with $\varphi = (x^1, ..., x^n)$. Then define $1 \leq i \leq n$, define the* ***ith coordinate tangent vector*** *to be the derivation*

$$\left.\frac{\partial}{\partial x^i}\right|_p := d(\varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right).$$

Then $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{i=1}^n$ is a basis of T_pM, called the ***coordinate basis*** *for T_pM (with respect to (U, φ)). In particular we may write $v \in T_pM$ as*

$$v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

and we call $(v^1, ..., v^n)$ the ***components*** *of v.*

Example 2.2.9. Let $F : M \to N$ be a smooth map, $p \in M$ and choose smooth charts (U, φ) containing p and (V, ψ) with components $\varphi = (x^1, ..., x^n)$ and $\psi = (y^1, ..., y^m)$. Then letting $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ and $\hat{p} = \varphi(p)$ and using that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$ and Lemma 2.2.4 we get

$$\begin{aligned}
& dF_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = dF_p \left(d(\varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}} \right) \right) = d(\psi^{-1})_{\hat{F}(\hat{p})} \left(d\hat{F}_{\hat{p}} \left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}} \right) \right) \\
\implies & dF_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = d(\psi^{-1})_{\hat{F}(\hat{p})} \left(\sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})} \right) \\
\implies & dF_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_{F(p)},
\end{aligned}$$

where $\hat{F}^j$ is the jth component of $\hat{F}$. In this way, we see that dF_p is represented in coordinates by the Jacobian matrix of $\hat{F}$. So the differential of a smooth map really is the generalisation of the Jacobian.

Example 2.2.10. Let M be a smooth manifold and let $J \subseteq \mathbb{R}$ be an interval. We can think of J as a 1-dimensional manifold with a single coordinate tangent vector $\left.\frac{d}{dt}\right|_{t_0}$. We define a *curve in* M to be a smooth map $\gamma : J \to M$. Then its *derivative at* $t_0 \in J$ is defined to be the derivation $\gamma'(t_0) := d\gamma_{t_0} \left(\left.\frac{d}{dt}\right|_{t_0} \right)$. In particular, given $f \in C^\infty(M)$ we have

$$\gamma'(t_0) f = d\gamma_{t_0} \left(\left.\frac{d}{dt}\right|_{t_0} \right) f = \left.\frac{d}{dt}\right|_{t=t_0} (f \circ \gamma) = (f \circ \gamma)'(t_0).$$

So we can think of $\gamma'(t_0)$ as a directional derivative! That is, $\gamma'(t_0)$ tells us how much f changes in the direction of γ at $t = t_0$.

In many applications both pure or otherwise, it is useful to associate tangent vectors to points on a manifold in a smooth way. To get a notion of smoothness, we need to produce a smooth manifold out of tangent spaces. This motivates the next definition.

Definition 2.2.11. Let M be a smooth manifold. The *tangent bundle of* M, denoted TM, is defined as

$$TM := \bigsqcup_{p \in M} T_pM.$$

We write elements of TM as the pair (p, v) where $p \in M$ and $v \in T_pM$. There is a natural projection map $\pi : TM \to M$ given by $\pi(p, v) = p$.

Proposition 2.2.12. *Let M be a smooth manifold. Then TM has a natural topology and smooth structure which make it a $2n$-dimensional smooth manifold. In particular, with respect to this smooth structure, the projection map $\pi : TM \to M$ is smooth.*

Proof. Given $(p, v) \in TM$, let (U, φ) be a smooth chart containing p such that $\varphi = (x^1, ..., x^n)$. Then we have a chart $(\pi^{-1}(U), \tilde{\varphi})$, where $\varphi : \pi^{-1}(\tilde{U}) \to \mathbb{R}^n$ is given by

$$\tilde{\varphi}\left(p, \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p\right) := (x^1, ..., x^n, v^1, ..., v^n).$$

These are called **natural coordinates** on TM. The proof that this is indeed a chart and that these charts are smoothly compatible is given in Riemannian Geometry IV and is Proposition 3.18 in [Lee12]. □

Tangent bundles now allow us to define the notion of vector fields, which associate a tangent vector to each point.

Definition 2.2.13. Let M be a smooth manifold. A function $X : M \to TM$ is a *rough vector field* if $\pi \circ X = \mathrm{id}_M$. We say that a rough vector field $X : M \to TM$ is a *smooth vector field* if it is a smooth map. We denote the set of smooth vector fields on M by $\mathfrak{X}(M)$. We write $X_p := X(p)$ and we often abuse notation and consider X_p to be the vector part, i.e. $X_p \in T_pM$.

Proposition 2.2.14. *Let M be a smooth manifold and $X : M \to TM$ a rough vector field. Then if (U, φ) is a smooth chart for M with $\varphi = (x^1, ..., x^n)$, define the **component functions** of X to be the functions $X^i : U \to \mathbb{R}$ such that the vector part of X can be written as*

$$X_p = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x^i}\Big|_p.$$

Then the restriction of X to U is smooth if and only if its component functions are smooth.

Proof. The proof is given in Riemannian Geometry IV and is Proposition 8.1 in [Lee12]. □

Example 2.2.15. Let (U, φ) be a smooth chart for a smooth manifold M with components $\varphi = (x^1, ..., x^n)$. Then for all $1 \leq i \leq n$ the vector field $\frac{\partial}{\partial x^i} : p \mapsto \frac{\partial}{\partial x^i}\big|_p$ for $p \in U$ is smooth since its component functions $X_j \equiv 0$ if $j \neq i$ and $X_i \equiv 1$ are smooth. We call this the *ith coordinate vector field*.

Given a smooth real map, we may use smooth vector fields to produce new smooth real maps as the next result shows.

Lemma 2.2.16. *Let M be a smooth manifold, (U, φ) be a smooth coordinate chart with $\varphi = (x^1, ..., x^n)$ and $f \in C^\infty(M)$. Then*

(a) The map $p \mapsto \frac{\partial}{\partial x^i}\big|_p f$ is a smooth real map on U.

(b) Let $X \in \mathfrak{X}(M)$. Then the map $(Xf) : p \mapsto X_p f$ is smooth on M.

Proof. Let $p \in U$. By Proposition 2.2.8 we have that $\frac{\partial}{\partial x^i}\big|_p f = \frac{\partial}{\partial x^i}\big|_{\varphi(p)}(f \circ \varphi^{-1}) = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$. So $p \mapsto \frac{\partial}{\partial x^i}\big|_p f$ is equal to the composition of smooth maps $\frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \circ \varphi$ and hence is smooth. For (b), we have that Proposition 2.2.14 implies that the component functions of X are smooth real maps on U and that

$$Xf(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x^i}\Big|_p f.$$

Then using (a) we see that Xf is the sum of products on smooth real maps on U. So since $C^\infty(U)$ is a ring, Xf is also a smooth map on U. Then since the smooth chart was arbitrary, this holds for any smooth chart and hence by Proposition 2.1.13 (a), Xf is smooth on M. □

Proposition 2.2.17. *Let M be a smooth manifold and let $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$. Define the rough vector fields $(fX)_p := f(p)X_p$ and $(X + Y)_p := X_p + Y_p$. Then fX and $X + Y$ are smooth vector fields. In particular, these operations turn $\mathfrak{X}(M)$ into a $C^\infty(M)$ module.*

The proof of this proposition will be given in more generality in Section 2.5.

Example 2.2.10 showed us that we can think of derivations as directional derivatives of smooth real maps. One might wonder whether we can define a notion of directional derivatives of vector fields. We can indeed do this but it is tricker than it might seem. For a vector field W on $\mathbb{R}^n$, we may compare its values at different points between each since we can easily view W as a smooth map $W : \mathbb{R}^n \to \mathbb{R}^n$. That is, the difference $W_{p+tv} - W_p$ is meaningful and allows us to define a difference quotient to define the directional derivative of W at $p \in \mathbb{R}^n$ in the direction on $v \in \mathbb{R}^n$. In the case of a smooth manifold M, $p + tv$ does not make sense since $p \in M$ and $v \in T_pM$. So we could replace it with a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$. But $W_{\gamma(t)}$ and $W_{\gamma(0)}$ still lie in different vector spaces, so $W_{\gamma(t)} - W_{\gamma(0)}$ makes no sense. To solve this issue, we produce a notion of derivative of a vector field in the direction of another vector field: the Lie derivative. The definition relies on the fact that vector fields generate a smooth $\mathbb{R}$-action on M, called a flow, the differential of which allows us to push forward tangent vectors at different points to one chosen point which ultimately allows us to compare them.

Theorem 2.2.18. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume V is tangent to ∂M. Then there exist unique open $\mathfrak{D} \subseteq \mathbb{R} \times M$ and maximal smooth $\theta : \mathfrak{D} \to M$ such that*

- *the set $\mathfrak{D}^{(p)} := \{t \in \mathbb{R} : (t, p) \in \mathfrak{D}\}$ is an open interval containing 0 for all $p \in M$,*
- *$\theta(0, p) = p$ for all $p \in M$,*
- *$\theta(t, \theta(s, p)) = \theta(t + s, p)$ for all $s \in \mathfrak{D}^{(p)}$ and $\mathfrak{D}^{\theta(s,p)}$ such that $s + t \in \mathfrak{D}$,*
- *for all $p \in M$, the curve $\theta^{(p)} : \mathfrak{D}^{(p)} \to M$ is the unique maximal curve $\theta^{(p)}(t) = \theta(t, p)$ such that $\theta^{(p)}(0) = p$ and $\theta^{(p)'}(0) = V_p$,*
- *if $s \in \mathfrak{D}^{(p)}$, then $\mathfrak{D}^{(\theta(s,p))}$ is the interval $\{t - s : s \in \mathfrak{D}^{(p)}\}$,*
- *for all $t \in \mathbb{R}$, the set $M_t := \{p \in M : (t, p) \in \mathfrak{D}\}$ is open in M and $\theta_t : M_t \to M_{-t}$, given by $\theta_t(p) = \theta(t, p)$, is a diffeomorphism which inverse θ_{-t}.*

*We call $\mathfrak{D}$ the **flow domain** and we call θ the flow generated by V. The flow θ is maximal in the sense that it cannot be extended to a larger flower domain and the curve $\theta^{(p)}$ is maximal in the sense that it cannot be extended to a curve with the same properties whose domain is larger.*

The proof relies on the theory of ODEs. See Theorems 9.12 and 9.34 in [Lee12].

Given a vector field V and its flow θ, we push forward tangent vectors at $\theta_t(p)$ to tangent vectors at $p = \theta_0(p)$ by applying $d(\theta_{-t})_{\theta_t(p)}$. This now allows us to compare tangent vectors at different points in a smooth way. We use this to compare the value of another vector field W at different points and see what happens when $t \to 0$.

Definition 2.2.19. Let M be a smooth manifold and let $V, W \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$, then V is tangent to ∂M. Let θ be the flow generated by V. Then we define the *Lie derivative of W with respect to V* to be the (rough) vector field

$$(\mathcal{L}_V W)_p = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right) - W_p}{t}$$

Remark 2.2.20. We can also write the Lie derivative as

$$(\mathcal{L}_V W)_p = \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right).$$

Lemma 2.2.21. *Let M be a smooth manifold and let $V, W \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$, then V is tangent to ∂M. Then $(\mathcal{L}_V W)_p$ exists for all $p \in M$ and $\mathcal{L}_V W$ is a smooth vector field.*

Proof. The proof relies on calculating the component functions of $d(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right)$ and showing that they are smooth in (t, p), then taking the derivative to imply that the Lie derivative is defined at $p \in M$ and is smooth. See Lemma 9.36 in [Lee12]. □

2.3 Immersions, Embeddings and Submanifolds

We would now like to produce a more general notion of submanifold than just our notion of open submanifolds. Unlike with open submanifolds, it is much harder to define a smooth structure on "lower dimensional" subsets of a smooth manifold which make the inclusion a smooth map. Instead, we make use of the more geometric idea that tangent spaces are a linear approximation of manifolds. We want to consider subsets of a smooth manifold which are topological manifolds and that can be "consistently" approximated by subspaces of the tangent spaces of M. This latter condition motivates the definition of a smooth immersion. Unless stated otherwise, what follows is adapted from [Lee12].

Definition 2.3.1. Let $F : M \to N$ be a smooth map. We say F is a *smooth immersion* if the differential $dF_p : T_pM \to T_{F(p)}M$ is injective for all $p \in M$.

Example 2.3.2. This is my own example. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth map given by $F(x, y) = (x, y, x^2 + y^2)$. Let $p = (x_0, y_0) \in \mathbb{R}^2$ Then using the formula in Example 2.2.9, we get that

$$dF_p\left(\frac{\partial}{\partial x}\bigg|_p\right) = 1\frac{\partial}{\partial x}\bigg|_{F(p)} + 2x_0\frac{\partial}{\partial z}\bigg|_{F(p)} \text{ and } dF_p\left(\frac{\partial}{\partial y}\bigg|_p\right) = 1\frac{\partial}{\partial y}\bigg|_{F(p)} + 2y_0\frac{\partial}{\partial z}\bigg|_{F(p)}.$$

So $\ker(dF_p) = \{0\}$ since the basis of $T_p\mathbb{R}^2$ is mapped to linearly independent vectors which are always non-zero. Hence the differential of F is injective for all $p \in \mathbb{R}^2$. So F is a smooth immersion.

Definition 2.3.3. Let $F : M \to N$ be a smooth immersion. We say F is a *smooth embedding* if F is also a homeomorphism onto its image $F(M) \subseteq N$.

Example 2.3.4. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be as in Example 2.3.2. Then F is also a homeomorphism with inverse $F^{-1}(x, y, x^2 + y^2) = (x, y)$. This is just the restriction of the projection map $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by $\pi((x, y), z) = (x, y)$, which is continuous by properties of the product topology. Combining this fact with Example 2.3.2, we get that F is actually a smooth embedding.

Definition 2.3.5. Let M be a smooth manifold and let $S \subseteq M$. We say that S is an *embedded submanifold* if it is a topological manifold in the subspace topology and it is equipped with a smooth structure with respect to which the inclusion $i : S \hookrightarrow M$ is a smooth embedding. We call the difference $\dim(M) - \dim(S)$ the *codimension of* S. Given a smooth manifold M and a submanifold $S \subseteq M$, we call the pair (M, S) a *pair of smooth manifolds*.

Example 2.3.6. Let M be a smooth manifold without boundary and let $U \subseteq M$. In Example 2.1.9 (c) we showed that U is a smooth manifold with $\dim(M) = \dim(U)$ and Proposition 2.1.14 gives that the inclusion $i : U \hookrightarrow M$ is smooth. Moreover, Lemma 2.2.4 gives us that $di_p : T_pU \to T_pM$ is an isomorphism for all $p \in U$, so di_p is certainly injective for all $p \in U$. So the inclusion is a smooth immersion. Lastly, the inclusion map itself is clearly injective and so is a homeomorphism onto its image. Hence U is an embedded submanifold of M of codimension 0. In fact, one can show that these are the only submanifolds of codimension 0, see Proposition 5.1 in [Lee12].

Proposition 2.3.7. *Let $F : M \to N$ be a smooth map and let $S \subseteq M$ be an embedded submanifold. Then $F|_S : S \to N$ is a smooth map.*

Proof. Since S is an embedded submanifold, the inclusion map $i : S \hookrightarrow M$ is smooth. Then $F|_S = F \circ i$ is the composition of smooth maps and is hence smooth by Proposition 2.1.14. □

2.4 Smooth Homotopies

Homotopies are key in algebraic topology but one limitation for our purposes is that currently they are only continuous maps. We would like to know when homotopies are actually smooth maps and what follows is a series of useful results related to this question. The theory to prove these results is deep and lengthy so it is omitted but relevant results from [Lee12] will be signposted. It is worth noting that the theory is split into two cases: when the codomain of the relevant maps is without

boundary and when it has boundary. The former is treated in Chapter 6 of [Lee12] and the latter requires the theory of flows and is treated in Chapter 9 of [Lee12].

To motivate our first result we look at continuous maps $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n, m \geq 0$. Since $\mathbb{R}^m$ is convex we can use a straight line homotopy to show that f is homotopic to any other continuous map [Mun14]. In particular, since smooth maps are continuous, every continuous map is homotopic to a smooth map. In general, manifolds are not convex or even contractible, yet we still have this result!

Theorem 2.4.1 (Whitney Approximation Theorem)**.** *Let $F : M \to N$ be a continuous map between smooth manifolds. Then there is a smooth map $\tilde{F} : M \to N$ which is homotopic to F. Moreover, if $\partial N = \varnothing$ and there exists a closed set $A \subset N$ on which F is smooth, then the homotopy is relative to A (See definition A.2.1).*

The proofs for when N does not have boundary and does have boundary are given in Theorems 6.26 and 9.27 in [Lee12] respectively.

Definition 2.4.2. Let M and N be smooth manifolds. A homotopy $H : M \times I \to N$ is a *smooth homotopy* if it can be extended to a smooth map of some neighbourhood of $M \times I$ in $M \times \mathbb{R}$. Two smooth maps $F, G : M \to N$ are *smoothly homotopic* if there exists a smooth homotopy between them.

As with standard homotopies, "smoothly homotopic to" is an equivalence relation. Symmetry and reflexivity are proven analogously to the normal case. One has to take more care to ensure when proving transitivity that the resulting homotopy is indeed smooth. The details of this are given in Lemma 6.28 in [Lee12].

The key question now that we have a notion of smooth homotopy is "when are homotopic smooth maps smoothly homotopic?" If we do not require the homotopy to be relative to some subset, then the answer is always yes. But relative homotopies are more delicate.

Theorem 2.4.3. *Let $F, G : M \to N$ be homotopic smooth maps. Then F and G are smoothly homotopic. Moreover, if $\partial N = \varnothing$ and F and G are homotopic relative to some closed subset $A \subset N$, then F and G are smoothly homotopic relative to A.*

The proofs for when M does not have boundary and does have boundary are given in Theorems 6.29 and 9.28 in [Lee12] respectively.

2.5 Vector Bundles

In Section 2.2, given a smooth manifold M we defined its tangent bundle (Definition 2.2.11) and vector fields on M (Definition 2.2.13). The former is another smooth manifold which locally looks like a product of M and $\mathbb{R}^n$. We use this as motivation to define the more general notion of a vector bundle of a topological space. Whilst we only care about the case of smooth vector bundles in this report, I present the more general theory and specialise when necessary. What follows is adapted from [Lee12] unless stated otherwise.

Definition 2.5.1. Let E and M be topological spaces and let $\pi : E \to M$ be a continuous surjection. We say that the triple (E, M, π), or simply just E, is a *(real) vector bundle of rank k over M* if the following are satisfied:

1. For each $p \in M$, the fibre $E_p := \pi^{-1}(p)$ is a real k-dimensional vector space.
2. For all $p \in M$, there exist an open neighbourhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a *local trivialisation*, which satisfies
   - $\text{proj}_U \circ \Phi = \pi$, where $\text{proj}_U : U \times \mathbb{R}^k$ is the projection map
   - for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

We call E the *total space*, M the *base space* and π the *projection map*. If E and M are smooth manifolds, the projection π is a smooth map and the local trivialisations are smooth maps, then we call E a *smooth vector bundle* over M and we call the local trivialisations *smooth local trivialisations*. For convenience we often say $\pi : E \to M$ is a (smooth) vector bundle. For $p \in M$ we write elements of the fibre E_p as v_p.

Remark 2.5.2. The first condition for local trivialisations can be summed up as the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{R}^k \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_U} & \\ & U & \end{array}$$

Proposition 2.5.3. *Let M be a smooth manifold and TM be its tangent bundle. Then with the natural projection $\pi : TM \to M$, TM is a smooth vector bundle of rank n over M, where $n = \dim(M)$.*

Proof. Let (U, φ) be a smooth chart for M with $\varphi = (x^1, ..., x^n)$. Then we define the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = (p, v^1, ..., v^n),$$

where we use Proposition 2.2.8 to write tangent vectors in local coordinates. Then for all $p \in U$, $\Phi|_{E_p}$ is the vector space isomorphism $\frac{\partial}{\partial x^i}\big|_p \mapsto (p, e_i)$, where e_i is the ith standard coordinate vector of $\mathbb{R}^n$. We also have for all $p \in U$ that

$$(\mathrm{proj}_U \circ \Phi)(p, v) = \mathrm{proj}_U \circ \Phi\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \mathrm{proj}_U(p, v^1, ..., v^n) = p = \pi(p, v).$$

Lastly, the composition

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathrm{id}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}$$

is equal to the smooth coordinate chart $\tilde{\varphi}$ for TM defined in Proposition 2.2.12. Since both $\tilde{\varphi}$ and $\varphi \times \mathrm{id}_{\mathbb{R}^n}$ are diffeomorphisms, we have that Φ must also be a diffeomorphism. Hence Φ is a smooth local trivialisation and so $\pi : TM \to M$ is indeed a smooth vector bundle. □

Recall from Definition 2.2.13 that a vector field is a map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. We now generalise this to vector bundles by defining sections.

Definition 2.5.4. Let $\pi : E \to M$ be a vector bundle. A *section* is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. If $\pi : E \to M$ is a smooth vector bundle, we say that $\sigma : M \to E$ is a *smooth section* if it is a section and is a smooth map. We denote the set of smooth sections of E by $\Gamma(E)$.

Example 2.5.5. Smooth vector fields of a smooth manifold M are smooth sections of the tangent bundle. That is $\mathfrak{X}(M) = \Gamma(TM)$.

We may also generalise the notion of coordinate vector fields (see Proposition 2.2.14) by first defining local sections and then our "local coordinates" will be a choice of local sections which form a basis on each fibre.

Definition 2.5.6. Let $\pi : E \to M$ be a vector bundle and let $U \subseteq M$ be open. A *local section* is a continuous map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $\pi : E \to M$ is a smooth vector bundle, a *smooth local section* is a section $\sigma : U \to E$ which is also smooth.

Definition 2.5.7. Let $\pi : E \to M$ be a vector bundle and let $U \subseteq M$ be open. We say a k-tuple of local sections $(\sigma_1, ..., \sigma_k)$ is *linearly independent* if $(\sigma_1(p), ..., \sigma_k(p))$ is a linearly independent k-tuple in E_p for all $p \in U$. We say $(\sigma_1, ..., \sigma_k)$ *span* E if $(\sigma_1(p), ..., \sigma_k(p))$ spans E_p for all $p \in U$. A *local frame for E over U* is an ordered k-tuple $(\sigma_1, ..., \sigma_k)$ of local sections which is linearly independent and span E. If $\pi : E \to M$ is a smooth vector bundle, we say a local frame is a *smooth local frame* if the local sections are smooth.

Analogous to Proposition 2.2.14 we can determine exactly when sections of a smooth vector bundle are smooth by considering the component functions in a smooth local frame.

Definition 2.5.8. Let $\pi : E \to M$ be a vector bundle of rank k. Let $U \subseteq M$ be open and let $(\sigma_1, ..., \sigma_k)$ be a local frame. Then a function $\tau : M \to E$ satisfying $\pi \circ \tau = \mathrm{id}_M$ can be written as $\tau(p) = \sum_{i=1}^{k} \tau^i(p)\sigma_i(p)$, where $\tau^i : U \to \mathbb{R}$ is uniquely determined since for all p, $(\sigma_1(p), ..., \sigma_k(p))$ is a basis of E_p. We call these τ^is the *component functions of* τ with respect to the given local frame.

