University of Glasgow

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Level-4 Honours project

Introductory Categorification

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Abstract

We give an introduction to the basics of category theory required to perform categorification: the notion of a category, key definitions pertaining to morphisms, and functors are all defined and discussed, with illustrative examples. A decategorification of the category **FinSet** together with the disjoint union operation is performed, resulting in the commutative monoid $(\mathbb{N}, +)$. As an approach to the process of categorification, we proceed to an overview of the theory of modules over associative algebras. The notion of a module is introduced with illustrative examples. The tensor product of two modules is carefully constructed, with simple computational examples for ideals of $\mathbb{Z}$. Associative algebras are then discussed, with examples. We then define and discuss the notion of the Grothendieck group, define naive categorification, and proceed to a simple example of weak categorification. We then perform a more sophisticated weak categorification: the categorification of the polynomial representation of the Weyl algebra. We conclude with a brief discussion of strong categorification.

Contents

1 Introduction

Volodymyr Mazorchuk in [1] introduces *categorification* as a term which refers to the process of replacing ideas in set theory with their analogues in the field of *category theory*, a highly abstract branch of mathematics which aims to view mathematical objects and the relationships between them as generally as possible. As will be seen, this process gives extra structure on the mathematical object in question, the study of which may provide unique insights. As a way of attaining intuition, it will be instructive to briefly consider the opposite process, which is known as *decategorification*.
The classic, everyday example of decategorification comes from counting. To count is simply to recognise that collections containing the same number of objects have something in common. In everyday language, we say that they have the same quantity. In the language of category theory, we say that they are *isomorphic in the category of finite sets*. If we choose to forget the categorical structure and treat all finite sets of the same cardinality as equal, what is left is simply $\mathbb{N}$. In a similar way, the disjoint union operation becomes nothing more than addition; we forget everything except the cardinality of the set. This is the basic method of all decategorification: we treat isomorphic objects as equal.
Because it is always easier to do away with information than it is to create more of it, the process of categorification is not so simple. There is no hard and fast rule for categorifying an object: it is done on a case-by-case basis.
In Section 2 and Section 3 we give overviews of basic category theory and the theory of modules over associative algebras, respectively. Having made these preparations, we proceed, in Section 4, to actual examples of categorification.

2 Basic category theory

In order to introduce examples of categorification, it will first be necessary to give an overview of the language and theory of categories. Correspondences are often noticed between different types of mathematical objects, as well as the maps between those objects; for example, the relationship between groups and group homomorphisms seems the same, intuitively, as the relationship between topological spaces and continuous functions. Category theory formalises this intuition.

2.1 Categories

Definition 2.1.1. A *category* $\mathcal{C}$, in part, consists of two collections, called *objects* and *morphisms*, denoted $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ respectively, together with three assignments, called domain, codomain and identity:

- Domain and codomain each assign every morphism f an object $\mathrm{dom}(f)$ (respectively, $\mathrm{cod}(f)$). When a morphism satisfies $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$, we write $f : A \to B$ or $A \xrightarrow{f} B$.
- Identity assigns every object A a morphism id_A such that $\mathrm{dom}(id_A) = A = \mathrm{cod}(id_A)$

To make this structure a category, there must also exist an operation called *composition* (denoted by $\circ$) on the collection of pairs of morphisms (f, g) such that $\mathrm{dom}(f) = \mathrm{cod}(g)$. This is known as the collection of *composable morphisms*. Composition gives a new morphism $f \circ g$ such that $\mathrm{dom}(f \circ g)$ $= \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
In addition, the following two conditions must hold:

1. *Associativity*: given a sequence of objects and morphisms of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $h \circ (g \circ f) = (h \circ g) \circ f$.

2. *Unit Law*: for any morphism $f : A \to B$, we have $id_B \circ f = f = f \circ id_A$.

Remark. Notice that care has been taken to avoid referring to Ob($\mathcal{C}$) or Mor($\mathcal{C}$) as sets. This is because, if they were defined to be sets, it would be impossible to construct certain categories without encountering difficulties such as Russell's Paradox. As will become apparent, this would make many interesting examples impossible. Ob($\mathcal{C}$) and Mor($\mathcal{C}$) are in fact examples of *classes*, which are collections of sets which it is possibly to identify by a shared property. A class which is not simply a set is known as a *proper class*. For a more in-depth discussion of the set-theoretic foundations of categories, see Herrlich and Strecker. [3]

It will be instructive at this point to consider a few examples.

Example 2.1.2. Let Ob($\mathcal{C}$) be the class of all sets, let Mor($\mathcal{C}$) be the class of functions between the sets, and let id_A be the standard identity function on the set A. Then $\mathcal{C}$ is a category, commonly denoted **Set**.

Example 2.1.3. **Grp** is the category whose objects are groups and whose morphisms are group homomorphisms. The notion of the identity morphism extends in a natural way from **Set**. This construction generalises easily: in a very similar manner, we can form such categories as **AGrp**, (abelian groups, abelian group homomorphisms) **Rng**, (rings, ring homomorphisms) and **Top** (topological spaces, continuous maps.) The intuitive notion is that the morphisms should preserve the structure of the objects which they map between.

The following example aims to demonstrate the value of the category-theoretic point of view.

Example 2.1.4. Recall the First Isomorphism Theorem for groups: let G and H be groups, and let $\varphi : G \to H$ be a group homomorphism. Then $\operatorname{Ker}\varphi$ is normal in G, $\operatorname{Im}\varphi$ is a subgroup of H. Because $\operatorname{Ker}\varphi$ is normal in G, we can form the quotient group $G/\operatorname{Ker}\varphi \neq \operatorname{Im}\varphi$. The First Isomorphism Theorem says that $\operatorname{Im}\varphi \cong G/\operatorname{Ker}\varphi$. Recall also that this generalises easily so that a version exists for rings. Going even further, versions exist for many other structures, including vector spaces, as well as algebras and modules, which will be discussed later. This is no accident: it pertains to properties shared by all of the categories involved. Thus, by using category-theoretic concepts, all of the versions of this useful theorem can be obtained at once, rather than laboriously considering each case individually.

Note. We will require the use of the First Isomorphism Theorem for modules later. The above example will be referred to at that stage.

In the examples discussed so far objects have been all been sets, with or without an extra structure, and the morphisms have all been functions whose domain and codomain have been the objects. As the following example, due to Simmons in [5], will demonstrate, this is not necessarily the case.

Example 2.1.5. Let Ob($\mathcal{C}$) be the class of all finite sets. Let an arbitrary morphism $A \xrightarrow{f} B$ be a function $f : A \times B \to \mathbb{R}$ with no other conditions imposed. For an object A, we define

$$\mathrm{id}_A : A \times A \to \mathbb{R}$$

by

$$\mathrm{id}_A(a_1, a_2) = 1 \quad \forall a_i \in A$$

For a given pair of composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ define

$$g \circ f : A \times C \to \mathbb{R}$$

by

$$(g \circ f)(a, c) = \sum \{f(a, b)g(b, c) \mid b \in B\}$$

for $a \in A$, $c \in C$.

It can be shown that these two classes of objects and morphisms give rise to a category.

Definition 2.1.6. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{S}$ of $\mathcal{C}$ consists of:

- A collection $\mathrm{Ob}(\mathcal{S})$ contained in $\mathrm{Ob}(\mathcal{C})$
- A collection $\mathrm{Mor}(\mathcal{S})$ contained in $\mathrm{Mor}(\mathcal{C})$

Such that the following conditions hold:

- Every $S \in \mathrm{Ob}(\mathcal{S})$ has identity $\mathrm{id}_S \in \mathrm{Mor}(\mathcal{S})$
- For all $f : S \to T \in \mathrm{Mor}(\mathcal{S})$, both S and T are in $\mathrm{Ob}(\mathcal{S})$
- For every pair of composable morphisms $g, h \in \mathrm{Mor}(\mathcal{S})$, $f \circ g$ is in $\mathrm{Mor}(\mathcal{S})$

Example 2.1.7. Let $(M, \circ)$ be a set together with a binary operation. If $\circ$ is associative and the structure has an identity e, it is known as a *monoid*. Now, consider a category $\mathcal{C}$ with a single object A. Then $\mathcal{C}$ is a monoid, because, from the definition of a category, composition of morphisms is associative, and there must be an identity morphism. This is in fact true for any single object in a category: the morphisms with the object as both domain and codomain form a monoid under composition.

We now give some key definitions pertaining to morphisms.

Definition 2.1.8. Let A and B be a pair of objects in a category. Then their *hom-set*, denoted $\mathrm{Hom}(A, B)$, is the collection of all morphisms with A as domain and B as codomain.