Example 2.5.9. Let $\pi : E \to M$ be a smooth vector bundle and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialisation. Let $e_1, ..., e_k$ be the standard basis for $\mathbb{R}^k$ and let $\tilde{e}_i : U \to U \times \mathbb{R}^k$ be the smooth map given by $\tilde{e}_i(p) = (p, e_i)$. Now define the maps $\sigma_i(p) := \Phi^{-1} \circ \tilde{e}_i(p)$. Then σ_i is smooth since the composition of smooth maps is smooth and Φ is a diffeomorphism so Φ^{-1} is smooth. Then $\mathrm{proj}_U \circ \Phi = \pi$ implies

$$\pi \circ \sigma_i(p) = \pi \circ \Phi^{-1}(p, e_i) = \mathrm{proj}_U(p, e_i) = p.$$

So each σ_i is a smooth section. Then since $\Phi|_{E_p}$ is an isomorphism $E_p \to \{p\} \times \mathbb{R}^k$, its inverse $\Phi|_{E_p}^{-1}$ maps bases of $\{p\} \times \mathbb{R}^k$ to bases of E_p. Then since $e_1, ..., e_k$ is a basis of $\mathbb{R}^n$, $(p, e_1), ..., (p, e_k)$ is a basis for $\{p\} \times \mathbb{R}^k$. Then since $\Phi^{-1}(p, e_i) = \sigma_i(p)$, $\sigma_1(p), ..., \sigma_k(p)$ is a basis of E_p, as required. We call $\sigma_1, ..., \sigma_k$ the *local framed associated to* Φ.

It turns out that given a local frame, we can construct a local trivialisation whose associated local frame is the one given. The proof is omitted for brevity and is given in Proposition 10.19 in [Lee12]. The result is stated formally below.

Proposition 2.5.10. *Every smooth local frame for a smooth vector bundle is associated with a smooth local trivialisation as in Example 2.5.9.*

Proposition 2.5.11. *Let $\pi : E \to M$ be a smooth vector bundle of rank k. Let $U \subseteq M$ be open and let $(\sigma_1, ..., \sigma_k)$ be a smooth local frame. Then a function $\tau : M \to E$ satisfying $\pi \circ \tau = \mathrm{id}_M$ is smooth on U if and only if its component functions with respect to $(\sigma_1, ..., \sigma_k)$ are smooth.*

Proof. Let $\Phi : U \to U \times \mathbb{R}^k$ be the local trivialisation associated to this smooth local frame. Then since Φ is a diffeomorphism, $\Phi \circ \tau$ is smooth on U if and only if τ is smooth on U. Then we have that for all $p \in U$

$$(\Phi \circ \tau)(p) = (\Phi \circ \tau)\left(\sum_{i=1}^{k} \tau^i(p)\sigma_i(p)\right).$$

Then since Φ is a linear map on each fibre, we have that

$$(\Phi \circ \tau)(p) = \sum_{i=1}^{k} \tau^i(p)\Phi(\sigma_i(p)).$$

Then since $(\sigma_1, ..., \sigma_k)$ is the local frame associated to Φ, $\Phi \circ \sigma = \Phi \circ \Phi^{-1} \circ \tilde{e}_i$. So we have

$$(\Phi \circ \tau)(p) = \sum_{i=1}^{k} \tau^i(p)\tilde{e}_i(p) = (p, \tau^1(p), ..., \tau^k(p)).$$

Then $\Phi \circ \tau$ is smooth if and only if the component functions $(\tau^1, ..., \tau^k)$ are smooth. □

Example 2.5.12. What follows is left as Exercise 10.9 in [Lee12] and as such is my own work. Let $\pi : E \to M$ be a vector bundle of rank k. We define the *zero section* to be the function $\Theta : M \to E$ given by $\Theta(p) := 0_p$, where $0_p \in E_p$ is the zero vector. We have that $\pi \circ \Theta(p) = \pi(0_p) = p$. We first show that Θ is continuous and hence a section. To this end, let $V \subseteq E$ be open and let $v_p \in V$. Then let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialisation with $p \in U$. We have that $(\Phi \circ \Theta)(p) = (p, 0)$ for all $p \in U$, so $\mathrm{proj}_U \circ (\Phi \circ \Theta)(p) = p = \mathrm{id}_U(p)$ and $\mathrm{proj}_{\mathbb{R}^k} \circ (\Phi \circ \Theta)(p) = 0$. Both of these maps are continuous since the identity and constant maps are continuous. So by properties of the product topology, $\Phi \circ \Theta$ is continuous on U. Then since both V and $\pi^{-1}(U)$ are open in E, $V \cap \pi^{-1}(U)$ is open

in E. So by continuity, $\Theta^{-1}(V \cap \pi^{-1}(U)) = \Theta^{-1}(V) \cap \Theta^{-1}(\pi^{-1}(U)) = \Theta^{-1}(V) \cap U$ is open in U. Then since U is open in M, $\Theta^{-1}(V) \cap U$ is also open in M. Hence $\Theta^{-1}(V) \cap U$ is an open neighbourhood of p which is completely contained in $\Theta^{-1}(V)$, so p is an interior point of $\Theta^{-1}(V)$. Since $v_p \in V$ was arbitrary, every $p \in \Theta^{-1}(V)$ is an interior point and so $\Theta^{-1}(V)$ is open, as required.

Now we show that if $\pi : E \to M$ is actually a smooth vector bundle, then the zero section is smooth. Let $p \in M$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a smooth local trivialisation with $p \in U$. Above, we calculated that $\Phi \circ \Theta(p) = (p, 0)$, so each of its component functions in the smooth local frame associated to Φ are the map $p \mapsto 0 \in \mathbb{R}$, which is constant and hence smooth. Thus by Proposition 2.5.11, Θ is smooth on U. Then since $p \in M$ was arbitrary, every point in M has an open neighbourhood $U \subseteq M$ such that $\Theta|_U$ is smooth. So by Proposition 2.1.13, Θ is smooth on M, as required.

We now show a general version of Proposition 2.2.17, i.e. the smooth sections of a smooth vector bundle form a $C^\infty(M)$-module. The proof is left as Exercise 10.11 in [Lee12] so what is presented is my own work.

Proposition 2.5.13. *Let $\pi : E \to M$ be a smooth vector bundle and let $f, g \in C^\infty(M)$ and $\sigma, \tau \in \Gamma(E)$. We define pointwise addition and scalar multiplication as follows. For $p \in M$ let*

$$(\sigma + \tau)(p) := \sigma(p) + \tau(p) \text{ and } (f\sigma)(p) := f(p)\sigma(p).$$

Then $f\sigma + g\tau$ is a smooth section. Moreover, $\Gamma(E)$ is a $C^\infty(M)$-module.

Proof. Let $p \in M$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialisation with $p \in U$. Then for all $q \in U$ we have

$$\Phi \circ (f\sigma + g\tau)(q) = (q, f(q)\sigma_1(q) + g(q)\tau_1(q), ..., f(q)\sigma_k(q) + g(q)\tau_k(q)).$$

Then since σ and τ are smooth sections, their component functions $\sigma_i, \tau_i : U \to \mathbb{R}$ are smooth for all $1 \leq i \leq k$ by Proposition 2.5.11. We also have that $f|_U$ and $g|_U$ are smooth since f and g are smooth. So $f, g, \sigma_i, \tau_i \in C^\infty(U)$ for all $1 \leq i \leq k$ and so since $C^\infty(U)$ is a ring (Lemma 2.1.18) we have that $f\sigma_i + g\tau_i$ is smooth on U for all $1 \leq i \leq k$. Then by Proposition 2.5.11, since the component functions of $f\sigma + g\tau$ are smooth on U, we have that $f\sigma + g\tau$ is smooth on U. So p has an open neighbourhood U on which $f\sigma + g\tau$ is smooth. Then since $p \in M$ was arbitrary, $f\sigma + g\tau$ is smooth on M be Proposition 2.1.13. This shows that $\Gamma(E)$ is closed under addition and scalar multiplication. Distributivity holds pointwise. The zero section is the additive identity of $\Gamma(E)$. Lastly, recall that $1_M : p \mapsto 1$ is the multiplicative identity of $C^\infty(M)$ and we have that $1_M\sigma(p) = 1_M(p)\sigma(p) = 1 \cdot \sigma(p) = \sigma(p)$ for all $p \in M$. Hence, $\Gamma(E)$ is a $C^\infty(M)$-module. □

Remark 2.5.14. We can view real numbers as constant smooth real maps. In this way, we see that the scalar multiplication above restricted to constant smooth real maps turns $\Gamma(E)$ into a real vector space.

We finish this section by proving a powerful result which gives us a way of ruling out spaces as potential vector bundles a given base space. Namely, we show that the projection map $\pi : E \to M$ is a homotopy equivalence. The proof is left as Problem 10-2 in [Lee12] so what follows is my own work.

Lemma 2.5.15. *Let $\pi : E \to M$ be a vector bundle over a topological space M. Then the projection π is a homotopy equivalence.*

Proof. The idea of the proof is to contract each fibre down to a point, e.g. the zero vector. In this way, we show that the zero section $\Theta : M \to E$ (see Example 2.5.12) is a homotopy inverse of the projection map $\pi : M \to E$. Firstly, since Θ is a section, we have that $\pi \circ \Theta = \mathrm{id}_M$ so certainly $\pi \circ \Theta \simeq \mathrm{id}_M$.

Next, to show that $\Theta \circ \pi \simeq \mathrm{id}_E$ consider the following candidate for a homotopy. Define $H : E \times I \to E$ by $H(v_p, t) := tv_p$ for all $p \in M$ and $v_p \in E_p$. Since each fibre E_p is a vector space, $tv_p \in E_p \subset E$ for all $t \in I$. So H is well-defined. We also have for $p \in M$ and $v_p \in E_p$ that $H(v_p, 0) = 0_p = \Theta \circ \pi(v_p)$

and $H(v_p, 1) = v_p = \mathrm{id}_E(v_p)$. It remains to show that H is continuous and we proceed similarly to example 2.5.12. Let $V \subseteq E$ be open, $v_p \in V$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialisation with $p \in U$. Define the map $f : \pi^{-1}(U) \to \mathbb{R}^k$ to be $f = \mathrm{proj}_{\mathbb{R}^k} \circ \Phi$. Then for $v_q \in \pi^{-1}(U)$, $(\Phi \circ H)(v_q, t) = (q, tf(v_q))$. So $\mathrm{proj}_U \circ (\Phi \circ H)(v_q, t) = q = \pi(v_q) = \pi \circ \mathrm{proj}_V(v_q, t)$, which is the composition of continuous functions so is continuous. We also have that $\mathrm{proj}_{\mathbb{R}^k} \circ (\Phi \circ H)(v_p, t) = tf(v_p)$, which we can represent as the following composition

$$\pi^{-1}(U) \times I \xrightarrow{(f, \mathrm{id}_I)} \mathbb{R}^k \times I \longrightarrow \mathbb{R}^k,$$

where the second map is scalar multiplication $(v, t) \mapsto t \cdot v$, which is continuous. So $\mathrm{proj}_{\mathbb{R}^k} \circ (\Phi \circ H)$ is the composition of continuous maps and is hence continuous. So by properties of the product topology, the map $\Phi \circ H$ is continuous on U. Then since V is open in E, $V \cap \pi^{-1}(U)$ is open in $\pi^{-1}(U)$. So since Φ is a homeomorphism, $\Phi(V \cap \pi^{-1}(U))$ is open in $U \times \mathbb{R}^k$. Then by continuity of $\Phi \circ H$, $(\Phi \circ H)^{-1}(\Phi(V \cap \pi^{-1}(U))) = H^{-1}(V \cap \pi^{-1}(U)) = H^{-1}(V) \cap H^{-1}(\pi^{-1}(U))$ is open in $\pi^{-1}(U) \times I$. Then since $E_q \subseteq \pi^{-1}(U)$ for $q \in U$ and each fibre is a vector space, $v_q \in \pi^{-1}(U) \implies tv_q \in \pi^{-1}(U)$ for all $t \in I$. So $H^{-1}(\pi^{-1}(U)) = \pi^{-1}(U) \times I$. So $H^{-1}(V) \cap (\pi^{-1}(U) \times I)$ is open in $\pi^{-1}(U) \times I$ and since $\pi^{-1}(U) \times I$ is open in $E \times I$ (since $\pi^{-1}(U)$ is open in E), we have that $H^{-1}(V) \cap (\pi^{-1}(U) \times I)$ is also open in $E \times I$. Then since $H^{-1}(V) \cap (\pi^{-1}(U) \times I) \subseteq H^{-1}(V)$ and contains (v_p, t) for all $t \in I$, each (v_p, t) is an interior point of $H^{-1}(V)$. Then since $v_p \in V$ was arbitrary, we have that every point in $H^{-1}(V)$ is an interior point. Hence $H^{-1}(V)$ is open. Thus H is continuous, as required. □

Corollary 2.5.16. *Let $\pi : E \to M$ be a vector bundle. Then $E \simeq M$.*

Example 2.5.17. Since the torus $T^2 := S^1 \times S^1$ is not homotopy equivalent to S^2, T^2 cannot be a vector bundle over S^2.

III. Linear Algebra and Further Constructions on Smooth Manifolds

3.1 Covectors and Tensors

In Section 2.5, we built up the theory of vector bundles and we saw that the tangent bundle of a smooth manifold is an example of a smooth vector bundle. We would now like to construct other smooth vector bundles on smooth manifolds. To do this, we build up the theory of tensors which allows us to construct new vector spaces out of old ones. We will then use this to construct new vector spaces associated to smooth manifolds by applying these constructions to each tangent space. In this chapter, we will consider only real vector spaces but this theory generalises naturally to vector spaces over any field. What follows is adapted from [Lee12] unless stated otherwise.

We begin this section by proving some results about the dual space. This will be our blueprint in generalising to covariant tensors. Recall that in Example 1.2.6 we defined the dual space of a (real) vector space V to be $V^* := \mathrm{Hom}(V, \mathbb{R})$. We will call elements of V^* *covectors*.

Proposition 3.1.1. *Let V be a finite dimensional vector space and let $(e_1, ..., e_n)$ be a basis for V. Define $\varepsilon^1, ..., \varepsilon^n \in V^*$ to be the covectors*

$$\varepsilon^i(e_j) := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

*Then $(\varepsilon^1, ..., \varepsilon^n)$ is a basis for V^*, called the **dual basis** of $(e_1, ..., e_n)$. Thus $\dim(V^*) = \dim(V)$.*

Proof. This proof was left as Exercise 11.2 in [Lee12] so what follows is my own work. First, let $\omega \in V^*$ and let $\omega_i := \omega(e_i)$. If $v = \sum_{i=1}^n v^i e_i \in V$, then by linearity

$$\omega(v) = \sum_{i=1}^n v^i \omega(e_i) = \sum_{i=1}^n v^i \omega_i = \sum_{i=1}^n \omega_i \varepsilon^i(v).$$

Since $v \in V$ was arbitrary, we have that $\omega = \sum_{i=1}^n \omega_i \varepsilon^i$. So $\omega \in \text{span}\{\varepsilon_1, ..., \varepsilon_n\}$. Since $\omega \in V^*$ was arbitrary, $(\varepsilon^1, ..., \varepsilon^n)$ span V^*.

Next, let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ be such that $\sum_{i=1}^n \lambda_i \varepsilon^i = 0$. Then for all $v \in V$, we have

$$0 = \sum_{i=1}^n \lambda_i \varepsilon^i(v) = \sum_{i=1}^n \lambda_i v^i.$$

Then since $v \in V$ is arbitrary, each $v^i \in \mathbb{R}$ is arbitrary. So for this to be true for all $v \in V$, we must have that $\lambda_i = 0$ for all $1 \leq i \leq n$. So $(\varepsilon^1, ..., \varepsilon^n)$ are linearly independent. Thus $(\varepsilon^1, ..., \varepsilon^n)$ is a basis for V^*. □

Notice that covectors are linear maps which take one vector as an input and output a real number. So to generalise, we consider multilinear maps which take in multiple vectors as an input and output a real number. This is not the most general type of tensor but is enough for our purposes.

Definition 3.1.2. Let V be a vector space and let $k \geq 1$. A *covariant k-tensor* on V, $\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$, is a map which satisfies

$$\alpha(v_1, ..., \lambda v_i + \mu v_i', ..., v_n) = \lambda \alpha(v_1, ..., v_i, ..., v_n) + \mu \alpha(v_1, ..., v_i', ..., v_n)$$

for all $\lambda, \mu \in \mathbb{R}$ and $v_1, ..., v_i, v_i', ..., v_n \in V$. We denote the space of covariant k-tensors on V by $T^k(V^*)$. It is convention for a 0-tensor to just be a real number. As is the case for covectors, $T^k(V^*)$ is a vector space when equipped with pointwise addition and scalar multiplication. We also refer to covariant k-tensors as *multilinear maps*.

Example 3.1.3. Let V be a real vector space and let $\omega, \eta \in V^*$. We define their *tensor product* to be the map $\omega \otimes \eta : V \times V \to \mathbb{R}$ given by $(\omega \otimes \eta)(v_1, v_2) = \omega(v_1)\eta(v_2)$. Bilinearity follows from linearity of ω and η. So $\omega \otimes \eta$ is a covariant 2-tensor on V.

More generally, let $A \in T^k(V^*)$ and $B \in T^j(V^*)$. Then we define their *tensor product* to be the $(k+j)$-linear map $A \otimes B : \underbrace{V \times \cdots \times V}_{k+j \text{ times}} \to \mathbb{R}$ given by

$$(A \otimes B)(v_1, ..., v_{k+j}) = A(v_1, ..., v_k)B(v_{k+1}, ..., v_{k+j}).$$

Proposition 3.1.4. *Let V be a real vector space and let $F, F' \in T^k(V^*)$, $G, G' \in T^j(V^*)$, $H \in T^i(V^*)$ and $\lambda, \mu \in \mathbb{R}$. Then*

(a) $(\lambda F + \mu F') \otimes G = \lambda(F \otimes G) + \mu(F' \otimes G)$.

(b) $F \otimes (\lambda G + \mu G') = \lambda(F \otimes G) + \mu(F \otimes G')$.

(c) $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

Proof. Let $v_1, ..., v_{k+j+i} \in V$ be arbitrary. Properties (a) and (b) are proven in a similar way so we only prove (a):

$$\begin{aligned}(\lambda F + \mu F') \otimes G(v_1, ..., v_{k+j}) &= (\lambda F + \mu F')(v_1, ..., v_k)G(v_{k+1}, ..., v_{k+j}) \\ &= \lambda(F(v_1, ..., v_k)G(v_{k+1}, ..., v_{k+j})) + \mu(F'(v_1, ..., v_k)G(v_{k+1}, ..., v_{k+j})) \\ &= \lambda\,(F \otimes G)(v_1, ..., v_{k+j}) + \mu\,(F' \otimes G)(v_1, ..., v_{k+j}).\end{aligned}$$

Now we prove property (c):

$$\begin{aligned}(F \otimes G) \otimes H(v_1, ..., v_{k+j+i}) &= (F \otimes G)(v_1, ..., v_{k+j})H(v_{k+j+1}, ..., v_{k+j+i}) \\ &= F(v_1, ..., v_k)G(v_{k+1}, ..., v_{k+j})H(v_{k+j+1,...,v_{k+j+i}}) \\ &= F(v_1, ..., v_k)(G \otimes H)(v_{k+1}, ..., v_{k+j+i}) \\ &= F \otimes (G \otimes H)(v_1, ..., v_{k+j+i}).\end{aligned}$$

□

A natural question now is "given an arbitrary covariant k-tensor, can we find a pair of lower rank tensors whose tensor product is equal to the given k-tensor?" It turns out this is not always the case but instead they are always a linear combination of such tensors. For finite dimensional vector spaces, this is a corollary of the next result.

Proposition 3.1.5. *Let V be a finite dimensional vector space with basis $(e_1, ..., e_n)$ and let $(\varepsilon^1, ..., \varepsilon^n)$ be its dual basis. Then*

$$\mathcal{B} := \{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_j \leq n \ \ \forall\, 1 \leq j \leq k\}$$

is a basis for $T^k(V^*)$. And so $\dim(T^k(V^*)) = n^k$.

Proof. The proof is analogous to the proof of Proposition 3.1.1. To check $\text{span}(\mathcal{B}) = T^k(V)$, given $F \in T^k(V^*)$ we define $F_{i_1,...,i_k} := F(e_{i_1}, ..., e_{i_k})$ and proceed analogously as in Proposition 3.1.1. To show linear independence, we analogously consider a linear combination of elements of $\mathcal{B}$ that equals 0 and conclude since any k-tuple of vectors in V is mapped to 0, the coefficients must be 0. For the explicit details, see Proposition 12.4 in [Lee12]. □

3.2 Alternating Tensors

Given a covariant k-tensor, one might wonder what happens to the output if we permuted its entries. In general there is not a pattern, however we can study two particular cases: permutations leave the output invariant and permutations multiply the output by -1 raised to the sign of the permutation. We call the former a *symmetric tensor* and the latter an *alternating tensor*. For our purposes, we only need to study alternating tensors but a treatment of symmetric tensors can be found in Chapter 12 of [Lee12]. Unless stated otherwise, what follows is adapted from [Lee12].

Lemma 3.2.1. *Let V be a finite dimensional vector space and let $\alpha \in T^k(V^*)$. Then the following are equivalent:*

(a) $\alpha(v_1, ..., v_j, ..., v_i, ..., v_k) = -\alpha(v_1, ..., v_i, ..., v_j, ..., v_k)$ for all $v_1, ..., v_i, ..., v_j, ..., v_k \in V$.

(b) $\alpha(v_1, ..., v_k) = 0$ whenever $(v_1, ..., v_k)$ is linearly dependent.

(c) α is 0 if two of its entries are equal.

Proof. ((a) $\implies$ (c)): Let $w \in V$. Then (a) implies $\alpha(v_1, ..., w, ..., w, ..., v_k) = -\alpha(v_1, ..., w, ..., w, ...v_k)$. So $\alpha(v_1, ..., w, ..., w, ..., v_k) = 0$.
((b) $\implies$ (c)): Let $w \in V$. Then the tuple $(v_1, ..., w, ..., w, ..., v_k)$ is linearly dependent, so (b) implies $\alpha(v_1, ..., w, ..., w, ..., v_k) = 0$.
((c) $\implies$ (a)): For $v_1, ..., v_k \in V$ (c) implies $0 = \alpha(v_1, ..., v_i + v_j, ..., v_i + v_j, ..., v_k)$. Then multilinearity implies

$$\begin{aligned}0 =& \alpha(v_1, ..., v_i, ..., v_i, ..., v_k) + \alpha(v_1, ..., v_i, ..., v_j, ..., v_k) \\ &+ \alpha(v_1, ..., v_j, ..., v_i, ..., v_k) + \alpha(v_1, ..., v_j, ..., v_j, ..., v_k).\end{aligned}$$

Then (c) implies the outer two terms are 0. So we have
$0 = \alpha(v_1, ..., v_i, ..., v_j, ..., v_k) + \alpha(v_1, ..., v_j, ..., v_i, ..., v_k)$, which implies (a).