Definition 2.1.9. Let A, B, and C be objects in a category. The morphism $f : A \to B$ is said to be an *epimorphism* if the following holds for all morphisms $g_1, g_2 : B \to C$

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

Definition 2.1.10. Let A, B, and C be as above. The morphism $f : A \to B$ is said to be a *monomorphism* if the following holds for all morphisms $g_1, g_2 : C \to A$

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

A morphism is said to be an *isomorphism* if it is both an epimorphism and a monomorphism. We also say, equivalently, that invertible morphisms are isomorphisms.

Remark. In most of the categories we have seen so far, the notion of an isomorphism is intuitive: isomorphisms are bijections which also preserve the structure of the objects of the category; so in **Set** the isomorphisms are bijections; in **Grp**, **Rng AGrp** and other categories of this type, they are the same as the traditional notion of isomorphisms for the algebraic structures in question; in **Top**, they are homeomorphisms.

Example 2.1.11. Let **FinSet** be the category whose objects are finite sets and whose morphisms are functions between finite sets. Further, let

$$\coprod : \mathbf{FinSet} \times \mathbf{FinSet} \to \mathbf{FinSet}$$
$$A \times B \mapsto A \coprod B$$

be the disjoint union operation. Note that, since the disjoint union of two finite sets is again a finite set, it is a binary operation. Then the decategorification of **FinSet** with this operation is the

commutative monoid $(\mathbb{N}, +)$.
To see this, first consider monomorphisms and epimorphisms. It can be shown that a morphism f is a monomorphism or an epimorphism if and only if it is an injective or surjective function, respectively. Then f is an isomorphism in **FinSet** if and only if it is a bijection. Sets with a bijection between them have the same cardinality; therefore an arbitrary ismorphism class of **FiniteSet** collapses all finite sets of the same cardinality into a single element, with 0 corresponding to $\emptyset$, 1 corresponding to all singleton sets, and so on. Therefore the underlying set of this decategorification is $\mathbb{N}$.
Now, consider the disjoint union operation on two finite sets, $A = \{a_1, \dots a_k\}$ and $B = \{b_1, \dots b_l\}$. Clearly $|A| = k$ and $|B| = l$. It is defined as follows:

$$A \coprod B = \{(a, 0) \,| a \in A\} \cup \{(b, 1) \,| b \in B\}.$$

Note. Here, 0 and 1 are members of an indexing set I: because we are only considering disjoint union as a binary operation, it only requires two elements. In fact this construction generalises to any family of sets A_i which are indexed by I.

So we have that $A \coprod B = \{(a_1, 0), (a_2, 0), \dots, (a_k, 0), (b_1, 1), (b_2, 1), \dots, (b_l, 1)\}$. This set has cardinality $k + l$. Also, trivially, $|A \coprod \emptyset| = |A|$. If we consider this only up to isomorphism, that is to say, forget everything about the sets in question except their cardinality, it is apparent that disjoint union corresponds to addition, with the empty set corresponding to the identity 0, and all other finite sets corresponding to their nonzero cardinalities.
Since the binary operation on a monoid must be associative, it must be verified that disjoint union is associative. Let A and B be as above, and let $C = \{c_1, \dots c_m\}$. First consider $A \coprod (B \coprod C)$:

$$\begin{aligned} &A \coprod (B \coprod C) \\ =&A \coprod \{(b_1, 1), (b_2, 1), \dots, (b_l, 1), (c_1, 2), (c_2, 2), \dots, (c_m, 2)\} \end{aligned}$$

The index here indicates only the originating set of the element, so we have:

$$A \coprod (B \coprod C) = \{(a_1, 0), (a_2, 0), \dots, (b_1, 1), (b_2, 1), \dots, (b_l, 1), (c_1, 2), (c_2, 2), \dots, (c_m, 2)\}$$

Similarly:

$$\begin{aligned} &(A \coprod B) \coprod C \\ =&\{(a_1, 0), (a_2, 0), \dots, (a_k, 0), (b_1, 1), (b_2, 1), \dots, (b_l, 1)\} \coprod C \\ =&\{(a_1, 0), (a_2, 0), \dots, (b_1, 1), (b_2, 1), \dots, (b_l, 1), (c_1, 2), (c_2, 2), \dots, (c_m, 2)\} \end{aligned}$$

Which is the same as $A \coprod (B \coprod C)$. Also, by a similar argument, it can be shown that $\coprod$ commutes. Therefore, the decategorification of $(\mathbf{FinSet}, \coprod)$ is the commutative monoid $(\mathbb{N}, +)$.