((c) $\implies$ (b)): Suppose $(v_1, ..., v_k)$ are linearly dependent in V, then up to relabelling $v_k = \sum_{j=1}^{k-1} a_j v_j$ for some $a_j \in \mathbb{R}$. Then by multilinearity we have

$$\alpha(v_1, ..., v_k) = \sum_{j=1}^{k-1} a_j \alpha(v_1, ..., v_k - 1, v_j).$$

Then v_j appears twice in the jth term, so (c) implies that each term is 0. Hence $\alpha(v_1, ..., v_k) = 0$. □

Definition 3.2.2. Let V be a finite dimensional vector space. We call $\alpha \in T^k(V^*)$ an *alternating k-tensor* on V if it satisfies any one of the properties in Lemma 3.2.1. We denote the set of alternating k-tensors on V by $\Lambda^k(V^*)$. We also call elements of $\Lambda^k(V^*)$ *k-covectors.*

Corollary 3.2.3. *Let V be a finite dimensional vector space and let $\alpha \in T^k(V^*)$. Then α is alternating if and only if for any permutation $\sigma \in S_k$ and $v_1, ..., v_k \in V$ we have*
$\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\alpha(v_1, ..., v_k)$.

Proof. The proof is left as Exercise 12.17 in [Lee12] so the proof given is my own. First suppose that $\alpha \in V$ is alternating and let $\sigma \in S_k$. Then since any permutation may be written as the product of 2-cycles $\sigma = \tau_1 \cdots \tau_n$, the action of σ on our entries is equivalent to swapping n pairs of entries. So since α is alternating, each swap multiplies the output by -1, resulting in a total multiplication by $(-1)^n$. Then since $\operatorname{sgn}(\sigma) = (-1)^n$, we retrieve $\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = (\operatorname{sgn}(\sigma))\alpha(v_1, ..., v_k)$.

On the other hand, suppose that $\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = (\operatorname{sgn}(\sigma))\alpha(v_1, ..., v_k)$ for all $v_1, ..., v_k \in V$ and $\sigma \in S_k$. Then swapping two entries can be represented by a 2-cycle $\tau \in S_k$. Then since $\operatorname{sgn}(\tau) = -1$ we retrieve property (a) in Lemma 3.2.1. So α is alternating. □

Proposition 3.2.4. *Let V be a finite dimensional vector space. Then $\Lambda^k(V^*)$ is a vector subspace of $T^k(V^*)$.*

Proof. The proof is omitted in [Lee12] so what follows is my own work. By the vector subspace criterion, it is sufficient to show that the linear combination of alternating k-tensors is alternating and that $\Lambda^k(V^*)$ is non-empty. Firstly, $\Lambda^k(V^*)$ is non-empty since the zero map $0 : (v_1, ..., v_k) \mapsto 0 \in \mathbb{R}$ is multilinear trivially multilinear and property (b) in Lemma 3.2.1 is trivially satisfied. Now suppose that $\alpha, \beta \in \Lambda^k(V^*)$ and $\lambda, \mu \in \mathbb{R}$ and suppose that $v_1, ..., v_k \in V$ are linearly dependent. Then we have that

$$(\lambda\alpha + \mu\beta)(v_1, ..., v_k) = \lambda\,\alpha(v_1, ..., v_k) + \mu\,\beta(v_1, ..., v_k).$$

Then since α and β are alternating, property (b) in Lemma 3.2.1 implies $\alpha(v_1, ..., v_k) = 0 = \beta(v_1, ..., v_k)$. Hence $(\lambda\alpha + \mu\beta)(v_1, ..., v_k) = 0$ and so is alternating. So $\Lambda^k(V^*)$ is closed under addition and scalar multiplication. □

Remark 3.2.5. Property (b) in Lemma 3.2.1 implies that if $k > \dim(V)$, then $\dim(\Lambda^k(V^*)) = 0$ since every k-tuple is necessarily linearly dependent. Covectors and real numbers satisfy property (c) vacuously and hence $\Lambda^1(V^*) = T^1(V^*) = V^*$ and $\Lambda^0(V^*) = T^0(V^*) = \mathbb{R}$.

We would now like to construct a basis for $\Lambda^k(V^*)$. To do this, we first introduce multi-indices.

Definition 3.2.6. Let $k \in \mathbb{N}$. A *multi-index of length k* is an ordered k-tuple of positive integers $I = (i_1, ..., i_k)$. If $\sigma \in S_k$, we define $I_\sigma := (i_{\sigma(1)}, ..., i_{\sigma(k)})$. We say a multi-index $I = (i_1, ..., i_k)$ is *increasing* if $i_1 < i_2 < \cdots < i_k$. Given two multi-indices $I = (i_1, ..., i_k)$ and $J = (j_1, ..., j_l)$, of length k and l respectively, we define a multi-index of length $k + l$ by $IJ := (i_1, ..., i_k, j_1, ..., j_l)$.

With multi-indices, we can now define elementary alternating tensors. These will not quite form a basis of $\Lambda^k(V^*)$ but they are the first step.

Definition 3.2.7. Let $(\varepsilon^1, ..., \varepsilon^n)$ be a basis for V^* and let $I = (i_1, ..., i_k)$ be a multi-index of length k with $1 \le i_1, ..., i_k \le n$. We define the covariant k-tensor ε^I by

$$\varepsilon^I(v_1, ..., v_k) := \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix}.$$

We call ε^I an *elementary alternating tensor*. Swapping two entries of ε^I results in swapping two columns in the matrix in its definition, which results in multiplication by -1 by properties of the determinant. So ε^I is indeed alternating.

Lemma 3.2.8. *Let $(e_1, ..., e_n)$ be a basis for a vector space V and let $(\varepsilon^1, ..., \varepsilon^n)$ be the associated dual basis for V^*. Let ε^I be an elementary alternating tensor. Then*

(a) If I has a repeated index, then $\varepsilon^I = 0$.

(b) If $J = I_\sigma$ for $\sigma \in S_k$, then $\varepsilon^I = \operatorname{sgn}(\sigma)\varepsilon^J$.

(c) Evaluating ε^I on a sequence of basis vectors results in

$$\varepsilon^I(e_{j_1}, ..., e_{j_k}) = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix} =: \delta^I_J,$$

where δ^a_b is the Kronecker delta.

Proof. If I has a repeated index, then the matrix in the definition of ε^I has at least two repeated rows and hence its determinant is 0, which proves (a). For (b), we have that σ permutes the rows of the matrix in the determinant which results in multiplication by $\operatorname{sgn}(\sigma)$. Lastly, to show (c) note that $\varepsilon^i(e_j) = \delta^i_j$. So each entry of the matrix will be a Kronecker delta. □

Before we give a basis for $\Lambda^k(V^*)$, we need one last simplifying notation.

Definition 3.2.9. We write

$$\sum_I{}' \alpha_I \varepsilon^I := \sum_{\{I : i_1 < \cdots i_k\}} \alpha_I \varepsilon^I$$

for a sum over increasing multi-indices of length k.

Proposition 3.2.10. *Let V be an n-dimensional vector space with basis $(e_1, ..., e_n)$ and let $(\varepsilon^1, ..., \varepsilon^n)$ be the associated dual basis for V^*. Then for $k \leq n$,*

$$\mathcal{E} = \{\varepsilon^I : I \text{ an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore $\dim(\Lambda^k(V^*)) = \binom{n}{k}$.

Proof. First suppose $\alpha \in \Lambda^k(V^*)$ and for each multi-index of length k, I, define the number $\alpha_I := \alpha(e_{i_1}, ..., e_{i_k})$. Then since α is alternating, $\alpha_I = 0$ if I has a repeated index and for $\sigma \in S_k$ we have $J = I_\sigma \implies \alpha_J = \operatorname{sgn}(\sigma)\alpha_I$. Then for any multi-index J, Lemma 3.2.8 (c) gives us that

$$\sum_I{}' \alpha_I \varepsilon^I(e_{j_1}, ...e_{j_k}) = \sum_I{}' \alpha_I \delta^I_J = \alpha_J = \alpha(e_{j_1}, ..., e_{j_k}).$$

Then by multi-linearity $\sum_I{}' \alpha_I \varepsilon^I(v_1, ..., v_k) = \alpha(v_1, ..., v_k)$ for all $v_1, ..., v_k \in V$. So $\sum_I{}' \alpha_I \varepsilon^I = \alpha$. So $\mathcal{E}$ spans $\Lambda^k(V^*)$. Now suppose that $\sum_I{}' \alpha_I \varepsilon^I = 0$. Then for any increasing multi-index J, Lemma 3.2.8 (c) implies

$$0 = \sum_I{}' \alpha_I \varepsilon^I(e_{j_1}, ..., e_{j_k}) = \sum_I{}' \alpha_I \delta^I_J = \alpha_J.$$

So each coefficient $\alpha_J = 0$. So $\mathcal{E}$ is linearly independent and is hence a basis for $\Lambda^k(V^*)$. The dimension follows from a counting argument on the number of increasing multi-indices. □

One question we might ask now is whether the tensor product of two alternating tensors is again an alternating tensor. In general the answer is no. For example, suppose V is a vector space with basis (e_1, e_2) with associated dual basis $(\varepsilon^1, \varepsilon^2)$. Then $\varepsilon^1 \otimes \varepsilon^2(e_1, e_2) = 1$ but $\varepsilon^1 \otimes \varepsilon^2(e_2, e_1) = 0$ so $\varepsilon^1 \otimes \varepsilon^2$ is not alternating. So we would like to construct another kind of product between alternating tensors which produces new alternating tensors: the wedge product. We do this by taking the tensor product then projecting onto the subspace of alternating tensors. We first define this projection.

Definition 3.2.11. Let V be a finite dimensional vector space. We define the *alternation* $\operatorname{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ by

$$(\operatorname{Alt}\alpha)(v_1, ..., v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma))\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}).$$

Proposition 3.2.12. *Let V be a finite dimensional vector space and let $\alpha \in T^k(V^*)$. Then*

(a) $\operatorname{Alt}\alpha$ *is alternating.*

(b) $\operatorname{Alt}\alpha = \alpha$ *if and only if* α *is alternating.*

(c) Alt *is a linear map.*

Proof. This was left as Exercise 14.4 in [Lee12] so what follows is my own work. To prove (a), let $v_1, ..., v_i, ..., v_j, ..., v_k \in V$ and let $\sigma \in S_k$. Define $\tau_\sigma \in S_k$ be the two cycle given by $\tau_\sigma(\sigma(i)) = \sigma(j)$, $\tau_\sigma(\sigma(j)) = \sigma(i)$ and $\tau_\sigma(\sigma(l)) = \sigma(l)$ for $l \neq i, j$. Then $\operatorname{sgn}(\tau_\sigma\sigma) = -\operatorname{sgn}(\sigma)$ and

$$\alpha(v_{\tau_\sigma\sigma(1)}, ..., v_{\tau_\sigma\sigma(j)}, ..., v_{\tau_\sigma\sigma(i)}, ..., v_{\tau_\sigma\sigma(k)}) = \alpha(v_{\sigma(1)}, ..., v_{\sigma(i)}, ..., v_{\sigma(j)}, ..., v_{\sigma(k)}).$$

Then we have

$$\begin{aligned}(\operatorname{Alt}\alpha)(v_1, ..., v_i, ..., v_j, ..., v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma))\alpha(v_{\tau_\sigma\sigma(1)}, ..., v_{\tau_\sigma\sigma(j)}, ..., v_{\tau_\sigma\sigma(i)}, ..., v_{\tau_\sigma\sigma(k)}) \\ &= -\frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}(\tau_\sigma\sigma))\alpha(v_{\tau_\sigma\sigma(1)}, ..., v_{\tau_\sigma\sigma(j)}, ..., v_{\tau_\sigma\sigma(i)}, ..., v_{\tau_\sigma\sigma(k)}).\end{aligned}$$

Since we are summing over all permutations, we may sum over $\tau_\sigma\sigma$ instead without affecting the result. So we may re-index the sum with $\rho = \tau_\sigma\sigma$ and we get

$$(\operatorname{Alt}\alpha)(v_1, ..., v_i, ..., v_j, ..., v_k) = -\frac{1}{k!} \sum_{\rho \in S_k} (\operatorname{sgn}(\rho))\alpha(v_{\rho(1)}, ..., v_{\rho(j)}, ..., v_{\rho(i)}, ..., v_{\rho(k)}).$$

This then gives us $(\operatorname{Alt}\alpha)(v_1, ..., v_i, ..., v_j, ..., v_k) = -(\operatorname{Alt}\alpha)(v_1, ..., v_j, ..., v_i, ..., v_k)$, so $\operatorname{Alt}\alpha$ is indeed alternating.

Part (a) gives us one direction of part (b). To show the other direction, suppose α is alternating. Then $\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = (\operatorname{sgn}(\sigma))\alpha(v_1, ..., v_k)$ by Corollary 3.2.3. So we have that

$$(\operatorname{Alt}\alpha)(v_1, ..., v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma))^2\alpha(v_1, ..., v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha(v_1, ..., v_k) = \frac{1}{k!} \cdot k!\, \alpha(v_1, ..., v_k),$$

since $\operatorname{sgn}(\sigma) \in \{1, -1\}$ and $|S_k| = k!$. So $\operatorname{Alt}\alpha = \alpha$.

Lastly, to show (c), note that parts (a) and (b) imply that Alt is a projection onto the subspace $\Lambda^k(V^*)$ and hence must be a linear map. □

Definition 3.2.13 (Wedge Product)**.** Let V be a finite dimensional vector space, $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$. We define their *wedge product* to be the $(k+l)$-covector

$$\omega \wedge \eta := \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\omega \otimes \eta).$$

The inclusion of this coefficient becomes apparent in the proof of the next result.

Lemma 3.2.14. *Let V be a finite dimensional vector space with basis $(e_1, ..., e_n)$ and let $(\varepsilon^1, ..., \varepsilon^n)$ be its associated dual basis for V^*. For any multi-indices $I = (i_1, ..., i_k)$ and $J = (j_1, ..., j_l)$ we have*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}.$$

Proof. By multilinearity, it is sufficient to show that

$$\varepsilon^I \wedge \varepsilon^J(e_{p_1}, ..., e_{p_{k+l}}) = \varepsilon^{IJ}(e_{p_1}, ..., e_{p_{k+l}}) \tag{3.1}$$

for any sequence of basis vectors $e_{p_1}, ..., e_{p_{k+l}}$. We proceed with 4 different cases.
Case 1: The multi-index $P = (p_1, ..., p_{k+l})$ has a repeated index. Then both sides of (3.1) are equal to 0 since both are alternating.
Case 2: P contains an index that does not appear in either I or J. So by Lemma 3.2.8 (c) evaluating the right hand side of (3.1) produces a matrix that contains at least one column of zeros. So the determinant is 0. Similarly, each term of the left hand side of (3.1) involves either ε^I or ε^J evaluated on a sequence of basis vectors whose indices are not a permutation of either I or J respectively. So each term is again 0 by Lemma 3.2.8.
Case 3: $P = IJ$ and contains no repeated indices. Then by Lemma 3.2.8 the right hand side of (3.1) is equal to 1. So we need to show that the left hand side is also 1. We have that

$$\varepsilon^I \wedge \varepsilon^J(e_{p_1}, ..., e_{p_{k+l}}) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}(\sigma)) \varepsilon^I(e_{p_{\sigma(1)}}, ..., e_{p_{\sigma(k)}}) \varepsilon^J(e_{p_{\sigma(k+1)}}, ..., e_{p_{\sigma(k+l)}}).$$

By Lemma 3.2.8 (c), the only non-zero terms are those for which σ permutes the first k indices and the last l indices of P separately. So we write $\sigma = \tau\rho$ where $\tau \in S_k$ acts on $\{1, ..., k\}$ and $\rho \in S_l$ acts on $\{k+1, ..., k+l\}$. Then $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\rho)$ and we have

$$\begin{aligned}
\varepsilon^I \wedge \varepsilon^J(e_{p_1}, ..., e_{p_{k+l}}) &= \frac{1}{k!\, l!} \sum_{\substack{\tau \in S_k \\ \rho \in S_l}} (\operatorname{sgn}(\tau)\operatorname{sgn}(\rho)) \varepsilon^I(e_{(p_{\tau(1)}}, ..., e_{p_{\tau(k)}}) \varepsilon^J(e_{p_{\rho(k+1)}}, ..., e_{p_{\rho(k+l)}}) \\
&= \left(\frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn}(\tau)) \varepsilon^I(e_{(p_{\tau(1)}}, ..., e_{p_{\tau(k)}}) \right) \left(\frac{1}{l!} \sum_{\rho \in S_l} (\operatorname{sgn}(\rho)) \varepsilon^J(e_{p_{\rho(k+1)}}, ..., e_{p_{\rho(k+l)}}) \right) \\
&= (\operatorname{Alt} \varepsilon^I)(e_{p_1}, ..., e_{p_k})(\operatorname{Alt} \varepsilon^J)(e_{p_{k+1}}, ..., e_{p_{k+l}}) \\
&= \varepsilon^I(e_{p_1}, ..., e_{p_k}) \varepsilon^J(e_{p_{k+1}}, ..., e_{p_{k+l}}) = 1.
\end{aligned}$$

Case 4: P is a permutation of IJ and has no repeated indices. We can apply a permutation to P to retrieve case 3. Then since permuting the entries of P multiplies both sides by the sign of that permutation, we still have equality. □

We finish this section by proving some properties of the wedge product.

Proposition 3.2.15. *Let $\omega, \omega', \eta, \eta'$ and λ be multi-covectors on a finite dimensional vector space V. Then*

(a) Bilinearity: For $a, a' \in \mathbb{R}$

$$\begin{aligned}
(a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta), \\
\omega \wedge (a\eta + a'\eta') &= a(\omega \wedge \eta) + a'(\omega \wedge \eta').
\end{aligned}$$

(b) Associativity: $\omega \wedge (\eta \wedge \lambda) = (\omega \wedge \eta) \wedge \lambda$.

(c) Anti-commutativity: if $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

(d) If $(\varepsilon^1, ..., \varepsilon^n)$ is a basis for V^* and $I = (i_1, ..., i_k)$ is a multi-index, then

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

Proof. Bilinearity follows from the fact that $\otimes$ is bilinear (Proposition 3.1.4) and that Alt is linear (Proposition 3.2.12 (c)). To show (b), let I, J and K be multi-indices. Then Lemma 3.2.14 implies

$$\varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K) = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^{IJK} = \varepsilon^{IJ} \wedge \varepsilon^K = (\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K.$$

Associativity in general then follows from bilinearity.

To show (c), let I and J be multi-indices of length k and l respectively and let $\tau \in S_{k+l}$ be the permutation which sends IJ to JI. Then by Lemma 3.2.14 and Corollary 3.2.3

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\mathrm{sgn}(\tau))\varepsilon^{JI} = (\mathrm{sgn}(\tau))\varepsilon^J \wedge \varepsilon^I.$$

Then since $k \cdot l$ many transpositions are required to permute IJ to JI, we have that $\mathrm{sgn}(\tau) = (-1)^{kl}$. So $\varepsilon^I \wedge \varepsilon^J = (-1)^{kl}\varepsilon^J \wedge \varepsilon^I$. The general case then follows from bilinearity.

To show (d), we proceed by induction. First, let $I_1 = (i_1)$. Then clearly $\varepsilon^{I_1} = \varepsilon^{i_1}$. Now suppose there exists an $k \in \mathbb{N}$ such that for a multi-index $I_k = (i_1, ..., i_k)$ (d) holds. Then the multi-index $I_{k+1} = (i_1, ..., i_{k+1})$ is equal to $I_k(i_{k+1})$. So by Lemma 3.2.14 and associativity we have that

$$\varepsilon^{I_{k+1}} = \varepsilon^{I_k} \wedge \varepsilon^{i_k+1} = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \wedge \varepsilon^{i_k}.$$

Hence (d) follows by induction. □

Corollary 3.2.16. *If k is odd and $\omega \in \Lambda^k(V^*)$, then $\omega \wedge \omega = 0$.*

Proof. Since k is odd, k^2 is also odd. So Proposition 3.2.15 (c) implies that $\omega \wedge \omega = (-1)\omega \wedge \omega$. So $\omega \wedge \omega = 0$. □

Remark 3.2.17. Let V be an n-dimensional vector space. We define the *exterior algebra* of V to be the vector space

$$\Lambda(V^*) := \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

Bilinearity and associativity of the wedge product implies that $\Lambda(V^*)$ with the wedge product is an associative algebra. Moreover, since $\Lambda^k(V^*) \wedge \Lambda^l(V^*) \subseteq \Lambda^{k+l}(V^*)$, the exterior algebra of V is said to be *graded*. Lastly, property (c) in Proposition 3.2.15 implies that the exterior algebra of V is an anti-commutative associative graded algebra.

3.3 Cotangent and Tensor Bundles

We have now developed enough of the theory of tensors to construct new vector bundles on smooth manifolds. In this section, we focus on the cotangent and tensor bundles and leave the discussion of bundles of alternating tensors to Chapter 4. What follows is adapted from [Lee12] unless stated otherwise.

Definition 3.3.1. Let M be a smooth manifold and let $p \in M$. We denote the *cotangent space* of M at p by $T_p^*M := (T_pM)^*$. Now we define the *cotangent bundle* of M by

$$T^*M := \bigsqcup_{p \in M} T_p^*M$$

and the *bundle of covariant k-tensors* on M by

$$T^kT^*M := \bigsqcup_{p \in M} T^k(T_p^*M).$$

Remark 3.3.2. Since $T^1(T_p^*M) = T_p^*M$ for all $p \in M$, we have that $T^1T^*M = T^*M$.

Lemma 3.3.3. *Let M be a smooth n-manifold. Then the cotangent bundle of M and the bundle of k-tensors on M are smooth manifolds and smooth vector bundles over M of rank n and n^k respectively.*

The proof is similar to the proofs that the tangent bundle is a smooth manifold and a smooth vector bundle over M and can be found in [Lee12] in Proposition 11.9 and Exercise 12.18 respectively.

Definition 3.3.4. Let M be a smooth manifold. We call sections of the cotangent bundle *covector fields* and sections of the bundle of covariant k-tensors on M *(covariant) k-tensor fields*. We also refer to smooth covector fields as *1-forms* and denote the set of 1-forms on M by $\Omega^1(M)$. That is $\Omega^1(M) := \Gamma(T^*M)$. We denote the set of smooth sections of T^kT^*M by $\mathcal{T}^k(M) := \Gamma(T^kT^*M)$. Given a smooth section of any of these bundles, A, we use the notation $A_p := A(p)$.

Remark 3.3.5. In Definition 3.1.2 we defined a 0-tensor to be a real number. So the sections of T^0T^*M are maps $M \to \mathbb{R}$. In particular, smooth sections are smooth real maps and so $C^\infty(M) = \Gamma(T^0T^*M)$. We also use the notation $\Omega^0(M) = C^\infty(M)$.