2.2 Functors

We now proceed to a discussion of *functors*: these are, roughly speaking, maps between categories, which assign objects to objects and morphisms to morphisms.

Definition 2.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a map which assigns to each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$, and assigns to each morphism $A \xrightarrow{f} B \in \mathrm{Mor}(\mathcal{C})$ a morphism $F(A) \xrightarrow{F(f)} F(B) \in \mathrm{Mor}(\mathcal{D})$ In addition, the following properties should be satisfied:

1. Identity morphisms are preserved: $F(id)_A = id_{F(A)} \;\; \forall A \in \mathrm{Ob}(\mathcal{C})$.

2. Composition is preserved: for an arbitrary pair of composable morphisms (f, g), $F(g \circ f) = F(g) \circ F(f)$.

Similarly to the notation for morphisms, when we mean that F is a functor from $\mathcal{C}$ to $\mathcal{D}$, we shall write $F : \mathcal{C} \to \mathcal{D}$ or $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

Definition 2.2.2. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories, together with functors F and G such that $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{E}$. Then we can define the *composite functor* $F \circ G$ as follows:

$$\begin{aligned} &\mathcal{C} \to \mathcal{E} \\ &A \mapsto F(G(A)) \\ &f \mapsto F(G(f)) \end{aligned}$$

for all $A \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(\mathcal{C})$.

Example 2.2.3. The map F : **Grp** $\to$ **Mon** which maps each group to the monoid consisting of the same set with the same operation and each group homomorphism to its underlying monoid homomorphism is an example of a functor: we refer to functors of this type as *forgetful* functors; in this case, we have forgotten the necessity for inverses, and hence, the group structure.

Example 2.2.4. Let F be as in the previous example, and let G : **Rng** $\to$ **Grp** be the forgetful functor mapping each ring to its underlying group, and each ring homomorphism to its underlying group homomorphism. Then we can define the composite functor $F \circ G$: **Rng** $\to$ **Mon** which assigns each ring to its underlying monoid and each ring homomorphism to its underlying monoid homomorphism. Here, we have forgetten the necessity for additive inverses, as well as the ring multiplication operation and its properties.

Example 2.2.5. The disjoint union operation in Example 2.1.11 is a functor.

Definition 2.2.6. A category $\mathcal{C}$ is called *small* if $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets which are not proper classes.

Example 2.2.7. Let $\mathcal{C}$ be a category. Then there exists an identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which assigns every object and morphism to itself. This, along with the fact the functor composition is associative, allows us to define **Cat**, the *category of small categories*, whose objects are small categories and whose morphisms are functors.

Remark. Since the objects and morphisms of a small category are sets, there is no logical difficulty here. However, because this is not the case for all categories, it is not possible to form a category of categories: in attempting to do so, a contradiction analogous to Russell's Paradox is encountered.

Example 2.2.8. ([2], Example 1.5.6) Let R and S be commutative rings. Recall $GL_n(R)$, the multiplicative group of all invertible $n \times n$ matrices whose entries are in R. Let $\phi : R \to S$ be an arbitrary ring homomorphism. Let the entry in the ith row and jth column of a member of this group be denoted by (a_{ij}). For an arbitrary matrix with entries denoted this way, define a map:

$$\begin{aligned} GL_n\phi : GL_n(R) &\to GL_n(S) \\ (a_{ij}) &\mapsto \phi[(a_{ij})] \end{aligned}$$

It can be easily shown that $GL_n\phi$ is a group homomorphism. In fact, this information defines an infinite family of functors; for every $n \in \mathbb{N}^+$ there exists a functor which maps from **CRng** (commutative rings with ring homomorphisms) to **Grp**, asigning each commutative ring R to $GL_n(R)$ and each ring homomorphism $\phi : R \to S$ to the group homomorphism $GL_n\phi : GL_n(R) \to GL_n(S)$.

Definition 2.2.9. All examples of functors so far have preserved the directions of morphisms; we say that such functors are *covariant*. However we shall continue to simply write "functor" in this case. On the other hand, a *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ reverses the directions; that is to say, it assigns every $f : A \to B \in \mathrm{Mor}(\mathcal{C})$ a morphism $F(f) : F(B) \to F(A) \in \mathrm{Mor}(\mathcal{D})$. Composable morphisms $f, g \in \mathrm{Mor}(\mathcal{C})$ satisfy $F(g \circ f) = F(f) \circ F(g)$. There are no other ways in which covariant and contravariant functors differ.