We would like to construct a local frame for these bundles. The local frame for the bundle of covariant tensors relies on some of the theory for the cotangent bundle so we will revisit it later in this section. This next result will provide us with a local frame for the cotangent bundle. We first define the candidate for a local frame of the cotangent bundle.

Definition 3.3.6. Let M be a smooth manifold and let (U, φ) be a smooth chart. Then for $p \in U$ define the covector $\lambda^i|_p \in T_p^*M$ by

$$\lambda^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) := \delta^i_j.$$

Then given a section ω, we define its *component functions* to be the maps $\omega_i : U \to \mathbb{R}$ given by $\omega_i(p) := \omega_p\left(\frac{\partial}{\partial x^i}\big|_p\right)$. In particular, $(\lambda^1|_p, ..., \lambda^n|_p)$ is the dual basis associated to the basis $\left(\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p\right)$ of T_pM. Then we have that $\omega_p = \sum_{i=1}^n \omega_i(p)\lambda^i|_p$ using linearity of ω. We then define the maps $\lambda^i : U \to T^*M$ by $\lambda^i(p) := \lambda^i|_p$. We call $(\lambda^1, ..., \lambda^n)$ the *coordinate coframe*.

Remark 3.3.7. Proposition 11.9 in [Lee12] provides us with a smooth chart for T^*M involving these $\lambda^i|_p$'s. Given a smooth chart of M, (U, φ) with $\varphi = (x^1, ..., x^n)$, we define the chart $\tilde{\varphi} : \sum_{i=1}^n \alpha_i\lambda^i|_p \mapsto (x_1, ..., x_n, \alpha_1, ..., \alpha_n)$.

Proposition 3.3.8. *Let M be a smooth manifold and let $\omega : M \to T^*M$ be a function such that $\pi \circ \omega = \mathrm{id}_M$, where $\pi : T^*M \to M$ is the smooth vector bundle projection. Then the following are equivalent:*

(a) ω is smooth

(b) In every smooth chart, the component functions of ω are smooth

(c) Each point of M is contained in some smooth chart on which ω has smooth component functions.

Proof. This is left as Exercise 11.12 in [Lee12] so the proof given is my own. To show that (a) $\implies$ (c) suppose ω is smooth and let (U, φ) be a smooth chart for M with $\varphi = (x^1, ..., x^n)$ and let $\tilde{\varphi}$ be the chart defined in Remark 3.3.7. Then $(\tilde{\varphi} \circ \omega)(p) = (x^1, ..., x^n, \omega_1(p), ..., \omega_n(p))$. Let π_{n+i}: $\mathbb{R}^{2n} \to \mathbb{R}$ be the projection onto the $(n+i)$th coordinate. Then $\omega_i = \pi_{n+i} \circ \tilde{\varphi} \circ \omega$ and so ω_i is the composition of smooth maps and is hence smooth.

To show (b) $\implies$ (c), let $p \in M$ and let (U, φ) be a smooth chart containing p. Then by assumption, the component functions of ω are smooth on U. Since $p \in M$ was arbitrary, (c) holds.

Lastly, to show (c) $\implies$ (a), let $p \in M$ and let (U, φ) be the smooth chart containing p on which the component functions of ω are smooth. Let $\tilde{\varphi}$ be the chart for T^*M defined in Remark 3.3.7. Then for all $q \in \varphi(U)$ we have that

$$\tilde{\varphi} \circ \omega \circ \varphi^{-1}(q) = \tilde{\varphi}(\omega_{\varphi^{-1}(q)}) = (q, (\omega_1 \circ \varphi^{-1})(q), ..., (\omega_n \circ \varphi^{-1})(q)).$$

Then since each component function is smooth, $\omega_i \circ \varphi^{-1}$ is the composition of smooth maps so is smooth. Then each component of $\tilde{\varphi} \circ \omega \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^{2n}$ is smooth, so it is a smooth map. Hence ω is smooth on U. Then since $p \in M$ was arbitrary, ω is smooth on M by Proposition 2.1.13. $\square$

Corollary 3.3.9. *Let M be a smooth manifold and (U, φ) be a smooth chart. Then the coordinate coframe $(\lambda^1, ..., \lambda^n)$ is a smooth local frame for T^*M.*

Proof. The component functions of each λ^i are constant maps and are hence smooth. So by Proposition 3.3.8 each λ^i is smooth. Then since $(\lambda^1|_p, ..., \lambda^n|_p)$ is a basis for T_p^*M for all $p \in U$, $(\lambda^1, ..., \lambda^n)$ is a smooth local frame. □

Given a smooth real map $f \in C^\infty(M)$, it is possible to define an associated 1-form which can be thought of as a generalisation of the gradient from multivariable calculus. It will also allow us to give a better notation for the coordinate coframe.

Proposition 3.3.10. *Let M be a smooth manifold and let $f \in C^\infty(M)$. Define the **differential of** f to be the map $df : M \to T^*M$ by $df_p(v) := vf$ for $v \in T_pM$. Then df is a smooth covector field on M.*

Proof. Whilst a proof is given in [Lee12], I have opted to present a slightly different proof of my own. Let (U, φ) be a smooth chart for M with $\varphi = (x^1, ..., x^n)$. Then the component functions of df are given by

$$(df)_i(p) := df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f.$$

Then by Lemma 2.2.15 (a), each $(df)_i$ is smooth. Since the chart was arbitrary, property (b) in Proposition 3.3.8 is satisfied. Hence df is smooth. □

Corollary 3.3.11. *Let M be a smooth manifold and let (U, φ) be a smooth chart for M with $\varphi = (x^1, ..., x^n)$. Let $(\lambda^1, ..., \lambda^n)$ be the coordinate coframe. Then $\lambda^i = dx^i$ for all $1 \leq i \leq n$. We call the dx^i's coordinate 1-forms and we call dx^i_p a coordinate covector.*

Proof. Let $p \in U$, then $dx^i\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta^i_j = \lambda^i|_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)$. □

Using the notation $\frac{\partial f}{\partial x^i} := \left.\frac{\partial}{\partial x^i}\right|_p f$, this allows us to write the differential in coordinates as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

It is in this form where the resemblance to the gradient is seen.

The next result demonstrates further that the differential of a smooth real map is a kind of derivative.

Proposition 3.3.12. *Let M be a smooth manifold and $f, g \in C^\infty(M)$. Then*

(a) Let $a, b \in \mathbb{R}$, then $d(a\,f + b\,g) = a\,df + b\,dg$.

(b) $d(fg) = f\,dg + g\,df$.

(c) If f is constant, then $df = 0$.

Proof. This is left as Exercise 11.21 in [Lee12] so what follows is my own work. Both (a) and (b) follow from the properties of derivations. For example let $p \in M$ and $v \in T_pM$, then $d(fg)_p(v) = v(fg) = f(p)\,vg + g(p)\,vf$. Then since $p \in M$ and $v \in T_pM$ were arbitrary, (b) follows. To prove (c), recall that Lemma 2.2.3 says that if $f \in C^\infty(M)$ is constant, then $vf = 0$ for all $v \in T_pM$ and $p \in M$. So $df_p(v) = vf = 0$ for all $v \in T_pM$ and $p \in M$. Hence $df = 0$. □

Proposition 3.3.13. *Let M be a smooth manifold and let $f \in C^\infty(M)$. Then f is constant on each connected component of M if and only if $df = 0$.*

Proof. It is sufficient to assume that M is connected. First, if f is constant we have that $df = 0$ by Proposition 3.3.12. Conversely, suppose $df = 0$, let $p \in M$ and let $C_p := \{q \in M : f(q) = f(p)\}$. If we show that $M = C_p$, we have shown that f is constant. Let $q \in C_p$ and let (U, φ) be a smooth chart. By definition of openness in $\mathbb{R}^n$ we can assume that U is homeomorphic to an open ball centred at $\varphi(q)$ (or a ball intersected with $\mathbb{H}^n$ in the case that $q \in \partial M$). Otherwise, let $B \subseteq \varphi(U)$ be an open ball containing q and make a new chart $(\varphi^{-1}(B), \varphi)$. Then each component of df is zero, so $\frac{\partial f}{\partial x^i} = 0$ on all of U. So by a result from multivariable calculus, f must be constant on U. Then $U \subseteq C_p$ and is an open neighbourhood of q and so q is an interior point. Then since $q \in C_p$ was arbitrary, we have that C_p is open. We also have that since $C_p = f^{-1}(\{p\})$ and singleton sets are closed in Hausdorff spaces, C_p is the continuous preimage of a closed set so is closed. Hence by the connectedness of M, C_p is either empty or is equal to M. But clearly $p \in C_p$, so $C_p = M$. □

To construct a local frame for the tensor bundles, recall that Proposition 3.1.5 gives us that the basis for $T^k(V^*)$, where V is a finite dimensional vector space, is given by k-fold tensor products of the dual basis for V^*. This means that at each point $p \in M$ in a smooth manifold and smooth chart (U, φ) containing p, a basis for $T^k(T^*M)$ is given by k-fold tensor products of coordinate 1-forms.

Definition 3.3.14. Let M be a smooth manifold and let $A : M \to T^kT^*M$ be a map such that $\pi \circ A = \mathrm{id}_M$ where $\pi : T^kT^*M \to M$ is the smooth vector bundle projection. Then given a smooth chart (U, φ) for M with $\varphi = (x^1, ..., x^n)$, we may write

$$A = \sum_{i_1,...,i_k} A_{i_1,...,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

We call $A_{i_1,...,i_k} : U \to \mathbb{R}$ the *component functions* of A.

We then have a result analogous to Proposition 3.3.8!

Proposition 3.3.15. *Let M be a smooth manifold and let $A : M \to T^kT^*M$ be a function such that $\pi \circ A = \mathrm{id}_M$, where $\pi : T^kT^*M \to M$ is the smooth vector bundle projection. Then the following are equivalent:*

(a) A is smooth

(b) In every smooth chart, the component functions of A are smooth

(c) Each point of M is contained in some smooth chart on which A has smooth component functions.

The proof is analogous to the proof of Proposition 3.3.8, see Proposition 12.19 in [Lee12]. Then as with the coordinate coframe for T^*M, we have that each k-fold tensor product of coordinate 1-forms has constant and thus smooth component functions and hence they form a smooth local frame.

As with tensors, we can produce new tensor fields via the tensor product!

Proposition 3.3.16. *Let M be a smooth manifold and let $A \in \mathcal{T}^k(M)$ and $B \in \mathcal{T}^j(M)$ and $f \in C^\infty(M)$. Then the tensor fields $(A \otimes B)_p := A_p \otimes B_p$ and $(fA)_p := f(p)A_p$ are smooth.*

Proof. This is left as Exercise 12.23 in [Lee12] so what follows is my own work. First, let (U, φ) be a smooth chart with $\varphi = (x^1, ..., x^n)$. Then on one hand, $A \otimes B$ in these coordinates is

$$A \otimes B = \sum_{i_1,...,i_{k+j}} (A \otimes B)_{i_1,...,i_{k+j}} dx^{i_1} \otimes \cdots \otimes dx^{i_{k+j}}.$$

On the other hand, for $p \in M$ and $v_1, ..., v_{k+j} \in T_pM$ we have that

$$(A \otimes B)_p(v_1, ..., v_{k+j}) = A_p(v_1, ..., v_k)B_p(v_{k+1}, ..., v_{k+j}).$$

So in coordinates we have

$$\begin{aligned} A \otimes B &= \left(\sum_{i_1,\dots,i_k} A_{i_1,\dots,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \right) \left(\sum_{i_{k+1},\dots,i_{k+j}} B_{i_{k+1},\dots,i_{k+j}} dx^{i_{k+1}} \otimes \cdots \otimes dx^{i_{k+j}} \right) \\ &= \sum_{i_1,\dots,i_{k+j}} A_{i_1,\dots,i_k} B_{i_{k+1},\dots,i_{k+j}} dx^{i_1} \otimes \cdots \otimes dx^{i_{k+j}}. \end{aligned}$$

Comparing the coefficients, we get that $(A \otimes B)_{i_1,\dots,i_{k+j}} = A_{i_1,\dots,i_k} B_{i_{k+1},\dots,i_{k+j}}$. Then since A and B are smooth, Proposition 3.3.15 (b) gives us that each component function $A_{i_1,\dots,i_k}, B_{i_{k+1},\dots,i_{k+j}} : U \to \mathbb{R}$ is smooth. Since $C^\infty(U)$ is a ring, products of these component functions are also smooth. Hence each component function of $A \otimes B$ is smooth. So by Proposition 3.3.15, $A \otimes B$ is smooth.

Similarly, let (U, φ) with $\varphi = (x^1, ..., x^n)$ be a smooth chart. Then in coordinates

$$fA = \sum_{i_1,\dots,i_k} (fA)_{i_1,\dots,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

We also have that for $p \in U$ and $v_1, ..., v_k \in T_pM$ that $(fA)_p(v_1, ..., v_k) = f(p)A_p(v_1, ..., v_n)$. So in coordinates, we have

$$(fA)_p = \sum_{i_1,\dots,i_k} f(p) A_{i_1,\dots,i_k}(p) dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p.$$

So comparing coefficients we get that $(fA)_{i_1,\dots,i_k}(p) = f(p)A_{i_1,\dots,i_k}(p)$ for all $p \in U$. Since A is smooth, Proposition 3.3.15 implies that each $A_{i_1,\dots,i_k}$ is smooth. Then each $(fA)_{i_1,\dots,i_k}$ is the product of smooth functions on U and is hence smooth. So by Proposition 3.3.15, fA is a smooth tensor field. □

Recall in Example 1.2.6 we showed that any linear map $f : V \to W$ has a dual map $f^* : W^* \to V^*$ given by $f^*(\varphi) = \varphi \circ f$ for $\varphi : W \to \mathbb{R}$ linear. Given a smooth map, $F : M \to N$, we have an associated linear map $dF_p : T_pM \to T_{F(p)}N$ at each tangent space to M. So given a covector $\omega \in T^*_{F(p)}N$, we can define a covector $(dF_p)^*(\omega) = \omega \circ dF_p \in T^*_pM$. This motivates the definition of the pullback of a tensor field.

Definition 3.3.17. Let $F : M \to N$ be a smooth map between smooth manifolds. Given a tensor field $A \in \mathcal{T}^k(N)$, we define the *pullback of A by F* to be the map $F^*A : M \to T^kT^*M$ defined by

$$(F^*A)_p(v_1, ..., v_k) := A_{F(p)}(dF_p(v_1), ..., dF_p(v_k)),$$

where $p \in M$ and $v_1, ..., v_k \in T_pM$. If $f \in \mathcal{T}^0(N)$, i.e. f is a smooth real map, then $F^*f := f \circ F$ and is smooth since the composition of smooth maps is smooth.

Proposition 3.3.18. *Let $F : M \to N$ be a smooth map and let $u \in C^\infty(M)$. Then $F^*(du) = d(u \circ F)$.*

Proof. Let $p \in M$ and $v \in T_pM$ be arbitrary. Then

$$\begin{aligned} (F^*du)_p(v) &= du_{F(p)}(dF_p(v)) \\ &= dF_p(v)u \text{ (by Lemma 2.2.4)} \\ &= v(u \circ F) \\ &= d(u \circ F)_p(v). \end{aligned}$$

□

Proposition 3.3.19. *Let $F : M \to N$ and $G : N \to P$ be smooth maps between smooth manifolds and let A and B be smooth covariant tensor fields on N. Then*

(a) Let $\lambda, \mu \in \mathbb{R}$. Then $F^*(\lambda A + \mu B) = \lambda F^*A + \mu F^*B$.

(b) Let $f \in C^\infty(N)$. Then $F^*(fA) = (f \circ F)F^*B$

(c) $F^*(A \otimes B) = (F^*A) \otimes (F^*B)$.

(d) F^*A *is a smooth tensor field on* A.

(e) If $C \in \mathcal{T}^k(P)$, *then* $(G \circ F)^*C = F^*(G^*C)$

(f) $(\mathrm{id}_N)^*B = B$

Proof. The proof is left as Exercise 12.26 in [Lee12] so what follows is my own work. For (a), (b) and (c) let $p \in M$ and $v_1, ..., v_{k+j} \in T_pM$ be arbitrary. Then (a) follows from

$$\begin{aligned}(F^*(\lambda A + \mu B))_p(v_1, ..., v_k) &= (\lambda A + \mu B)_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) \\ &= \lambda A_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) + \mu B_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) \\ &= \lambda (F^*A)_p(v_1, ..., v_k) + \mu (F^*B)_p(v_1, ..., v_k),\end{aligned}$$

(b) follows from

$$(F^*(fA))_p(v_1, ..., v_k) = (fA)_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) = (f \circ F)(p)\,(F^*A)_p(v_1, ..., v_k),$$

and (c) follows from

$$\begin{aligned}(F^*(A \otimes B))_p(v_1, ..., v_{k+j}) &= (A \otimes B)_{F(p)}(dF_p(v_1), ..., dF_p(v_{k+j})) \\ &= A_{F(p)}(dF_p(v_1), ..., dF_p(v_k))B_{F(p)}(dF_p(v_{k+1}), ..., dF_p(v_{k+j})) \\ &= (F^*A)_p(v_1, ..., v_k)(F^*B)_p(v_{k+1}, ..., v_{k+j}) = (F^*A \otimes F^*B)_p(v_1, ..., v_{k+j}).\end{aligned}$$

For (d) let (U, φ) and (V, ψ) be smooth charts for M and N respectively with $F(U) \subseteq V$ and $\psi = (y^1, ..., y^k)$. Then on U, using parts (a)-(c) and Proposition 3.3.18, we can write F^*A as

$$F^*A = F^*\left(\sum_{i_1,...,i_k} A_{i_1,...,i_k} dy^{i_1} \otimes \cdots \otimes dy^{i_k}\right) = \sum_{i_1,...,i_k} (A_{i_1,...,i_k} \circ F) d(y^{i_1} \circ F) \otimes \cdots \otimes d(y^{i_k} \circ F).$$

Then since A is smooth, Proposition 3.3.15 implies each $A_{i_1,...,i_k}$ is smooth on V. So each $A_{i_1,...,i_k} \circ F$ is smooth on U. We also have that each $y^{i_j} \circ F$ is smooth so its differential is also smooth by Proposition 3.3.10. So on U, F^*A is a $C^\infty(U)$-linear combination of smooth covector fields and hence must be smooth on U since the set of smooth sections is a $C^\infty(U)$-module (Proposition 2.5.13). Then since (U, φ) was an arbitrary chart, Proposition 3.3.15 implies that F^*A is smooth on M.

For (e), let $p \in M$ and $v_1, ..., v_k \in T_pM$. Then we use Lemma 2.2.4 (b) to get

$$\begin{aligned}((G \circ F)^*C)_p(v_1, ..., v_k) &= C_{G(F(p))}(d(G \circ F)_p(v_1), ..., d(G \circ F)_p(v_k)) \\ &= C_{G(F(p))}(dG_{F(p)}(dF_p(v_1)), ..., dG_{F(p)}(dF_p(v_k))) \\ &= (G^*C)_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) = (F^*(G^*C))_p(v_1, ..., v_k).\end{aligned}$$

Lastly, to prove (f), let $p \in N$ and $v_1, ..., v_k \in T_pN$. Then by Lemma 2.2.4 (c) we have

$$((\mathrm{id}_N)^*B)_p(v_1, ..., v_k) = B_{\mathrm{id}_N(p)}(d(\mathrm{id}_N)_p(v_1), ..., d(\mathrm{id}_N)_p(v_1)) = B_p(v_1, ..., v_k).$$

□

IV. Differential Forms

4.1 Differential Forms

In Chapter 3 we developed the theory of tensors and in particular of alternating tensors and we concluded by talking about the cotangent bundle and the bundle of covariant k-tensors. What remains is to talk about the bundle of alternating k-tensors. The smooth sections of this bundle are what we will call differential k-forms. This section mirrors Section 3.3, whilst the subsequent sections develop further the theory of differential forms. What follows is adapted from [Lee12] unless stated otherwise.

Definition 4.1.1. Let M be a smooth manifold and $k \in \mathbb{Z}_{\geq 0}$. The *bundle of alternating k-tensors* on M is defined to be

$$\Lambda^k T^*M := \bigsqcup_{p \in M} \Lambda^k(T_p^*M).$$

We call smooth sections of $\Lambda^k T^*M$ *(differential) k-forms* and we denote the set of k-forms by $\Omega^k(M) := \Gamma(\Lambda^k T^*M)$.

Remark 4.1.2. Following Remark 3.2.5 we see that $\Lambda^0 T^*M = T^0 T^*M$ and $\Lambda^1 T^*M = T^1 T^*M$ so our notation for the set of 0- and 1-forms is consistent. Furthermore, Remark 3.2.5 also implies that if $k > \dim(T_pM) = \dim(M)$, $\Lambda^k(T_p^*M)$ is trivial. So for $k > \dim(M)$ we have that $\Omega^k(M)$ is trivial. That is, we have no non-trivial k-forms when $k > \dim(M)$.

We also have that $\Omega^k(M)$ is a $C^\infty(M)$-module and a real vector space by Proposition 2.5.13 and Remark 2.5.14 respectively.

Analogous to the discussion of local frames for the bundle of covariant tensors, we have that at each point in a smooth chart (U, φ), a k-form is the linear combination of k-fold wedge products of coordinate covectors:

$$\omega_p = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Complementary to the notation for the elementary alternating tensors and Proposition 3.2.15 we define $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ for a multi-index $I = (i_1, ..., i_k)$. Then analogous to the case for the bundle of covariant k-tensors, $\{dx^I : I \text{ an increasing multi-index of length } k\}$ is a smooth local frame for $\Lambda^k T^*M$. Then by Proposition 2.5.11 we see that a form ω is smooth if and only if its component functions ω_I are smooth in any smooth chart.

We can extend the wedge product and its properties to the level of differential forms. That is Lemma 3.2.14 holds pointwise on any chart U and Proposition 3.2.15 holds pointwise on all of M.

Proposition 4.1.3. *Let M be a smooth manifold, $\omega \in \Omega^k(M)$, and $\eta \in \Omega^l(M)$. Then the map $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$ is smooth and hence is a $(k+l)$-form.*

Proof. The proof is omitted in [Lee12] so what follows is my own work. Let (U, φ) be a smooth chart with $\varphi = (x^1, ..., x^n)$. Then we can write

$$\omega = \sum_I{}' \omega_I dx^I \text{ and } \eta = \sum_J{}' \eta_J dx^J,$$

where the component functions ω_I, η_J are smooth. Then using the properties of the wedge product, we have that on U

$$\omega \wedge \eta = \left(\sum_I{}' \omega_I dx^I \right) \left(\sum_J{}' \eta_J dx^J \right) = \sum_{I,J} \omega_I \eta_J dx^I \wedge dx^J = \sum_{I,J} \omega_I \eta_J dx^{IJ}.$$

Then each component function of $\omega \wedge \eta$ is $\omega_I \eta_J$, which is the product of smooth functions on U and is hence also smooth on U. So by Proposition 2.5.11, $\omega \wedge \eta$ is smooth on M. □

Remark 4.1.4. Analogous to Remark 3.2.17, we can define the *exterior algebra* of M by

$$\Omega^*(M) := \bigoplus_{k=1}^{n} \Omega^k(M).$$

The wedge product turns $\Omega^*(M)$ into an associative anticommutative graded algebra.

In Section 3.3, we defined the pullback of a tensor field by a smooth map. Since differential forms are also tensor fields, the definition of a pullback works for differential forms too. In particular, Proposition 3.3.19 holds for differential forms. Moreover, given a smooth map $F : M \to N$ and a k-form on N, $\omega \in \Omega^k(N)$, we have that $(F^*\omega)_p$ is an alternating tensor on T_pM for all $p \in M$ since $\omega_{F(p)}$ is an alternating tensor on $T_{F(p)}N$. So $F^*\omega$ is in fact a differential k-form on M.

Lemma 4.1.5. *Let $F : M \to N$ be a smooth map between smooth manifolds, $\omega \in \Omega^k(N)$ and $\eta \in \Omega^l(N)$. Then*

(a) $F^* : \Omega^k(N) \to \Omega^l(M)$ is a linear map over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart, we can write

$$F^*\omega = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

Proof. The proof is left as Exercise 14.17 in [Lee12] so what follows is my own. (a) follows from Proposition 3.3.19 and the discussion above which shows the pullback of a differential form is a differential form. (c) follows in an analogous way to the proof of (d) in Proposition 3.3.19 using parts (a) and (b) of this lemma. It remains to show (b). First we will show that given $A \in \mathcal{T}^k(N)$, we have $F^*(\operatorname{Alt} A) = \operatorname{Alt}(F^*A)$. Indeed, let $p \in M$ and $v_1, ..., v_k \in T_p$ be arbitrary. Then

$$\begin{aligned}
\operatorname{Alt}(F^*A)_p(v_1, ..., v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(F^*A)_p(v_{\sigma(1)}, ..., v_{\sigma(k)}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{F(p)}(dF_p(v_{\sigma(1)}), ..., dF_p(v_{\sigma(k)})) \\
&= (\operatorname{Alt} A)_{F(p)}(dF_p(v_1), ..., dF_p(v_k)) = (F^*(\operatorname{Alt} A))_p(v_1, ..., v_k).
\end{aligned}$$

Then combining this fact with the fact the pullback is linear over $\mathbb{R}$ and respects tensor products, Proposition 3.3.19 (c), we get that

$$F^*(\omega \wedge \eta) = F^*\left(\frac{(k+l)!}{k!\,l!} \operatorname{Alt}(\omega \otimes \eta)\right) = \frac{(k+l)!}{k!\,l!} \operatorname{Alt}((F^*\omega) \otimes (F^*\eta)) = (F^*\omega) \wedge (F^*\eta).$$

□

Corollary 4.1.6. *Let M be a smooth manifold. Then the assignments $\Omega^k : M \mapsto \Omega^k(M), F \mapsto F^*$ and $\Omega^* : M \mapsto \Omega^*(M), F \mapsto F^*$ are contravariant functors to the category of real vector spaces and real associative anticommutative graded algebras respectively.*

Proof. This is not mentioned in [Lee12] and so the proof is my own. Firstly, Lemma 4.1.5 (a) and (b) imply that the pullback is actually a graded algebra homomorphism. Then Proposition 3.3.19 (e) and (f) imply that the assignments are indeed functorial. □

Remark 4.1.7. One might be pleased that we have constructed some functors from the category of smooth manifolds to some categories of algebraic structures. However, spaces of differential forms are not in general finite dimensional vector spaces so actually trying to calculate and classify $\Omega^k(M)$ or $\Omega^*(M)$ is too much work to be beneficial for our purposes. In the next section, we will introduce an operation on differential forms which will provide us with much richer structure that we can take advantage of for the purposes of algebraic topology.

4.2 The Exterior Derivative

In Section 3.3 we defined the differential of a smooth real map. That is, given a smooth real map $f : M \to \mathbb{R}$, we can define an associated 1-form $df \in \Omega^1(M)$. Proposition 3.3.12 (a) then implies that we can think of the differential $d : \Omega^0(M) \to \Omega^1(M)$ as a real linear map which maps 0-forms to 1-forms. The goal of this section is to extend this operation to arbitrary k-forms by constructing the exterior derivative. We first define it on $\mathbb{R}^n$ and then use smooth charts and their pullbacks to show that the exterior derivate exists on all manifolds. What follows is adapted from [Lee12] unless stated otherwise.

Definition 4.2.1. Let $\omega = \sum_J{}' \omega_J \, dx^J$ be a differential k-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$. We define its *exterior derivative*, $d\omega$, to by the following $(k+1)$-form:

$$d\left(\sum_J{}' \omega_J \, dx^J\right) := \sum_J{}' d\omega_J \wedge dx^J.$$

More specifically,

$$d\left(\sum_J{}' \omega_J \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \sum_J{}' \sum_{i=1}^n \frac{\partial \omega_J}{\partial x^i} \, dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}. \tag{4.1}$$

Example 4.2.2. This example does not appear in [Lee12] and is my own. Let $\omega = (y^2+2x)\,dx + x^2\,dy$ be a 1-form on $\mathbb{R}^2$. Then

$$d\omega = d((y^2+2x)\,dx + x^2\,dy) = 2\,dx \wedge dx + 2y\,dy \wedge dx + 2x\,dx \wedge dy + 0\,dy \wedge dy.$$

Then using Proposition 3.2.15 and Corollary 3.2.16 we have that $dy \wedge dx = -dy \wedge dx$ and $dx \wedge dx = 0 = dy \wedge dy$. So

$$d\omega = -2y\,dx \wedge dy + 2x\,dx \wedge dy = 2(x-y)\,dx \wedge dy.$$

Remark 4.2.3. The standard smooth frame for $\Lambda^0 T^*M$ is the smooth map $1_M : M \to \mathbb{R}$ given by $1_M(p) = 1$ for all $p \in M$ since any smooth real map $f \in C^\infty(M) = \Omega^0(M)$ can be written as the product $f\,1_M$. So the exterior derivative of $f \in C^\infty(\mathbb{R}^n)$ is given by $df = df \wedge 1_{\mathbb{R}^n} = df$, since the wedge product with a 0-form is regular pointwise multiplication. So the exterior derivative is in fact an extension of the differential.

Proposition 4.2.4. *Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and V and open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$.*

(a) d is linear over $\mathbb{R}$.

(b) If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \, \omega \wedge d\eta$.

(c) $d \circ d \equiv 0$.

(d) If $F : U \to V$ is a smooth map and $\omega \in \Omega^k(V)$, then $F^*(d\omega) = d(F^*\omega)$.

Proof. The linearity of d follows from the linearity of the differential and the bilinearity of the wedge product. To show (b), by linearity, it is sufficient to only consider expressions of the form $u\,dx^I \in \Omega^k(U)$ and $v\,dx^J \in \Omega^l(U)$ for smooth maps $u, v : U \to \mathbb{R}$. First given any multi-index I, we show that $d(u\,dx^I) = du \wedge dx^I$. If I has repeating indices, $dx^I = 0$ so by linearity $d(u\,dx^I) = 0$ and $du \wedge dx^I = 0$. Otherwise, let σ be the permutation which sends I to an increasing multi-index J. Then using Lemma 3.2.8 (b), we have

$$d(u\,dx^I) = (\operatorname{sgn}(\sigma)) d(u\,dx^J) = (\operatorname{sgn}(\sigma))\,du \wedge dx^J = du \wedge dx^I.$$

Then using the above calculation and Propositions 3.3.12 and 3.2.15 we get that

$$\begin{aligned} d((u\,dx^I) \wedge (v\,dx^J)) &= d(uv\,dx^I \wedge dx^J) = d(uv) \wedge dx^I \wedge dx^J = (u\,dv + v\,du) \wedge dx^I \wedge dx^J \\ &= u\,dv \wedge dx^I \wedge dx^J + v\,du \wedge dx^I \wedge dx^J \\ &= (-1)^k\,(u\,dx^I) \wedge (dv \wedge dx^J) + (du \wedge dx^I) \wedge (v\,dx^J) \\ &= (-1)^k (u\,dx^I) \wedge d(v\,dx^J) + d(u\,dx^I) \wedge (v\,dx^J). \end{aligned}$$

To show (c), we first consider the case for 0-forms. Let $u \in \Omega^0(U) = C^\infty(U)$. Then

$$d(du) = d\left(\sum_{i=1}^n \frac{\partial u}{\partial x^i}\,dx^i\right) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial x^i \partial x^j}\,dx^i \wedge dx^j = \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} dx^i \wedge dx^j\right) = 0,$$

since $u : U \to \mathbb{R}$ is smooth so its mixed second partial derivatives are equal. Now suppose $\omega \in \Omega^k(U)$. Then using part (b) we get that

$$\begin{aligned} d(d\omega) &= d\left(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \\ &= \sum_J{}' d(d\omega_J) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J{}' \sum_{i=1}^{k} (-1)^i \, d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge d(dx^{j_i}) \wedge \cdots \wedge dx^{j_k}. \end{aligned}$$

Then each term contains a form $d(du)$ where u is a smooth real function. So each term and hence the whole sum is equal to 0. So $d(d\omega) = 0$. Lastly, to prove (d), by linearity is it sufficient to consider $\omega = u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. By Lemma 4.1.5 (c) we have

$$F^*\omega = F^*(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = (u \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

So by Proposition 3.3.18, Lemma 4.1.5 (b) and the fact that $d \circ d \equiv 0$ we have that

$$\begin{aligned} d(F^*\omega) &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) = F^*(du) \wedge F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}) \\ &= F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = F^*(d\omega). \end{aligned}$$

□

We can now extend the exterior derivative to a general smooth manifold!

Theorem 4.2.5 (Existence and Uniqueness of the Exterior Derivative)**.** *Let M be a smooth manifold. Then there exist unique maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all $k \geq 0$, called the exterior derivative, which satisfy*

(a) d is linear over $\mathbb{R}$.

(b) If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\, \omega \wedge d\eta$.

(c) $d \circ d \equiv 0$.

(d) For $f \in \Omega^0(M)$, df is the differential of f as defined in Proposition 3.3.10.

In any smooth chart, d is given by (4.1).

Proof. Let $\omega \in \Omega^k(M)$. Since we want $d\omega$ to be given by (4.1) on each smooth chart (U, φ) we pullback ω by φ^{-1} to give us a k-form on an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. That is, we first calculate $d((\varphi^{-1})^*\omega)$. Then we pullback this $(k+1)$-form to produce a $(k+1)$-form on M. That is, we define $d\omega := \varphi^* d((\varphi^{-1})^*\omega)$. To show that the exterior derivative is well-defined, we need to check that it does not depend on the choice of smooth chart. Let (V, ψ) be another smooth chart for M with $U \cap V \neq \varnothing$. Then $\varphi \circ \psi^{-1}$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$ so Proposition 4.2.4 implies

$$(\varphi \circ \psi^{-1})^* d((\varphi^{-1})^*\omega) = d((\varphi \circ \psi^{-1})^*(\varphi^{-1})^*\omega).$$

Then using the functoriality of the pullback (Proposition 3.3.19), we get that

$$\begin{aligned} (\psi^{-1})^* \circ \varphi^* d((\varphi^{-1})^*\omega) &= d((\psi^{-1})^*\varphi^*(\varphi^{-1})^*\omega) \\ \implies \varphi^* d((\varphi^{-1})^*\omega) &= \psi^* d((\psi^{-1})^*\omega). \end{aligned}$$

So the exterior derivative is indeed well-defined. Then (a)-(d) follow from Proposition 4.2.4. The proof of uniqueness uses an argument involving smooth bump functions (see Lemma 2.1.24) and is given in Theorem 14.24 in [Lee12]. □

We also have that the exterior derivative commutes with pullbacks in general!

Proposition 4.2.6 (Naturality of d)**.** *Let $F : M \to N$ be a smooth map. Then for all $\omega \in \Omega^k(N)$ we have*

$$F^*(d\omega) = d(F^*\omega).$$

Proof. Let (U, φ) and (V, ψ) be smooth charts for M and N respectively. Then $\psi \circ F \circ \varphi^{-1}$ is a smooth map between open subsets of Euclidean spaces so we can apply Proposition 4.2.4 (d):

$$\begin{aligned} F^*(d\omega) &= F^*\psi^* d((\psi^{-1})^*\omega) = \varphi^*(\varphi^{-1})^* F^*\psi^* d((\psi^{-1})^*\omega) = \varphi^*(\psi \circ F \circ \varphi^{-1})^* d((\psi^{-1})^*\omega) \\ &= \varphi^* d((\psi \circ F \circ \varphi^{-1})^*(\psi^{-1})^*\omega) = \varphi^* d((\varphi^{-1})^* F^*\psi^*(\psi^{-1})^*\omega) \\ &= \varphi^* d((\varphi^{-1})^*(F^*\omega)) = d(F^*\omega). \end{aligned}$$

□

Remark 4.2.7. Recall that Corollary 4.1.6 gave us that Ω^k is a contravariant functor for each $k \geq 0$. In this perspective, Proposition 4.2.6 implies that the exterior derivative is a natural transformation from Ω^k to Ω^{k+1} (see Definition 1.2.11).

Definition 4.2.8. Let M be a smooth manifold and let $\omega \in \Omega^k(M)$. We say ω is a *closed k-form* if $d\omega = 0$ and we say it is an *exact k-form* if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.

The fact that $d \circ d \equiv 0$ implies that all exact forms are closed but in general it is not the case that all closed forms are exact. We will revisit this in Chapter 5 when we discuss de Rham Cohomology.

4.3 Other Operations with Differential Forms

In the previous section we studied the exterior derivative, but this is only one of many operations we can define for differential forms. In this section, we will briefly study two such operations that will be useful to us: interior multiplication and the Lie derivative. Since we only make little use of these operations in this report, many of the details of proofs are omitted and can be found in Chapters 12 and 14 of [Lee12]. What follows is adapted from [Lee12] unless stated otherwise.

Definition 4.3.1. Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. We define the *interior multiplication of ω by X* to be the $(k-1)$-form

$$(X \lrcorner \omega)_p(v_1, ..., v_{k-1}) := \omega_p(X_p, v_1, ...v_{k-1}).$$

If $k = 0$, the convention is that $X \lrcorner \omega = 0$.

One can show that given that X and ω are both smooth, $X \lrcorner \omega$ is indeed also smooth. Moreover, one can also show that the map $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ given by $i_X(\omega) = X \lrcorner \omega$ is linear over $C^\infty(M)$ (see Exercise 14.22 in [Lee12]). We also have $i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X(\eta))$ (see Lemma 14.13 in [Lee12]).

We now extend the Lie derivative of vector fields (see Definition 2.2.19) to differential forms. In fact, this definition holds for tensor fields in general.

Definition 4.3.2. Let M be a smooth manifold, $V \in \mathfrak{X}(M)$ with associated flow θ such that if $\partial M \neq \varnothing$ then V is tangent to ∂M, and $\omega \in \Omega^k(M)$. The *Lie derivative of ω with respect to V* is defined to be the form

$$(\mathcal{L}_V\omega)_p := \left.\frac{d}{dt}\right|_{t=0} (\theta_t^*\omega)_p,$$

provided the derivative exists.

Proposition 4.3.3. *Let M be a smooth manifold and $V \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$ then V is tangent to ∂M. Then*

(a) If $f \in C^\infty(M)$, then $\mathcal{L}_V f = Vf$.

(b) If $f \in C^\infty(M)$, then $\mathcal{L}_V(df) = d(\mathcal{L}_V f)$.

(c) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta)$.

For the proofs of (a), (b) and (c) see Proposition 12.32, Corollary 12.34 and Proposition 14.33 in [Lee12] respectively.

Proposition 4.3.4. *Let M be a smooth manifold and $V \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$ then V is tangent to ∂M. Let θ be the flow associated to V. Then for $\omega \in \Omega^k(M)$ and pair (t_0, p) in the domain of θ, we have that*

$$\left.\frac{d}{dt}\right|_{t=t_0} (\theta_t^*\omega)_p = (\theta_{t_0}^*(\mathcal{L}_V\omega))_p.$$

The proof involves a computation manipulating the pullbacks of the flow using properties in Theorem 2.2.18 and can be found in Proposition 12.36 in [Lee12].

We now prove Cartan's Magic Formula which gives us a relation between the Lie derivative and interior multiplication and the exterior derivative.

Theorem 4.3.5 (Cartan's Magic Formula)**.** *Let M be a smooth manifold and $V \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$ then V is tangent to ∂M. Then for any differential form $\omega \in \Omega^k(M)$ we have*

$$\mathcal{L}_V\omega = V \lrcorner d\omega + d(V \lrcorner \omega). \tag{4.2}$$

Proof. We proceed by induction. Firstly, let $f \in \Omega^0(M)$ be a 0-form on M. Then

$$(V \lrcorner df)_p + d(V \lrcorner f))_p = (V \lrcorner df)_p = df_p(V_p) = Vf(p) = (\mathcal{L}_V f)_p.$$

Now let $k \geq 1$ and suppose that (4.2) holds for $(k-1)$-forms. Let $\omega \in \Omega^k(M)$ be written as $\omega = {\sum_I}' \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ in local coordinates. Then let $u = x^{i_1}$ and $\beta = \omega_I \, dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ and so each term in the above sum can be written as $du \wedge \beta$, where u is a smooth function and β is a $(k-1)$-form. By Proposition 4.3.3 we have $\mathcal{L}_V(du) = d(\mathcal{L}_V u) = d(Vu)$. Combining this with Proposition 4.3.3 (c) and the induction hypothesis we get

$$\mathcal{L}_V(du \wedge \beta) = \mathcal{L}_V(du) \wedge \beta + du \wedge \beta = d(Vu) \wedge \beta + du \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)).$$

On the other hand, we have that

$$\begin{aligned} V \lrcorner d(du \wedge \beta) + d(V \lrcorner (du \wedge \beta)) &= V \lrcorner (-du \wedge d\beta) + d((Vu)\beta - du \wedge (V \lrcorner \beta)) \\ &= -(Vu)\, d\beta + du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + (Vu) d\beta + du(V \lrcorner \beta) \\ &= du \wedge (V \lrcorner d\beta) + du \wedge d(V \lrcorner \beta) + d(Vu) \wedge \beta \\ &= d(Vu) \wedge \beta + du \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)) = \mathcal{L}_V(du \wedge \beta). \end{aligned}$$

Thus, by linearity of the Lie derivative, we have (4.2), as required. □

Corollary 4.3.6. *Let M be a smooth manifold and $V \in \mathfrak{X}(M)$ such that if $\partial M \neq \varnothing$ then V is tangent to ∂M. Then for any differential form $\omega \in \Omega^k(M)$ we have*

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega).$$

Proof. We use Cartan's Magic Formula to compute

$$\begin{aligned} \mathcal{L}_V(d\omega) &= V \lrcorner d(d\omega) + d(V \lrcorner \omega) = d(V \lrcorner d\omega); \\ d(\mathcal{L}_V\omega) &= d(V \lrcorner d\omega) + d(d(V \lrcorner d\omega)) = d(V \lrcorner d\omega). \end{aligned}$$

□

4.4 Line Integrals

We conclude this chapter by generalising line integrals from multivariable calculus to smooth manifolds by defining the integral of a 1-form over a piecewise smooth curve. It is possible to define the integral of arbitrary k-forms but we do not need that for this report and details can be found in Chapters 16 and 18 of [Lee12]. What follows is adapted from [Lee12] unless stated otherwise.

We proceed by first defining the integral of a 1-form on a compact interval $[a, b] \subset \mathbb{R}$ and then define line integrals in general by pulling back the 1-form by the smooth curve.

Definition 4.4.1. Let $[a, b] \subset \mathbb{R}$ be a compact interval and let $\omega \in \Omega^1([a, b])$. If t is the standard coordinate on $\mathbb{R}$, we can write $\omega_t = f(t)dt$ for some smooth map $f : [a, b] \to \mathbb{R}$. We define the *integral of ω over $[a, b]$* to be

$$\int_{[a,b]} \omega := \int_a^b f(t)dt.$$

Definition 4.4.2. Let M be a smooth manifold and $\gamma : [a, b] \to M$ be continuous. We say γ is a *piecewise smooth curve* if there exists a finite partition of $[a, b]$, $a = a_0 < a_1 < \cdots < a_n = b$, such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for all $1 \leq i \leq n$.

Definition 4.4.3. Let M be a smooth manifold, $\gamma : [a, b]$ be a smooth curve and $\omega \in \Omega^1(M)$. We define the *line integral of ω over γ* to be the real number

$$\int_\gamma \omega := \int_{[a,b]} \gamma^*\omega.$$

If γ is a piecewise smooth curve, we define

$$\int_\gamma \omega := \sum_{i=1}^n \int_{[a,b]} (\gamma|_{[a_{i-1},a_i]})^*\omega,$$

where $[a_{i-1}, a_i]$, $1 \leq i \leq n$, are the intervals on which γ is smooth.

Proposition 4.4.4. *Let M be a smooth manifold and $\gamma : [a, b] \to M$ be a piecewise smooth curve and $\omega, \omega_1, \omega_2 \in \Omega^1(M)$. Then*

(a) For $c_1, c_2 \in \mathbb{R}$ we have

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) If γ is a constant map, then

$$\int_\gamma \omega = 0.$$

(c) If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ for $a < c < b$, then

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) If $F : M \to N$ is a smooth map and $\eta \in \Omega^1(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Proof. The proof is left as Exercise 11.35 in [Lee12] so what follows is my own work. We assume here that γ is smooth, then the results for piecewise smooth curves follow. Linearity of line integrals, (a), follows from the fact that the pullback is a linear map (Proposition 3.3.19) and the fact that real integrals are linear. For (b), suppose that $\gamma : [a, b] \to M$ is a constant map. Then for $t \in [a, b]$ and $v \in T_t[a, b]$ we have $(\gamma^*\omega)_t(v) = \omega_{\gamma(t)}(d\gamma_t(v))$. Then since the derivation of a constant smooth map

is always 0 (Lemma 2.2.3), we have that $d\gamma_t(v)f = v(f \circ \gamma) = 0$ for all $f \in C^\infty(M)$. So $d\gamma_t(v) = 0$ and hence $(\gamma^*\omega)_t(v) = \omega_{\gamma(t)}(0) = 0$. Then (b) follows. For (c), suppose that $(\gamma^*\omega)_t = f(t)dt$ for some smooth map $f : [a, b] \to \mathbb{R}$. Then $(\gamma_1^*\omega)_t = f|_{[a,c]}(t)dt$ and $(\gamma_2^*\omega)_t = f|_{[c,b]}(t)dt$. So we have

$$\int_{\gamma_1} \omega + \int_{\gamma_2} \omega = \int_a^c f|_{[a,c]}(t)dt + \int_c^b f|_{[c,b]}(t)dt = \int_a^b f(t)dt = \int_\gamma \omega.$$

Lastly, for (d) we use the fact that $\gamma^*F^*\eta = (F \circ \gamma)^*\eta$:

$$\int_\gamma F^*\eta = \int_{[a,b]} \gamma^*F^*\eta = \int_{[a,b]} (F \circ \gamma)^*\eta = \int_{F\circ\gamma} \eta.$$

□

Proposition 4.4.5. *Let $\gamma : [a, b] \to M$ be a piecewise smooth curve. Then*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt.$$

Proof. We first assume that γ is smooth and that its image is contained in the domain of a single smooth chart (U, φ) with $\varphi = (x^1, ..., x^n)$. Then we may write $\gamma = (\gamma^1, ..., \gamma^n)$ and $\omega = \sum_{i=1}^n \omega_i\, dx^i$ in these coordinates. Then we have

$$\omega_{\gamma(t)}(\gamma'(t)) = \sum_{i=1}^n \omega_i(\gamma(t))(\gamma^i)'(t).$$

On the other hand, using Lemma 4.1.5 (c) we have

$$(\gamma^*\omega)_t = \sum_{i=1}^n (\omega_i \circ \gamma)\, d(x^i \circ \gamma)_t = \sum_{i=1}^n (\omega_i \circ \gamma)\, d(\gamma^i)_t = \sum_{i=1}^n (\omega_i \circ \gamma)\, (\gamma^i)'(t)dt = \omega_{\gamma(t)}\gamma'(t)dt.$$

So by definition of the integral, we have

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt.$$

Now if γ is an arbitrary smooth curve, its image is compact since γ is continuous and $[a, b]$ is compact. So there is a finite cover of $\gamma([a, b])$ by smooth charts and a finite partition of $[a, b]$, $a = a_0 < \cdots < a_n = b$ such that each $\gamma|_{[a_{i-1},a_i]}$ is contained in a single smooth chart. We then do the above calculation on each subinterval $[a_{i-1}, a_i]$. Lastly, if γ is piecewise smooth, we do the above calculation on each subinterval on which γ is smooth. □

We now show that the Fundamental Theorem of Calculus extends to line integrals!