Definition 2.2.10. Let $\mathcal{C}$ be a category whose hom-sets are sets and not proper classes. For any objects $A, B \in \mathrm{Ob}(\mathcal{C})$ we can define a covariant functor $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$ as follows:

- $X \in \mathrm{Ob}(\mathcal{C}) \mapsto \mathrm{Hom}(A, X)$
- Each morphism $f : X \to Y$ is mapped to the function

$$\begin{aligned} \mathrm{Hom}(A, f) : \mathrm{Hom}(A, X) &\to \mathrm{Hom}(A, Y) \\ g &\mapsto f \circ g \end{aligned}$$

  for each g in $\mathrm{Hom}(A, X)$.

A contravariant functor $\mathrm{Hom}(-, B) : \mathcal{C} \to \mathbf{Set}$ is defined analogously. These functors are known as *hom-functors*.

Remark. We have written above that the hom-functors map simply to **Set**, for conciseness. In fact, if the hom-classes in question have additional structure, the hom-functors can map to the category with those structures as objects. For example, if the hom-sets of A can be furnished with a group structure, it will be the case that $\mathrm{Hom}(A, -)$ will map from $\mathcal{C}$ to **Grp**.

3 Modules over associative algebras

Many examples of categorification are based on the theory of modules over associative algebras. We now give an overview of the topic, beginning with the definition of a *module*, which can be thought of intuitively as a generalization of the notion of a vector space.

3.1 Introduction to modules

Definition 3.1.1. Given a ring R with unity, not necessarily commutative, a *left R-module* is an abelian group $(M, +)$ together with a product $R \times M \to M$ (analogous to scalar multiplication for a vector space) satisfying the following axioms:

$\forall\, r_k \in R$ and $a_k \in M$:

1. $1_R a = a$
2. $(r_1 r_2)a = r_1(r_2 a)$
3. $(r_1 + r_2)a = r_1 a + r_2 a$
4. $r(a_1 + a_2) = ra_1 + ra_2$

Note. A *right R-module* is defined similarly, with scalar multiplication on the right, and the axioms changed accordingly.

Example 3.1.2. Any vector space over a field $\mathbb{F}$ is also both a left and right $\mathbb{F}$-module: because $\mathbb{F}$ is a commutative ring, there is no difference between left and right modules. In this case they are simply called $\mathbb{F}$-modules. (Or R-modules, for a general commutative ring R.)

Example 3.1.3. Let G be an abelian group. Let n be an integer, and let g be an element of G. Then G is a $\mathbb{Z}$-module with the action:

$$ng = \begin{cases} \underbrace{g + \cdots + g}_{\text{n times}} & n > 0 \\ 0 & n = 0 \end{cases}$$

In the case that $n < 0$, take ng to be the inverse element of $-ng$; that is to say, the inverse element of $\underbrace{g + \cdots + g}_{\text{n times}}$.

Example 3.1.4. For a given ring R with a left ideal I, I is a left R-module, where the scalar multiplication is simply the multiplication operation belonging to the ring. The same is true of right ideals and right R-modules.

Definition 3.1.5. Let M be a left or right R-module with a subgroup N. Then N is a *submodule* of M, if, $\forall n \in N$ and $r \in R$, the following holds:

$$\begin{cases} rn \in N & \text{when M is a left module.} \\ nr \in N & \text{when M is a right module.} \end{cases}$$

Definition 3.1.6. Let M_i be a family of left R-modules indexed by $i \in I$. Then their *direct sum*, denoted $\oplus_i M_i$, is comprised by all a_i with $a_i \in M_i$ and $i = 0$ for confinitely many i. Addition and scalar multiplication are defined component-wise: $(a_i) + (b_i) = (a_i + b_i), r(a_i) = (ra_i)$. The familiar notions of direct sums of vector spaces and abelian groups are special cases of this construction.

Definition 3.1.7. A module M is said to be *simple* if it contains no non-zero proper submodules.

Definition 3.1.8. A module M is said to be *indecomposable* if it is non-zero, and, in addition, cannot be written as a direct sum of two non-zero submodules.

Remark. All simple modules are indecomposable, but the converse is not true.

We now give a few definitions, leading up to the Jordan-Hölder theorem for modules, which we will make use of later.