Theorem 4.4.6 (Fundamental Theorem of Line Integrals)**.** *Let M be a smooth manifold, $\gamma : [a, b] \to M$ a piecewise smooth curve and $f \in C^\infty(M)$. Then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Proof. First suppose that γ is smooth. In a similar way to Example 2.2.10, we have that $df_{\gamma(t)}(\gamma'(t)) = (f \circ \gamma)'(t)$ (see Proposition 11.23 in [Lee12]). So by Proposition 4.4.5 and the Fundamental Theorem of Calculus, we have that

$$\int_\gamma df = \int_a^b df_{\gamma(t)}(\gamma'(t))\, dt = \int_a^b (f \circ \gamma)'(t)dt = f(\gamma(b)) - f(\gamma(a)).$$

If γ is only piecewise smooth, we apply the above calculation on each subinterval on which γ is smooth to get

$$\int_\gamma \omega = \sum_{i=1}^n (f(\gamma(a_i)) - f(\gamma(a_{i-1}))) = f(\gamma(b)) - f(\gamma(a)),$$

since the inner terms all cancel out. □

Proposition 4.4.7. *Let M be a smooth manifold, $\gamma : [a,b] \to M$ a piecewise smooth curve, $\omega \in \Omega^1(M)$ and a diffeomorphism $\varphi : [c,d] \to [a,b]$. Then*

$$\int_{\gamma\circ\varphi} \omega = \begin{cases} \int_\gamma \omega \text{ if } \varphi \text{ is increasing,} \\ - \int_\gamma \omega \text{ if } \varphi \text{ is decreasing.} \end{cases}$$

The proof relies on a similar fact for integral of 1-forms over compact intervals in $\mathbb{R}$. See Propositions 11.31 and 11.37 and Exercise 11.32 in [Lee12] for the details.

We now provide a sufficient condition in terms of integration for a 1-form to be exact!

Definition 4.4.8. Let M be a smooth manifold. We say a 1-form $\omega \in \Omega^1(M)$ is *conservative* if $\int_\gamma \omega = 0$ for any closed piecewise smooth curve $\gamma : [a,b] \to M$. We say ω's line integrals are *path-independent* if $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$ for any two piecewise smooth curves γ, $\tilde{\gamma}$ whose start and end points are the same.

Theorem 4.4.9. *Let M be a smooth manifold and let $\omega \in \Omega^1(M)$. Then the following are equivalent:*

(a) ω is conservative.

(b) The line integrals of ω are path-independent.

(c) ω is exact.

Proof. The proof that (a) $\iff$ (b) is left as Exercise 11.41 in [Lee12] so the proof given is my own. The proof of (a) $\iff$ (c) is given in Theorem 11.42 in [Lee12]. First, suppose ω is conservative and let $\gamma_1, \gamma_2 : [0,1] \to M$ be piecewise smooth curves with $\gamma_1(i) = \gamma_2(i)$ for $i \in \{0,1\}$. By Proposition 4.4.7 it is sufficient to only consider such curves since we may reparameterise an arbitrary curve into one of this form. Define the piecewise smooth curve $\Gamma : [0,2] \to M$ by

$$\Gamma(t) := \begin{cases} \gamma_1(t), & t \in [0,1], \\ \gamma_2(2-t), & t \in [1,2]. \end{cases}$$

Then using the fact that ω is conservative, we have by Propositions 4.4.7 and 4.4.4 (c) that

$$0 = \int_\Gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega.$$

So the line integral is indeed path independent and thus (a) $\implies$ (b). On the other hand, suppose the line integrals of ω are path-independent and let $\gamma : [0,2] \to M$ be a piecewise smooth closed curve (again, by Proposition 4.4.7 it is sufficient to only consider this case). Then let $\gamma_1, \gamma_2 : [0,1] \to M$ be the piecewise smooth curves given by $\gamma_1(t) := \Gamma|_{[0,1]}(t)$ and $\gamma_2(t) := \Gamma|_{[1,2]}(2-t)$. Then $\gamma_1(0) = \Gamma(0) = \Gamma(2) = \gamma_2(0)$ and $\gamma_1(1) = \Gamma(1) = \gamma_2(1)$. So by Propositions 4.4.7 and 4.4.4 (c) and path-independence we have

$$\int_\Gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega = 0.$$

So ω is conservative and thus (b) $\implies$ (a).

For (c) $\implies$ (a), note that the Fundamental Theorem of Line Integrals (Theorem 4.4.6) implies that the line integral of df for $f \in C^\infty(M)$ only depends on the value of f at the end points of the piecewise smooth curve, so is path-independent. Since we have already shown that path-independence of line integrals implies a 1-form is conservative, df is conservative. Hence if ω is exact, it is conservative. The proof that conservative implies exact is much lengthier so the details are omitted for brevity. The key is to pick a base point $p_0 \in M$ and use path-independence construct map $f : M \to \mathbb{R}$ given by $f(p) := \int_\gamma \omega$ where $\gamma : [a,b] \to M$ is any piecewise smooth curve with $\gamma(0) = p_0$ and $\gamma(1) = p$. What remains is to show that f is smooth and that $df = \omega$. The details are given in Theorem 11.42 in [Lee12]. □

We finish by stating a further useful property of the line integral of closed 1-forms.

Theorem 4.4.10. *Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be closed. Then if $\gamma_0, \gamma_1 : [a, b] \to M$ are path-homotopic piecewise smooth curves, then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

The proof requires the theory of integration of arbitrary k-forms and so is omitted but can be found in Theorem 16.26 in [Lee12]. The idea is to pullback ω by a homotopy $H : I \times I \to M$ between the paths and integrate the form over $I \times I$ and using Stokes' Theorem, which is the generalisation of the Fundamental Theorem of Line Integrals to higher dimensional integrals.

Corollary 4.4.11. *Let M be a simply connected smooth manifold. Then every closed 1-form on M is exact.*

Proof. Let $\omega \in \Omega^1(M)$ be closed. If $\gamma : [a, b] \to M$ is any piecewise smooth closed curve, then since M is simply connected γ is path-homotopic to a constant curve $\varepsilon : [a, b] \to M$. So by Theorem 4.4.10 and Proposition 4.4.4 (b) we have that

$$\int_{\gamma} \omega = \int_{\varepsilon} \omega = 0.$$

So ω is conservative. Hence by Theorem 4.4.9, ω is exact. □

V. De Rham Cohomology

5.1 The de Rham Cochain Complex

In Chapter 4 we introduced differential forms and in Section 4.2 we defined a notion of differentiating them: the exterior derivative. We saw that applying the exterior derivative twice always resulted in 0 and this is what allows us to define a notion of cohomology for smooth manifolds. In this section, we study the construction of de Rham cohomology and a few first results. What follows is adapted from [Lee12] but is reframed in the language of homological algebra (see Section 1.3).

Proposition 5.1.1. *Let M be a smooth manifold. Then we have the following cochain complex over $\mathbb{R}$:*

$$\Omega^\bullet(M) : \ 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \cdots \tag{5.1}$$

*We call $\Omega^\bullet(M)$ the **de Rham (cochain) complex** of M.*

Proof. By Remark 2.5.14, each $\Omega^k(M)$ is a real vector space and by Theorem 4.2.5 (a) and (c), the exterior derivative is a real linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ such that $d \circ d \equiv 0$. So d is a coboundary map and thus $\Omega^\bullet(M)$ is indeed a cochain complex over $\mathbb{R}$. □

This now allows us to define de Rham cohomology!

Definition 5.1.2. Let M be a smooth manifold and let $\Omega^\bullet(M)$ be the de Rham complex of M. For $k \geq 0$ we define the *kth de Rham cohomology group* of M, $H^k_{dR}(M)$, to be kth cohomology group of $\Omega^\bullet(M)$ as in Definition 1.3.5. That is

$$H^k_{dR}(M) := H^k(\Omega^\bullet(M)).$$

Remark 5.1.3.

- Despite being called "groups", the de Rham cohomology groups are in fact real vector spaces.

- The coboundaries in $\Omega^\bullet(M)$ are precisely exact forms and the cocycles are precisely closed forms. So the kth de Rham cohomology group can be seen as the space of closed k-forms modulo the space of exact k-forms. In this way, we can interpret de Rham cohomology to represent the failure of closed forms to be exact. For example, $H^k_{dR}(M) = 0$ if and only if all closed k-forms on M are exact.

Recall that Proposition 4.2.6 gave us that the pullback of a smooth map commutes with the exterior derivative. In the context of cochain complexes, this means the pullback of a smooth map is in fact a cochain map (see Definition 1.3.7).

Lemma 5.1.4. *Let $F : M \to N$ be a smooth map between smooth manifolds and let $F^* : \Omega^k(N) \to \Omega^k(M)$ be its pullback for $k \geq 0$. Then the following diagram commutes*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^{k-1}(N) & \xrightarrow{d} & \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
\cdots & \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \longrightarrow & \cdots
\end{array}$$

Hence $F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is a cochain map.

Proof. Firstly, Proposition 4.2.6 gives us that $d \circ F^* = F^* \circ d$, so the diagram indeed commutes. Furthermore, Proposition 4.1.5 (a) gives us that F^* is linear over $\mathbb{R}$ for each $k \geq 0$ so F^* is indeed a cochain map. □

Corollary 5.1.5. *Let M be a smooth manifold. Then the assignment $\Omega^\bullet(-) : M \mapsto \Omega^\bullet(M)$, $F \mapsto F^*$ is a contravariant functor.*

Proof. Corollary 4.1.6 implies that for each $k \geq 0$, the assignment of F to its pullback is functorial. So for each $k \geq 0$, $(\mathrm{id}_M)^* = \mathrm{id}_{\Omega^k(M)}$ and $(F \circ G)^* = G^* \circ F^*$. Comparing this to the definitions of the identity cochain map and the composition of cochain maps in Lemma 1.3.8, we see that as cochain maps $(\mathrm{id}_M)^* = \mathrm{id}_{\Omega^\bullet(M)}$ and $(F \circ G)^* = G^* \circ F^*$. So the assignment is indeed functorial. □

Corollary 5.1.6. *Let $F : M \to N$ be a smooth map between smooth manifolds. Then there is a well-defined real linear map $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$ for each $k \geq 0$ given by*

$$F^*[\omega] := [F^*\omega].$$

Moreover, the assignment $H^k_{dR}(-) : M \mapsto H^k_{dR}(M)$, $F \mapsto F^*$ a contravariant functor.

Proof. Lemma 5.1.4 implies that the pullback of F, $F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$, is a cochain map. Then Lemma 1.3.9 implies that the map $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$ given by $F^*[\omega] := [F^*\omega]$ is a well-defined real linear map. Moreover, Corollary 1.3.10 implies that the assignment of the pullback to its induced map in cohomology is functorial. Combining this with Corollary 5.1.5 we see that $H^k_{dR}(-)$ is the following composition of functors:

$$H^k_{dR}(-) : \ \mathbf{Diff} \xrightarrow{\Omega^\bullet(-)} \mathbf{Ch} \xrightarrow{H^k(-)} \mathbf{Vect}_{\mathbb{R}}.$$

□

Corollary 5.1.7 (Diffeomorphism Invariance of de Rham Cohomology)**.** *Let M and N be diffeomorphic smooth manifolds. Then for all $k \geq 0$ we have*

$$H^k_{dR}(M) \cong H^k_{dR}(N).$$

Proof. Corollary 5.1.6 implies that $H^k_{dR}(-)$ is a functor. So since diffeomorphisms are the isomorphisms in **Diff**, Theorem 1.2.8 implies that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. □

We finish this section by proving some elementary results about de Rham cohomology.

Proposition 5.1.8. *Let M be an n-dimensional smooth manifold. Then*

(a) $H^k_{dR}(M) = 0$ for all $k > n$.

(b) If M is connected, then $H^0_{dR}(M) \cong \mathbb{R}$.

(c) Additivity: If M is the disjoint union of countably many smooth manifolds, denote them $\{M_j\}_{j\in\mathbb{N}}$, then for all $k \geq 0$

$$H^k_{dR}(M) \cong \prod_{j=1}^{\infty} H^k_{dR}(M_j).$$

(d) If $p = \#\{\text{connected components of } M\}$, then $H^0_{dR}(M) \cong \mathbb{R}^p$.

(e) If M is simply connected, then $H^1_{dR}(M) = 0$.

Proof. First recall in Remark 4.1.2 we saw that $\dim(\Omega^k(M)) = 0$ for $k > n$, so the cohomology of the de Rham complex is trivial for $k > n$. For (b), we have that there are no exact 0-forms since we have no "(-1)-forms". So $H^0_{dR}(M) = \{\text{closed 0-forms on } M\}$. Recall also that Remark 4.1.2 gave us that 0-forms are precisely the smooth real maps on M. Then letting $f \in \Omega^0(M)$, we have that by Proposition 3.3.13 since M is connected f is closed $\iff$ f is constant. So for each $a \in \mathbb{R}$, there is a constant map $f_a : M \to \mathbb{R}$ with $f_a(p) = a$ for all $p \in M$. So the space of closed forms on M, and hence $H^0_{dR}(M)$, is isomorphic to $\mathbb{R}$.

For (c), let $i_j : M_j \hookrightarrow M$ be the inclusion map. Then we actually have an isomorphism for each $k \geq 0$, $\mathcal{I} : \Omega^k(M) \to \Omega^k\left(\bigsqcup_{j=1}^{\infty} M_j\right)$, given by $\mathcal{I}(\omega) = (i_1^*\omega, i_2^*\omega, i_3^*\omega, ...) = (\omega|_{M_1}, \omega|_{M_2}, \omega|_{M_3}, ...)$. $\mathcal{I}$ is injective since if the restriction of a k-form to each component of M is zero, then it must be 0 on all of M. $\mathcal{I}$ is surjective since giving an arbitrary k-form on each component of M defines a k-form on all of M. Then since isomorphisms are preserved by functors (Theorem 1.2.8), $\mathcal{I}$ induces an isomorphism in cohomology.

For (d), label each connected component of M by M_j. Then $M = \bigsqcup_{i=1}^{p} M_j$. Then (b) implies that $H^0_{dR}(M_j) \cong \mathbb{R}$ for all $1 \leq j \leq p$ and so additivity implies $H^0_{dR}(M) \cong \prod_{i=1}^{p} \mathbb{R} = \mathbb{R}^p$.

Lastly, if M is simply connected Corollary 4.4.11 implies that every closed 1-form is exact. So $H^1_{dR}(M) = 0$. □

Corollary 5.1.9. *Let M be a 0-manifold. Then M is the disjoint union of p map points. So*

$$H^k_{dR}(M) \cong \begin{cases} \mathbb{R}^p, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. Proposition 5.1.8 (d) and (a) give the results for $k = 0$ and $k > 0$ respectively. □

5.2 Eilenberg-Steenrod Axioms for de Rham Cohomology

Proposition 5.1.8 (d) and (e) hint that there is some connection between the de Rham cohomology of a smooth manifold and its topology. A question we might then ask is "how far does this connection go?" One angle that we can take to answer this question is to ask whether de Rham cohomology satisfies the Eilenberg-Steenrod axioms (Definition 1.3.16) since these axioms capture important aspects of how cohomology groups interact with the topology of spaces.

5.2.1 Dimension

The dimension axiom is the easiest of the four to prove. In fact, we have actually already proven it!

Lemma 5.2.1 (Dimension Axiom)**.** *Let $\mathcal{P}$ be the one point space. Then*

$$H^k_{dR}(\mathcal{P}) \cong \begin{cases} \mathbb{R}, & k = 0, \\ 0 & k > 0. \end{cases}$$

Proof. This is Corollary 5.1.9 with $p = 1$. □

5.2.2 Homotopy Invariance

Perhaps surprisingly, despite only using the smooth structure on the manifold to construct de Rham cohomology, we actually have that it is invariant under homotopy equivalences! Out of the four axioms, this one gives the strongest evidence that really de Rham cohomology provides us with topological information.

The key step in proving homotopy invariance is to construct a cochain homotopy (see Definition 1.3.11) between i_0^* and i_1^* where $i_t : M \to M \times I$ is defined by $i_t(p) = (p, t)$ for $p \in M$. However there is an issue when M is a smooth manifold with boundary: $M \times I$ does not have the smooth structure of a smooth manifold with or without boundary. The way to fix this is to introduce *smooth manifolds with corners*. Ignoring smooth structure, smooth manifolds with corners are topological manifolds with boundary. However their smooth structure is different from that of a smooth manifold with boundary. The charts are instead homeomorphisms $\varphi : M \to \overline{\mathbb{R}^n_+}$, where $\overline{\mathbb{R}^n_+} := \{(x^1, ..., x^n) \in \mathbb{R}^n : x^1 \geq 0, ..., x^n \geq 0\}$. The theory of smooth manifolds can be extended to smooth manifolds with corners and the details can be found in Chapter 16 of [Lee12]. In what follows, if M is a smooth manifold with boundary, then $M \times I$ has the smooth structure of a smooth manifold with corners. What follows, unless stated otherwise, is adapted from [Lee12].

Lemma 5.2.2. *Let M be a smooth manifold. Then there exists a cochain homotopy between the cochain maps $i_0^*, i_1^* : \Omega^\bullet(M \times I) \to \Omega^\bullet(M)$, where i_0 and i_1 are defined as above.*

Proof. For each $k \geq 0$, we need to find a linear map $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$ such that for all $\omega \in \Omega^k(M \times I)$ we have $h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega$. We observe that the right hand side of this equation looks similar to the result of the Fundamental Theorem of Calculus. So our approach is to construct h in a way such that $h(d\omega) + d(h\omega)$ is the integral from 0 to 1 of the derivative of $t \mapsto i_t^*\omega$ and apply the Fundamental Theorem of Calculus.

First, we define the vector field $S \in \mathfrak{X}(M \times \mathbb{R})$ by $S_{(p,s_0)} = \left(0, \frac{\partial}{\partial s}\big|_{s_0}\right)$, where s is the standard coordinate of $\mathbb{R}$ and we have identified $T_{(p,s)}(M \times \mathbb{R}) \cong T_pM \times T_s\mathbb{R}$. Then given $\omega \in \Omega^k(M \times I)$, we define $h\omega$ by

$$h\omega := \int_0^1 i_t^*(S \lrcorner \omega)\, dt.$$

That is for $p \in M$, $(h\omega)_p = \int_0^1 i_t^*((S \lrcorner \omega)_{(p,s)})\, dt$, where the integrand is a function $I \to \Lambda^k(T_p^*M) \cong \mathbb{R}^{\binom{n}{k}}$. Then given any smooth chart (U, φ) of M, the components of the integrand are smooth functions of $(p, t) \in U \times I$. So this integral defines a smooth $k-1$-form on M. Differentiating under the integral sign then gives that

$$d(h\omega) = \int_0^1 d(i_t^*(S \lrcorner \omega)).$$

So using Cartan's Magic Formula (Theorem 4.3.5) and the naturality of d (Proposition 4.2.6) we get

$$\begin{aligned} h(d\omega) + d(h\omega) &= \int_0^1 i_t^*(S \lrcorner d\omega) + d(i_t^*(S \lrcorner \omega))\, dt \\ &= \int_0^1 i_t^*(S \lrcorner d\omega) + i_t^*(d(S \lrcorner \omega))\, dt \\ &= \int_0^1 i_t^*(\mathcal{L}_S\omega)\, dt. \end{aligned}$$

Now if $\partial M \neq \varnothing$, S is tangent to $\partial(M \times \mathbb{R}) = \partial M \times \mathbb{R}$. So Theorem 2.2.18 implies there always exists a unique flow θ which is generated by S. Explicitly, $\theta_t(p, s) = (q, t + s)$ and so $(\theta_t \circ i_0)(p) = \theta_t(p, 0) = (p, t) = i_t(p)$. So using Proposition 4.3.4 we have that

$$i_t^*(\mathcal{L}_S\omega) = i_0^*(\theta_t^*(\mathcal{L}_S\omega)) = i_0^*\left(\frac{d}{dt}\theta_t^*\omega\right) = \frac{d}{dt}(i_0^*\theta_t^*\omega) = \frac{d}{dt}i_t^*\omega.$$

Thus by the Fundamental Theorem of Calculus

$$h(d\omega) + d(h\omega) = \int_0^1 \frac{d}{dt}(i_t^*\omega)\, dt = i_1^*\omega - i_0^*\omega.$$

□

We can now prove homotopy invariance of de Rham cohomology!

Lemma 5.2.3 (Homotopy Axiom)**.** *Let $F, G : M \to N$ be two homotopic smooth maps between smooth manifolds. Then for $k \geq 0$, the induced maps in cohomology $F^*, G^* : H^k_{dR}(N) \to H^k_{dR}(M)$ are equal.*

Proof. By Theorem 2.4.3, F and G are smoothly homotopic, so there exists a smooth homotopy $H : M \times I \to N$ from F to G. In particular, $F = H \circ i_0$ and $G = H \circ i_1$. Then since i_0^* and i_1^* are cochain homotopic (Lemma 5.2.2), their induced maps in cohomology are equal by Lemma 1.3.12. So we have

$$F^* = (H \circ i_0)^* = i_0^* \circ H^* = i_1^* \circ H^* = (H \circ i_1)^* = G^*.$$

□

Remark 5.2.4. The full homotopy axiom requires that homotopic maps of pairs induce equal maps in cohomology. However, for smooth manifolds with boundary, it is not guaranteed that relative homotopic smooth maps are relatively smoothly homotopic. So for maps of pairs, the above proof does not work.

Theorem 5.2.5 (Homotopy Invariance of de Rham Cohomology)**.** *Let M and N be homotopy equivalent smooth manifolds. Then for all $k \geq 0$ we have*

$$H^k_{dR}(M) \cong H^k_{dR}(N).$$

Proof. Let $F : M \to N$ be a homotopy equivalence and let $G : N \to M$ be its homotopy inverse. Then by the Whitney Approximation Theorem (Theorem 2.4.1) we have that there exist smooth maps $\tilde{F} : M \to N$ and $\tilde{G} : N \to M$ homotopic to F and G respectively. Then since F and G are homotopy inverses and homotopies respect compositions, we have $F \circ G \simeq \mathrm{id}_N \implies \tilde{F} \circ \tilde{G} \simeq \mathrm{id}_N$ and $G \circ F \simeq \mathrm{id}_M \implies \tilde{G} \circ \tilde{F} \simeq \mathrm{id}_M$. Moreover, since $\tilde{F}$ and $\tilde{G}$ are smooth, their compositions are also smooth. So we may apply Lemma 5.2.3 and use the functoriality of $H^k_{dR}(-)$ to get that

$$\begin{aligned} \tilde{G}^* \circ \tilde{F}^* = (\tilde{F} \circ \tilde{G})^* = (\mathrm{id}_N)^* = \mathrm{id}_{H^k_{dR}(N)} \\ \text{and } \tilde{F}^* \circ \tilde{G}^* = (G \circ F)^* = (\mathrm{id}_M)^* = \mathrm{id}_{H^k_{dR}(M)}. \end{aligned}$$

So $\tilde{F}^*$ and $\tilde{G}^*$ are inverses of each other and hence are isomorphisms. So for each $k \geq 0$,

$$H^k_{dR}(M) \cong H^k_{dR}(N).$$

□

5.2.3 Relative de Rham Cohomology and Exactness

Before we can prove the exactness axiom, we first must define the notion de Rham cohomology of a pair of smooth manifolds. Recall from Definition 2.3.5 that a pair of smooth manifolds (M, S) is a smooth manifold M and an embedded submanifold $S \subseteq M$. We would like to produce a sensible notion of "forms on M modulo forms on S" however since $\Omega^k(S)$ is not a subspace of $\Omega^k(M)$, $\Omega^k(M)/\Omega^k(S)$ makes no sense. Instead, we can take the approach of considering closed forms on M whose restriction to S is exact so that in cohomology, the contribution of S is trivial [Jos95]. Since the pullback of the inclusion map is just the restriction, we can generalise this to arbitrary smooth maps $F : N \to M$ and consider closed forms on M whose pullback to N is exact. This will produce a notion of de Rham cohomology for smooth maps and then the cohomology of the pair will be the cohomology of the inclusion map! [BT82].

Definition 5.2.6. [BT82] Let $F : N \to M$ be a smooth map between smooth manifolds and let $k \geq 0$. We define $\Omega^k(F)$ to be the real vector space

$$\Omega^k(F) := \Omega^k(M) \oplus \Omega^{k-1}(N),$$

with the convention $\Omega^0(F) = \Omega^0(M)$. We then define the map $d : \Omega^k(F) \to \Omega^{k+1}(F)$ by

$$d(\omega, \theta) := (d\omega, F^*\omega - d\theta).$$

We call a pair $(\omega, \theta) \in \Omega^k(F)$ *closed* if $d(\omega, \theta) = 0$ and *exact* if there exists $(\eta, \rho) \in \Omega^{k-1}(F)$ such that $(\omega, \theta) = d(\eta, \rho)$.

Proposition 5.2.7. *Let $F : N \to M$ be a smooth map. Then $(\Omega^\bullet(F), d)$ is a cochain complex.*

Proof. This is adapted from the proof in [Jos95] where F is the inclusion map. First, the linearity of d follows from the linearity of the exterior derivative and of the pullback F^*. To show that $d \circ d \equiv 0$, let $(\omega, \theta) \in \Omega^k(F)$, then using the naturality of d (Proposition 4.2.6), we have

$$d(d(\omega, \theta)) = d(d\omega, F^*\omega - d\theta) = (d(d\omega), F^*(d\omega) - d(F^*\omega) + d(d\theta)) = (0, 0).$$

□

Definition 5.2.8. [BT82] Let $F : N \to M$ be a smooth map. For $k \geq 0$, we define the *kth de Rham cohomology* of F to be the kth cohomology group of $\Omega^\bullet(F)$. That is $H^k_{dR}(F) := H^k(\Omega^\bullet(F))$. We denote the cohomology class of $(\omega, \theta) \in \Omega^k(F)$ by $[\omega, \theta] \in H^k_{dR}(F)$.

Remark 5.2.9. If $(\omega, \theta) \in \Omega^k(F)$ is closed, then $d(\omega, \theta) = (d\omega, F^*\omega - d\theta) = 0$. So $d\omega = 0$ and $F^*\omega = d\theta$. So ω is closed on M and its pullback by F is exact on N, which is what we wanted!

There are two obvious maps to consider: $\alpha : \Omega^{k-1}(N) \to \Omega^k(F)$ and $\beta : \Omega^k(F) \to \Omega^k(M)$ given by $\alpha(\theta) = (0, \theta)$ and $\beta(\omega, \theta) = \omega$. Both α and β are clearly real linear maps.

Proposition 5.2.10. *[Jos95] For α and β defined as above, we have*

$$d(\alpha(\theta)) = -\alpha(d\theta) \text{ and } d(\beta(\omega, \theta)) = \beta(d(\omega, \theta)).$$

Proof. Firstly, for $\theta \in \Omega^{k-1}(N)$, we have $d(\alpha(\theta)) = d(0, \theta) = (0, -d\theta) = -(0, d\theta) = -\alpha(d\theta)$. Next, for $(\omega, \theta) \in \Omega^k(F)$, we have $\beta(d(\omega, \theta)) = \beta(d\omega, F^*\omega - d\theta) = d\omega = d(\beta(\omega, \theta))$. □

This implies that β is a cochain map so induces a linear map $\beta^* : H^k_{dR}(F) \to H^k_{dR}(M)$ by Lemma 1.3.9 but α is not quite a cochain map. However, the proof of Lemma 1.3.9 still holds if instead we assume that $d \circ f = -f \circ d$ (once we include this extra factor of -1). That is, α still induces a well-defined real linear map in cohomology $\alpha^* : H^{k-1}_{dR}(N) \to H^k_{dR}(F)$ given by $\alpha^*[\theta] := [\alpha(\theta)]$ [Jos95]. By Corollary 5.1.6, the smooth map $F : N \to M$ also induces a map in cohomology $F^* : H^k_{dR}(M) \to H^k_{dR}(N)$. Together, these maps give us a long exact sequence in cohomology!

Lemma 5.2.11. *[BT82] Let $F : N \to M$ be a smooth map between smooth manifolds. Then there is a long exact sequence*

$$\cdots \longrightarrow H^k_{dR}(F) \xrightarrow{\beta^*} H^k_{dR}(M) \xrightarrow{F^*} H^k_{dR}(N) \xrightarrow{\alpha^*} H^{k+1}_{dR}(F) \longrightarrow \cdots \tag{5.2}$$

Proof. The following proof is adapted from the proof of Lemma 5.3.1 in [Jos95] which only shows the result for when F is an inclusion map. First for exactness at $H^k_{dR}(F)$ let $[\omega, \theta] \in H^k_{dR}(F)$ then $[\omega, \theta] \in \ker(\beta^*) \iff \beta^*[\omega, \theta] = 0 \iff [\omega] = 0$. On the other hand, $[\omega, \theta] \in \text{im}(\alpha^*) \iff [0, \rho] = \alpha^*[\theta] = [\omega, \theta] \iff [\omega, \theta] - [0, \rho] = [0, 0] \iff (\omega, \theta - \rho) = d(\eta, \nu)$. So $d\eta = \omega$ and $[\omega] = 0$. So we have that $[\omega, \theta] \in \text{im}(\alpha^*) \implies [\omega] = 0$. Conversely, if $[\omega] = 0$, then there exists $\eta \in \Omega^{k-1}(M)$ with $d\eta = \omega$. We want to show that $[\omega, \theta] \in \text{im}(\alpha^*)$. Since (ω, θ) represents a cohomology class, it is closed. So $(d\omega, F^*\omega - d\theta) = d(\omega, \theta) = 0$. So $F^*\omega - d\theta = 0$. Then using the fact that F^* commutes with the exterior derivative and $\omega = d\eta$ we have that $F^*(d\eta) - \theta = d(F^*\eta - \theta) = 0$. So $\theta - F^*\eta$ and hence $(0, \theta - F^*\eta)$ are

closed. Then $(\omega,\theta)-(0,\theta-F^*\eta)=(\omega,F^*\eta)=(d\eta,F^*\eta)=d(\eta,0)$. So $[\omega,\theta]=[0,\theta-F^*\eta]=\alpha^*[\theta-F^*\eta]$ and so $[\omega,\theta]\in\mathrm{im}(\alpha^*)$, as required. Hence $[\omega,\theta]\in\mathrm{im}(\alpha^*) \iff [\omega]=0 \iff [\omega,\theta]\in\ker(\beta^*)$. Thus $\mathrm{im}(\alpha^*)=\ker(\beta^*)$.

For exactness at $H^k_{dR}(M)$, let $[\omega]\in H^k_{dR}(M)$. Then $[\omega]\in\ker(F^*) \iff [F^*\omega]=0 \iff$ $\exists\,\eta\in\Omega^{k-1}(N) : F^*\omega=d\eta$. Then since ω is closed, we have $d(\omega,\eta)=(d\omega,F^*\eta-d\eta)=(0,0)$, so (ω,η) is closed and thus represents a cohomology class. Moreover, $\beta^*[\omega,\eta]=[\omega]$. So $[\omega]\in\mathrm{im}(\beta^*)$. Conversely, if $[\omega]\in\mathrm{im}(\beta^*)$ there exists $[\omega,\eta]\in H^k_{dR}(F)$ such that $\beta^*[\omega,\eta]=[\omega]$. Then $d(\omega,\eta)=(0,0)$ and so $F^*\omega-d\eta=0$. Hence $F^*[\omega]=0$ and $[\omega]\in\ker(F^*)$. Thus $\mathrm{im}(\beta^*)=\ker(F^*)$.

Lastly, for exactness at $H^k_{dR}(N)$, let $[\theta]\in H^k_{dR}(N)$. Then $[\theta]\in\ker(\alpha^*) \iff [0,\theta]=0 \iff$ $\exists\,(\eta,\nu)\in\Omega^k(F) : d(\eta,\nu)=(0,\theta) \iff d\eta=0$ and $F^*\eta-d\nu=\theta \iff d\eta=0$ and $F^*\eta-\theta=d\nu \iff$ $F^*[\eta]=[F^*\eta]=[\theta] \iff [\theta]\in\mathrm{im}(F^*)$. Hence $\mathrm{im}(F^*)=\ker(\alpha^*)$. □

Definition 5.2.12. [BT82] Let M be a smooth manifold and let $S\subseteq M$ be an embedded submanifold with inclusion $i_S : S\hookrightarrow M$. We define the *kth relative de Rham cohomology* to be the kth de Rham cohomology group of the inclusion. That is, $H^k_{dR}(M,S):=H^k_{dR}(i_S)$. We also define $\Omega^\bullet(M,S):=\Omega^\bullet(i_S)$.

To show that Lemma 5.2.11 indeed gives us the exactness axiom, we need to show that α^* is a natural transformation and that β is induced by the inclusion $j : M=(M,\varnothing)\hookrightarrow(M,S)$. To do this we first need a notion of smooth maps of pairs and their induced maps in cohomology. The former is adapted from the definition of maps of pairs given in [ES52].

Definition 5.2.13. Let (M,S) and (N,T) be pairs of smooth manifolds. A *smooth map of pairs* $F:(M,S)\to(N,T)$ is a smooth map $F:M\to N$ such that $F(S)\subseteq T$.

Lemma 5.2.14. *Let $F:(M,S)\to(N,T)$ be a smooth map of pairs. Then for all $k\geq 0$ there is a well-defined real linear map $F^* : H^k_{dR}(N,T)\to H^k_{dR}(M,S)$ induced by the map $F^*(\omega,\theta):=(F^*\omega,F|_S^*\theta)$. Moreover, the assignment $H^k_{dR}(-):(M,S)\mapsto H^k_{dR}(M,S)$, $F\mapsto F^*$ is a contravariant functor.*

Proof. This result is my own. We show that on the level of cochain complexes, $F^* : \Omega^\bullet(N,T)\to\Omega^\bullet(M,S)$ is a cochain map and then the result follows by Lemma 1.3.9. First note that the linearity of F^* follows from the linearity of the pullback of smooth maps. Next, since $F(S)\subseteq T$, we have that $F\circ i_S=i_T\circ F|_S$, so using functoriality of the pullback, we get $i_S^*\circ F^*=F|_S^*\circ i_T^*$. Then using the naturality of the exterior derivative, we have that

$$\begin{aligned} d(F^*(\omega,\theta)) &= d(F^*\omega,F|_S^*\theta)=(d(F^*\omega),i_S^*(F^*\omega)-d(F|_S^*\theta)) \\ &= (F^*(d\omega),F|_S^*(i_T^*\omega)-F|_S^*(d\theta))=F^*(d\omega,i_T^*\omega-d\theta)=F^*(d(\omega,\theta)). \end{aligned}$$

Functoriality then follows from the fact that the assignment of a smooth map to its pullback is functorial and from Corollary 1.3.10 in a similar way to Corollary 5.1.6. □

Proposition 5.2.15 (Naturality of α^*)**.** *Let $F:(M,S)\to(N,T)$ be a smooth map of pairs and let α_M and α_N be the maps $\alpha_M:\Omega^k(S)\to\Omega^{k+1}(M,S)$ and $\alpha_N:\Omega^k(T)\to\Omega^{k+1}(N,T)$ be defined as α is above. Then $F^*\circ\alpha_N^*=\alpha_M^*\circ F|_S^*$. That is the following diagram commutes*

$$\begin{array}{ccc} H^k_{dR}(T) & \xrightarrow{\alpha_N^*} & H^{k+1}_{dR}(N,T) \\ {\scriptstyle F|_S^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ H^k_{dR}(S) & \xrightarrow{\alpha_M^*} & H^{k+1}_{dR}(M,S) \end{array}$$

Proof. This result is my own. Let $[\theta]\in H^k_{dR}(T)$. Then $\alpha_M^*(F|_S^*[\theta])=[0,F|_S^*\theta]=F^*[0,\theta]=F^*(\alpha_N^*[\theta])$. □

Lemma 5.2.16 (Exactness Axiom)**.** *Let (M,S) be a pair of smooth manifolds with inclusion map $i_S : S\hookrightarrow M$ and let $j : M\hookrightarrow(M,S)$ be the inclusion. Then there is a natural transformation $\alpha^* : H^k_{dR}(S)\to H^k_{dR}(M,S)$ for each $k\geq 0$ and a long exact sequence*

$$\cdots\longrightarrow H^k_{dR}(M,S)\xrightarrow{j^*}H^k_{dR}(M)\xrightarrow{i_S^*}H^k_{dR}(S)\xrightarrow{\alpha^*}H^{k+1}_{dR}(M,S)\longrightarrow\cdots \tag{5.3}$$

Proof. First, the inclusion $j : M = (M, \varnothing) \hookrightarrow (M, S)$ is simply given by the identity since $\mathrm{id}_M(\varnothing) = \varnothing \subseteq S$. Defining $\Omega^k(\varnothing) := 0$ for all $k \geq 0$, we see that $\Omega^k(M) \cong \Omega^k(M) \oplus 0 = \Omega^k(M, \varnothing)$ for all $k \geq 0$. Then the pullback $j^* : \Omega^k(M, S) \to \Omega^k(M, \varnothing)$ is given by $j^*(\omega, \theta) = (\mathrm{id}_M^* \omega, 0) = (\omega, 0)$ since the only map $\Omega^{k-1}(S) \to \Omega^{k-1}(\varnothing)$ is trivial, so the second component must be 0. Then identifying $\Omega^k(M) \cong \Omega^k(M, \varnothing)$ as above, we see that β is the composition

$$\Omega^k(M, S) \xrightarrow{j^*} \Omega^k(M, \varnothing) \xrightarrow{\cong} \Omega^k(M).$$

Then combining this with Lemma 5.2.11 applied to the inclusion map and Proposition 5.2.15 we retrieve (5.3). □

5.2.4 Excision

Theorem 5.2.17 (Excision Axiom)**.** *[Jos95] Let $U \subseteq S \subseteq M$ be smooth manifolds with S an open submanifold of M and $\overline{U} \subset \mathring{S}$. Then the inclusion $i : (M\backslash U, S\backslash U) \hookrightarrow (M, S)$ induces an isomorphism $i^* : H^k_{dR}(M, S) \to H^k_{dR}(M\backslash U, S\backslash U)$ for all $k \geq 0$.*

Proof. We would like to construct an inverse to i^*: $\gamma : H^k_{dR}(M\backslash U, S\backslash U) \to H^k_{dR}(M, S)$. To do this, we derive a way of extending closed and exact forms in $\Omega^k(M\backslash U, S\backslash U)$ to closed and exact forms in $\Omega^k(M, S)$ respectively. First, note that $M\backslash S$ is closed and $\overline{U} \subset \mathring{S} \implies M\backslash S \subset X\backslash\overline{U}$. Moreover, $X\backslash\overline{U}$ is open, so by Lemma 2.1.24 there exists a smooth bump function $\varphi : M \to \mathbb{R}$ for $M\backslash S$ that is supported on $X\backslash\overline{U}$. First, given $\sigma \in \Omega^{k-1}(S\backslash U)$ define the $(k-1)$-form

$$\tilde{\sigma}_p := \begin{cases} \varphi(p)\sigma_p, & p \in Y\backslash U, \\ 0, & p \in U, \end{cases}$$

which is smooth by the Gluing Lemma (Proposition 2.1.13 (b)). That is, $\tilde{\sigma} \in \Omega^{k-1}(S)$. Now let $(\mu, \sigma) \in \Omega^k(M\backslash U, S\backslash U)$ be closed. Then $d\mu = 0$ and $\mu|_{S\backslash U} = d\sigma$. We then define

$$\tilde{\mu}_p := \begin{cases} \mu_p, & p \in M\backslash S, \\ d\tilde{\sigma}_p, & p \in S. \end{cases}$$

On $\partial S \cap \partial(M\backslash S)$ we have that $\sigma = \tilde{\sigma}$ and $\mu = d\sigma$ so $\tilde{\mu}$ is well-defined and again is smooth by the Gluing Lemma. So $\tilde{\mu} \in \Omega^k(M)$ and hence $(\tilde{\mu}, \tilde{\sigma}) \in \Omega^k(M, S)$ with $d(\tilde{\mu}, \tilde{\sigma}) = 0$.

Similarly, if $(\mu, \sigma) = d(\nu, \rho)$ we define

$$\tilde{\rho}_p := \begin{cases} \varphi(p)\rho_p, & p \in Y\backslash U, \\ 0, & p \in U; \end{cases} \text{ and } \tilde{\nu}_p := \begin{cases} \nu_p, & p \in M\backslash S, \\ \varphi(p)\sigma_p + d\tilde{\rho}_p, & p \in S. \end{cases}$$

These are again smooth via the Gluing Lemma and $(\tilde{\mu}, \tilde{\sigma}) = d(\tilde{\nu}, \tilde{\rho})$. Thus $(\mu, \sigma) \mapsto (\tilde{\mu}, \tilde{\sigma})$ is a map which sends closed forms to closed forms and exact forms to exact forms. Moreover, this extension is also linear since the exterior derivative is linear. So we have a cochain map and hence by Lemma 1.3.9 it induces a well-defined map in cohomology which we denote by $\gamma : H^k_{dR}(M\backslash U, S\backslash U) \to H^k_{dR}(M, S)$. Then since $i^* \circ \gamma$ leaves forms on $M\backslash S$ invariant, we have that $i^* \circ \gamma = \mathrm{id}_{H^k_{dR}(M\backslash U, S\backslash U)}$. On the other hand, given $[\omega, \theta] \in H^k_{dR}(M, S)$, define

$$\overline{\theta}_p := \begin{cases} 0, & p \in M\backslash S, \\ (1 - \varphi(p))\theta_p, & p \in S. \end{cases}$$

Then

$$\begin{aligned} d(\overline{\theta}, 0) &= \begin{cases} 0, & \text{on } M\backslash S, \\ (d\theta - d(\varphi\theta), \theta - \varphi\theta), & \text{on } S. \end{cases} \\ &= (\omega, \theta) - (\widetilde{\omega|_{M\backslash U}}, \tilde{\theta}|_{S\backslash U}). \end{aligned}$$

Thus $[\omega, \theta] = [\widetilde{\omega|_{M\backslash U}}, \tilde{\theta}|_{S\backslash U}] = \gamma \circ i^*[\omega, \theta]$. So $\gamma \circ i^* = \mathrm{id}_{H^k_{dR}(M,S)}$. Thus i^* is indeed an isomorphism. □

5.3 The Mayer-Vietoris Exact Sequence

Another important feature of cohomology theories is the Mayer-Vietoris exact sequence which allows us to calculate the cohomology of a space in terms of the cohomology of two smaller spaces whose union is the whole space. We now show that there is a Mayer-Vietoris exact sequence for de Rham cohomology.

Theorem 5.3.1 (Mayer-Vietoris)**.** *[Lee12] Let M be a smooth manifold with an open cover $\{U, V\}$ and let the following maps be inclusions: $i : U \cap V \hookrightarrow U$, $j : U \cap V \hookrightarrow V$, $k : U \hookrightarrow M$, and $l : V \hookrightarrow M$. Define the maps $(k^* \oplus l^*)\omega := (k^*\omega, l^*\omega)$ and $(i^* - j^*)(\omega, \eta) := i^*\omega - j^*\eta$. Then for all $k \geq 0$, there is a linear map $\delta : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M)$ such that the following sequence is exact*

$$\cdots \longrightarrow H^k_{dR}(M) \xrightarrow{k^* \oplus l^*} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i^* - j^*} H^k_{dR}(U \cap V) \xrightarrow{\delta} H^{k+1}_{dR}(M) \longrightarrow \cdots \tag{5.4}$$

Proof. We proceed by showing that the following sequence is an exact sequence of cochain complexes. Then we retrieve (5.4) as a consequence of the Snake Lemma (Lemma 1.3.15). That is we show that

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0$$

is exact for all $k \geq 0$. For exactness at $\Omega^k(M)$, we need to show that $k^* \oplus l^*$ is injective. Let $\omega \in \Omega^k(M)$ with $(k^* \oplus l^*)\omega = 0$, then $(k^*\omega, l^*\omega) = (\omega|_U, \omega|_V) = (0, 0)$. So the restrictions of ω to U and to V are both 0 and since U and V cover M, ω is 0 on all of M. So $k^* \oplus l^*$ is indeed injective.

For exactness at $\Omega^k(U) \oplus \Omega^k(V)$, first observe that for $\omega \in \Omega^k(M)$ we have that

$$(i^* - j^*) \circ (k^* \oplus l^*)\omega = (i^* - j^*)(\omega|_U, \omega|_V) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0.$$

So $\operatorname{im}(k^* \oplus l^*) \subseteq \ker(i^* - j^*)$. On the other hand, suppose that $(\eta, \eta') \in \ker(i^* - j^*)$. Then we have $(i^* - j^*)(\eta, \eta') = \eta|_{U\cap V} - \eta'|_{U\cap V} = 0$. So $\eta|_{U\cap V} = \eta'|_{U\cap V}$. Hence by the Gluing Lemma (Proposition 2.1.13 (b)), we have a global k-form $\sigma \in \Omega^k(M)$ such that $\sigma|_U = \eta$ and $\sigma|_V = \eta$. Then $(\eta, \eta') = (\sigma|_U, \sigma|_V) = (k^* \oplus l^*)\sigma$. So $(\eta, \eta') \in \operatorname{im}(k^* \oplus l^*)$. Hence $\ker(i^* - j^*) \subseteq \operatorname{im}(k^* \oplus l^*)$ and thus $\operatorname{im}(k^* \oplus l^*) = \ker(i^* - j^*)$.

Lastly, for exactness at $\Omega^k(U \cap V)$ we need to show that $i^* - j^*$ is surjective. That is, given $\omega \in \Omega^k(U \cap V)$, we need to find $(\eta, \eta') \in \Omega^k(U) \oplus \Omega^k(V)$ such that $\omega = i^*\eta - j^*\eta' = \eta|_{U\cap V} - \eta'|_{U\cap V}$. To find η and η' we shall make use of the existence of partitions of unity. Let $\{\varphi, \psi\}$ be the smooth partition of unity subordinate to the cover $\{U, V\}$, which exists by Theorem 2.1.23, and define $\eta \in \Omega^k(U)$ and $\eta' \in \Omega^k(V)$ as follows:

$$\eta := \begin{cases} \psi\,\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus \operatorname{supp}(\psi) \end{cases} \quad \text{and } \eta' := \begin{cases} -\varphi\,\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus \operatorname{supp}(\varphi) \end{cases}$$

Then on the sets $(U \cap V) \setminus \operatorname{supp}(\psi)$ and $(U \cap V) \setminus \operatorname{supp}(\varphi)$ where the definitions of η and η' overlap respectively, both definitions give 0. So η and η' are well-defined and are indeed smooth by the Gluing Lemma (Proposition 2.1.13 (b)). Then since partitions of unity sum to 1, we have that

$$\eta|_{U\cap V} - \eta'|_{U\cap V} = \psi\,\omega - (-\varphi\,\omega) = (\psi + \varphi)\,\omega = \omega,$$

as required. □

5.4 Further Calculations and Examples

We conclude this report with a few results and examples of calculating the de Rham cohomology groups of certain spaces. What follows is adapted from [Lee12] unless stated otherwise.

We first consider contractible manifolds (see Definition A.2.9 for the definition of a contractible space).

Lemma 5.4.1. *Let M be a contractible smooth manifold. Then*

$$H^k_{dR}(M) \cong \begin{cases} \mathbb{R}, & k = 0, \\ 0 & k > 0. \end{cases}$$

Proof. If M is contractible, then it is homotopy equivalent to the one point space. Thus the result follows from combining the dimension axiom (Lemma 5.2.1) and homotopy invariance (Theorem 5.2.5). □

Corollary 5.4.2 (The Poincaré Lemma)**.** *If U is a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then $H^k_{dR}(U) = 0$ for all $k > 0$.*

Proof. Star-shaped sets are contractible, so the result follows from Lemma 5.4.1. □

Remark 5.4.3. In particular, both $\mathbb{R}^n$ and $\mathbb{H}^n$ are star-shaped and hence contractible. So Lemma 5.4.1 gives us the de Rham cohomology of $\mathbb{R}^n$ and $\mathbb{H}^n$.

Corollary 5.4.4. *Let M be a smooth manifold. Then each point $p \in M$ has an open neighbourhood on which every closed form is exact.*

Proof. Recall from the proof of Proposition 3.3.13 that every point is contained in a chart (U, φ) where U is homeomorphic to an open ball in $\mathbb{R}^n$ or and open half-ball in $\mathbb{H}^n$. In either case, $\varphi(U)$ is star-shaped, so by the Poincaré Lemma (Corollary 5.4.2) we have $H^k_{dR}(\varphi(U)) = 0$ for all $k > 0$. Then since $\varphi : U \to \varphi(U)$ is a diffeomorphism, we have that $H^k_{dR}(U) = 0$ for all $k \geq 0$ by diffeomorphism invariance of de Rham cohomology (Corollary 5.1.7). Then as in Remark 5.1.3 we have that $H^k_{dR}(U) = 0$ implies that every closed k-form on U is exact for all $k > 0$. □

We now present another consequence of homotopy invariance.

Proposition 5.4.5. *Let $\pi : E \to M$ be a smooth vector bundle. Then $H^k_{dR}(M) \cong H^k_{dR}(E)$ for all $k \geq 0$.*

Proof. This is my own example. Recall that Corollary 2.5.16 gave is that M and E are homotopy equivalent. So the result follows from homotopy invariance of de Rham cohomology (Theorem 5.2.5). □

Recall Proposition 5.1.8 (e) said that simply connected smooth manifolds had trivial first de Rham cohomology group. This hints at a potential relation between H^1_{dR} and the fundamental group which we expand upon in this next result.

Theorem 5.4.6. *Let M be a connected smooth manifold without boundary. Then for each $p \in M$ there is a well-defined injective linear map $\Phi : H^1_{dR}(M) \to \mathrm{Hom}(\pi_1(M, p), \mathbb{R})$ given by*

$$\Phi[\omega] : [\gamma] \mapsto \int_{\tilde{\gamma}} \omega,$$

where $\tilde{\gamma}$ is a piecewise smooth curve that is path homotopic to γ.

Proof. Firstly, given any path homotopy class $[\gamma] \in \pi_1(M, p)$, the Whitney Approximation Theorem (Theorem 2.4.1) implies that there is a piecewise smooth curve $\tilde{\gamma}$ which is path homotopic to γ, i.e. $[\gamma] = [\tilde{\gamma}]$. So every path homotopy class is represented by a piecewise smooth curve so the definition of Φ makes sense and we proceed assuming the representative is as such. Moreover, by Theorem 4.4.10 the integral of a 1-form only depends on the path homotopy class of the piecewise smooth curve, so $\Phi[\omega][\gamma_1] = \Phi[\omega][\gamma_2]$ when $[\gamma_1] = [\gamma_2]$. Now given two closed forms, $\omega, \tilde{\omega} \in \Omega^1(M)$ such that $[\omega] = [\tilde{\omega}]$ we have that there exists a smooth map $f \in C^\infty(M)$ such that $\omega - \tilde{\omega} = df$. So by linearity and the Fundamental Theorem of Line Integrals (Theorem 4.4.6), we have

$$\Phi[\omega][\gamma] - \Phi[\tilde{\omega}][\gamma] = \int_\gamma \omega - \int_\gamma \tilde{\omega} = \int_\gamma df = f(\gamma(p)) - f(\gamma(p)) = 0.$$

Thus Φ is well-defined.

Next we show that $\Phi[\omega] \in \mathrm{Hom}(\pi_1(M,p),\mathbb{R})$. That is, we show that $\Phi[\omega]$ is a group homomorphism for each $[\omega] \in H^1_{dR}(M)$. Let $[\gamma_1],[\gamma_2] \in \pi_1(M,q)$. Then by Proposition 4.4.4 (c) we have that

$$\int_{\gamma_1 * \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

So $\Phi[\omega][\gamma_1 * \gamma_2] = \Phi[\omega][\gamma_1] + \Phi[\omega][\gamma_2]$, as required.

Next, the linearity of Φ is a direct consequence of the linearity of line integrals.

Lastly, we show that Φ is injective. Suppose $[\omega] \in \ker(\Phi)$, then $\Phi[\omega] : \pi_1(M,p) \to \mathbb{R}$ is the zero homomorphism. So for all piecewise smooth closed curves, γ, based at p, we have $\int_\gamma \omega = 0$. The aim is to show that $[\omega] = 0$, that is, we want to show ω is exact. To do this, we will show that it is conservative. Then by Theorem 4.4.9 we will have that ω is exact and hence $\ker(\Phi) = 0$ and Φ is injective. It remains, then, to show that ω is conservative. Let σ be any piecewise smooth closed curve based at $q \in M$. Then since M is connected, we can choose a piecewise smooth curve α from p to q. Then letting $\overline{\alpha}$ be the piecewise smooth curve from q to p achieved by going along α in the opposite direction, we get that $\alpha * \sigma * \overline{\alpha}$ is a piecewise smooth closed curve based at p. Then using Propositions 4.4.4 (c) and 4.4.7, we get that

$$0 = \int_{\alpha*\sigma*\overline{\alpha}} \omega = \int_\alpha \omega + \int_\sigma \omega - \int_\alpha \omega = \int_\sigma \omega.$$

Then since σ and $q \in M$ were arbitrary, the integral of ω over any piecewise smooth curve in M is always 0. Thus ω is indeed conservative, as required. □

Corollary 5.4.7. *Let M be a connected smooth manifold without boundary with finite fundamental group. Then $H^1_{dR}(M) = 0$.*

Proof. There are no non-trivial group homomorphisms from a finite group to the additive group of reals so $\mathrm{Hom}(\pi_1(M,p),\mathbb{R}) = 0$ for all $p \in M$. So by Theorem 5.4.6 there is an injective linear map $\Phi : H^1_{dR}(M) \to 0$, hence $H^1_{dR}(M) = 0$. □

Example 5.4.8. Since for $n \geq 2$ the fundamental group of $\mathbb{RP}^n$ for any point $p \in \mathbb{RP}^n$ is $\pi_1(\mathbb{RP}^n, p) \cong \mathbb{Z}/2$ [Hat02], Corollary 5.4.7 implies that $H^1_{dR}(\mathbb{RP}^n) = 0$ for $n \geq 2$.

We now use Mayer-Vietoris to compute the de Rham cohomology of S^n for $n \geq 1$. Recall that S^0 is the union of two points, so its cohomology is given by Corollary 5.1.9.

Theorem 5.4.9 (de Rham Cohomology of S^n)**.** *For $n \geq 1$ the de Rham cohomology groups of S^n are given by*

$$H^k_{dR}(S^n) \cong \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.5}$$

Proof. The cases for $k > n$ and $k = 0$ are given by Proposition 5.1.8 (a) and (b) respectively so it remains to consider the case $1 \leq k \leq n$. We proceed by an induction on n. This proof is largely adapted from Theorem 17.21 in [Lee12], however we instead adapt the $n = 1$ case from Example 2.9 in [BT82].

For $n = 1$, we cover S^1 by the sets $U := S^1 \backslash \{(0,1)\}$ and $V := S^1 \backslash \{(0,-1)\}$. Then both U and V are homeomorphic to I and are hence contractible so by Lemma 5.4.1 we have that $H^0_{dR}(U) \cong H^0_{dR}(V) \cong \mathbb{R}$ and for $k > 0$ $H^k_{dR}(U) \cong H^k_{dR}(V) = 0$. We also have that $U \cap V$ is homotopy equivalent to S^0 so $H^k_{dR}(U \cap V) \cong H^k_{dR}(S^0)$ for all $k \geq 0$. Then by the Mayer-Vietoris Theorem (Theorem 5.3.1) we have the following exact sequence:

$$0 \longrightarrow H^0_{dR}(S^1) \xrightarrow{k^* \oplus l^*} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{i^* - j^*} H^0_{dR}(U \cap V) \xrightarrow{\delta} H^1_{dR}(S^1) \longrightarrow 0.$$

So substituting in our known groups, we get the exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{k^* \oplus l^*} \mathbb{R}^2 \xrightarrow{i^* - j^*} \mathbb{R}^2 \xrightarrow{\delta} H^1_{dR}(S^1) \longrightarrow 0.$$

Then by exactness, $k^* \oplus l^*$ is injective, $\text{im}(k^* \oplus l^*) = \ker(i^* - j^*)$, $\text{im}(i^* - j^*) = \ker(\delta)$, and δ is surjective. Then using the Rank-Nullity Theorem and exactness, we get $\dim(\text{im}(k^* \oplus l^*)) = \dim(\mathbb{R}) - \dim(\ker(k^* \oplus l^*)) = 1$, so $\dim(\ker(i^* - j^*)) = \dim(\text{im}(k^* \oplus l^*)) = 1$. Then $\dim(\text{im}(i^* - j^*)) = \dim(\mathbb{R}^2) - \dim(\ker(i^* - j^*)) = 2 - 1 = 1$, so $\dim(\ker(\delta)) = \dim(\text{im}(i^* - j^*)) = 1$. Then since δ is surjective, $\text{im}(\delta) = H^1_{dR}(S^1)$, so $\dim(H^1_{dR}(S^1)) = \dim(\mathbb{R}^2) - \dim(\ker(\delta)) = 2 - 1 = 1$. Thus $H^1_{dR}(S^1) \cong \mathbb{R}$.

Now suppose that for some $n \geq 2$ that (5.5) is true for S^{n-1}. First, since S^n is simply connected for $n \geq 2$, Proposition 5.1.8 (e) gives us that $H^1_{dR}(S^n) = 0$. Next, we cover S^n in an analogous way to the cover of S^1 above: let $U := S^n \backslash \{(0, ..., 0, 1)\}$ and $V := S^n \backslash \{(0, ..., 0, -1)\}$. Then both U and V are homeomorphic to $\mathbb{R}^n$ via stereographic projection and hence are contractible so by homotopy invariance, their cohomology groups are given by Lemma 5.4.1. We also have that $U \cap V$ is homeomorphic to $\mathbb{R}^n \backslash \{0\}$ so is homotopy equivalent to S^{n-1}, so by homotopy invariance $H^k_{dR}(U \cap V) \cong H^k_{dR}(S^{n-1})$ for $k \geq 0$. Then the Mayer-Vietoris exact sequence is

$$\cdots \longrightarrow H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \longrightarrow H^{k-1}_{dR}(U \cap V) \longrightarrow H^k_{dR}(S^n) \longrightarrow H^k_{dR}(U) \oplus H^k_{dR}(V) \longrightarrow \cdots$$

Then since $H^k_{dR}(U)$ and $H^k_{dR}(V)$ are both trivial for $k \geq 1$ and $H^k_{dR}(U \cap V) \cong H^k_{dR}(S^{n-1})$ for $k \geq 0$, we have that

$$0 \longrightarrow H^{k-1}_{dR}(S^{n-1}) \longrightarrow H^k_{dR}(S^k) \longrightarrow 0$$

is an exact sequence for all $2 \leq k \leq n$. Hence by exactness $H^{k-1}_{dR}(S^{n-1}) \cong H^k_{dR}(S^k)$ for $2 \leq k \leq n$. Thus by the induction hypothesis, (5.5) holds for S^n. Hence by induction, (5.5) is true for all $n \geq 1$. $\square$

Example 5.4.10. Comparing Theorem 5.4.9 and Remark 5.4.3 we see that $\mathbb{R}^n$ and S^m have differing de Rham cohomology groups for $n, m \geq 0$. By the contrapositive of the homotopy invariance of de Rham cohomology, we get that $\mathbb{R}^n$ and S^m are not homotopy equivalent for all $n, m \geq 0$.

We conclude with some final remarks.

Homotopy invariance of de Rham cohomology suggests that there should be some way to calculate the de Rham cohomology groups in a purely topological way. Theorem 5.4.9 actually shows that the de Rham cohomology and the singular cohomology with real coefficients of the spheres are isomorphic! In fact, this is an example of a more general theorem which we state below. The theory to build up to the proof as well as the proof itself are far too lengthy to be included here, but the details can be found in Chapter 18 in [Lee12].

Theorem 5.4.11 (de Rham)**.** *Let M be a smooth manifold without boundary. Then for all $k \geq 0$*

$$H^k_{dR}(M) \cong H^k(M; \mathbb{R}),$$

where $H^*(M; \mathbb{R})$ is the singular cohomology of M with real coefficients.

Remark 5.4.12. Recall from Remark 4.1.7 that in general $\Omega^k(-)$ was not a useful functor for distinguishing spaces because $\Omega^k(M)$ is in general infinite dimensional and we do not have any tools to calculate it. Thus far we have provided a plethora of computational tools for de Rham cohomology but the question of whether these groups are finite dimensional still remains. It turns out that for compact smooth manifolds, this is always the case. To prove this we require much more theory. In particular, one can define a cochain complex by restricting to only compactly supported differential forms (see Chapter 17 in [Lee12]). In the case of compact smooth manifolds, all forms are compactly supported so we retrieve the usual de Rham cohomology groups but now armed with more powerful results. One such result is Poincaré duality, which we then use to show that the de Rham cohomology groups of compact smooth manifolds are finite dimensional (see Problems 18-7 and 18-8 in [Lee12]).

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I. Definitions From Topology

A.1 General Topology

Here we shall recall some basic definitions from general topology. What follows is adapted from *Topology* by James Munkres [Mun14].

Definition A.1.1. A *topological space* is a pair $(X, \mathcal{T})$ consisting of a set X and a collection, $\mathcal{T}$, of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$.
2. If $(U_i)_{i\in I}$ is a collection of elements in $\mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
3. If $(U_i)_{i=1}^n$ is a finite collection of elements in $\mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call $\mathcal{T}$ a *topology* on X and elements of $\mathcal{T}$ *open sets* (in X). Often we just write X for the topological space.

Definition A.1.2. Let X and Y be topological spaces and let $f : X \to Y$ be a function between them. We say f is *continuous* if for any open set $U \subseteq Y$ in Y, its preimage $f^{-1}(U) \subseteq X$ is open in X. It is common to refer to a continuous function simply as a *map*.

Definition A.1.3. Let X and Y be topological spaces. We say a continuous map $f : X \to Y$ is a *homeomorphism* if it is bijective and its inverse $f^{-1} : Y \to X$ is also continuous. If such a homeomorphism exists between two spaces X and Y, then we say X is *homeomorphic* to Y and we write $X \cong Y$.

Definition A.1.4. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a subcollection of $\mathcal{T}$. We say $\mathcal{B}$ is a *basis* for the topology on X if the following are satisfied

1. $\forall x \in X\ \exists B \in \mathcal{B}$ such that $x \in B$.
2. If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

We call elements of $\mathcal{B}$ *basis elements* and we say that $\mathcal{B}$ *generates* the topology on X.

Definition A.1.5. Let X be a topological space. We say that X is *second countable* if there exists a basis for the topology on X that is countable.

Definition A.1.6. Let X be a topological space. We say that X is *Hausdorff* if for all $x, y \in X$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Definition A.1.7. Let $A \subseteq X$ be a subset of a topological space X. We say a collection $(U_i)_{i\in I}$ of subsets of X is a *cover* of A if $A \subseteq \bigcup_{i\in I}$. We say a cover of A, $(U_i)_{i\in I}$, is an *open cover* if U_i is open in X for all $i \in I$. A *subcover* of a cover of A, $(U_i)_{i\in I}$, is a collection $(U_j)_{j\in J}$ with $J \subseteq I$ that is also a cover of A. We say A is *compact* in X if any open cover of A has a finite subcover.

Definition A.1.8. Let X be a topological space. We say X is *connected* if given two open sets $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V = \varnothing$, one of U and V must be empty.

Definition A.1.9. Let X be a topological space and let $I := [0, 1]$ be equipped with the subspace topology as a subspace of $\mathbb{R}$. Let $x_0, x_1 \in X$. A *path* from x_0 to x_1 is a continuous map $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition A.1.10. Let X be a topological space. We say X is *path connected* if given any two points $x_0, x_1 \in X$ there exists a path $\gamma : I \to X$ from x_0 to x_1.

A.2 Homotopy and the Fundamental Group

Here we shall recall some definitions from homotopy theory. Unless indicated otherwise, everything that follows is adapted from *Topology* by James Munkres [Mun14].

Definition A.2.1. Let $f, g : X \to Y$ be two continuous maps. A *homotopy* from f to g is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say f is *homotopic* to g and write $f \simeq g$. Let $A \subset X$. We say f and g are *homotopic relative to* A if the exists a homotopy $H : X \times I \to Y$ such that $f(x) = H(x, t) = g(x)$ for all $(x, t) \in A \times I$.

Definition A.2.2. Let $\gamma, \sigma : I \to X$ be paths from $x_0 \in X$ to $x_1 \in X$. A *path homotopy* from γ to σ is a homotopy $H : I \times I \to X$ such that $H(0, t) = x_0$ and $H(1, t) = x_1$ for all $t \in I$. If such a path homotopy exists, we say that γ is *path homotopic* to σ and write $\gamma \simeq \sigma$.

Lemma A.2.3. *The relations "homotopic to" and "path homotopic to" are equivalence relations.*

Proof. See Lemma 51.1 in *Topology* by Munkres [Mun14]. □

Definition A.2.4. [BBT20] Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. We denote the equivalence class of f to be $[f]$ under the relation $\simeq$ and we call it the *homotopy class* of f. The set of all homotopy classes of maps from X to Y is denoted $[X, Y]$.

Definition A.2.5. Let $\gamma, \sigma : I \to X$ be paths from x_0 to x_1 and x_1 to x_2 respectively. We define the *product* of these paths to be the path $\gamma * \sigma : I \to X$ given by

$$(\gamma * \sigma)(s) := \begin{cases} \gamma(2s), & x \in \left[0, \frac{1}{2}\right] \\ \sigma(2s - 1), & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

One can show that this product induces a well defined associative product on homotopy classes of paths: $[\gamma] * [\sigma] := [\gamma * \sigma]$. See Theorem 51.2 from *Topology* by Munkres for details [Mun14].

Definition A.2.6. Let X be a topological space and let $x_0 \in X$. A *loop* based at x_0 is a path $\gamma : I \to X$ such that $\gamma(0) = x_0 = \gamma(1)$. We denote the set of homotopy classes of loops based at x_0 by $\pi_1(X, x_0)$. We call $(\pi_1(X, x_0), *)$ the *fundamental group* of X relative to the base point x_0.

One can show that $(\pi_1(X, x_0), *)$ is indeed a group. See Theorem 51.2 and page 327 in *Topology* by Munkres for details [Mun14].

Theorem A.2.7. *Let X be path connected and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. See Theorem 52.1 and Corollary 52.2 in *Topology* by Munkres for details [Mun14]. □

Definition A.2.8. Let $f : X \to Y$ be a continuous map. We say f is a *homotopy equivalence* if there exists another continuous map $g : Y \to X$, which we call a *homotopy inverse of* f, such that

$$f \circ g \simeq \mathrm{id}_Y \quad \text{and} \quad g \circ f \simeq \mathrm{id}_X .$$

If such a homotopy equivalence exists, we say X and Y are *homotopy equivalent* and we write $X \simeq Y$.

Definition A.2.9. We say a topological space X is *contractible* if it is homotopy equivalent to the one point space.

Theorem A.2.10. *Let X and Y be topological spaces and $x_0 \in X$ and $y_0 \in Y$, Then*

$$X \simeq Y \implies \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

Proof. See Theorem 58.7 in *Topology* by Munkres [Mun14]. □