Definition 3.1.9. An R-module M is said to be *Noetherian* if every sequence $M_1 \subset M_2 \subset M_3 \subset \cdots$ of submodules of M stabilizies; that is to say, there is some $n \in \mathbb{N}$ such that $M_n = M_{n+1} = M_{n+2} = \cdots$, and so on.
Analogously, M is said to be *Artinian* if every sequence of submodules of M of the form $\cdots \subset M_3 \subset M_2 \subset M_1$ eventually stabilizies.

Definition 3.1.10. An R-module M is said to be of *finite length* if it is both Artinian and Noetherian, or, equivalently, if there exists a series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of M, such that, for all $i = 0, \ldots, n$, the quotient module (constructed in the same way as quotient groups or quotient rings) M_i/M_{i-1} is a simple R-module. Series such as this are known as *composition series*, the natural number n is known as the *length* of the series, and the simple quotient submodules are known as the series' *quotient factors*.

Theorem 3.1.11 (Jordan-Hölder)**.** *Let M be a module of finite length and let*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$
$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset M_n = M$$

each be composition series for M. Then $m = n$ and the quotient factors are the same up to permutation.

Proof. See [6].

The following subsection introduces the notion of the *tensor product* of two modules, which, as will be seen, is crucial for categorification.

3.2 Tensor products

The following construction of the tensor product of two modules is due to Dauns in [7]. Before proceeding, we require a definition.

Definition 3.2.1. An R-module M is said to be *free* if it has a basis; that is, a set $B \subseteq M$ such that the following hold:

1. B generates M: every element in M can be written as a sum of elements of B with coefficients in R. (The multiplication can be on the left or right, depending on the type of R-module in question. We write multiplication on the left here, but the case for a right R-module is completely analogous.)

2. B is linearly independent: for $e_1, \dots, e_n \in B$ and $r_1, \dots, r_n \in R$, $r_1 e_1 + \cdots + r_n e_n = 0_M \implies r_1 = \cdots = r_n = 0_R$.

We now define some notation which will be useful for the construction of the tensor product.

Notation. Let R be a ring: whether or not it has identity is immaterial. Further, let A be a right R-module and let B be a left R-module. Now, let S denote the free $\mathbb{Z}$-module with generating set $A \times B$. We will make use of standard abbreviations and pointwise operations: $p = \{p_{(a,b)} | (a, b) \in A \times B\} = (p_{(a,b)}), q = \{q_{(a,b)} | (a, b) \in A \times B\} = (q_{(a,b)}); j \in \mathbb{Z}; jp = pj = (jp_{(a,b)}) \in S, p - q = (p_{(a,b)} - q_{(a,b)}) \in S$. We can visualize elements of S in the following ways:

- Firstly, every element in S can be written in a unique way as a finite sum

$$\sum_{i=1}^{n} p_i(a_i, b_i)$$

  with the conditions that $p_1, \dots, p_n \in \mathbb{Z}\backslash\{0\}$ and $(a_j, b_j) \neq (a_i, b_i) \in A \times B$ if $j \neq i$.

- Secondly, every element of S can be written as a finite sum of the form $\Sigma_i \varepsilon(i)(a_i, b_i)$ where $\varepsilon(i) = \pm 1$, and where we relax the second condition above: repetitions in $(a_i, b_i) \in A \times B$ are permitted.

- Lastly, we can write an element of S as a formal sum $p = \Sigma p_{(a,b)}(a, b)$ where we allow (a, b) to be any element of $A \times B$, but also stipulate that the number of integers $p_{(a,b)} \in \mathbb{Z}$ which are zero should be cofinite. In this representation, algebraic operations are easy. The addition of $p, q \in S$ and the multiplication of $p \in S$ by $j \in \mathbb{Z}$ are carried out as follows:

$$\Sigma p_{(a,b)}(a, b) + \Sigma q_{(a,b)}(a, b) = \Sigma(p_{(a,b)} + q_{(a,b)})(a, b)$$
$$j\Sigma p_{(a,b)}(a, b) = \Sigma j p_{(a,b)}(a, b)$$

Here, the empty sum is always $0_S \in S$. Since there are, in this case, four other zero elements ($0_A \in A, 0_B \in B, 0_R \in R, 0_{\mathbb{Z}} = 0 \in \mathbb{Z}$) all of which are simply written as 0 when there is no ambiguity, care must be taken. Define $(a, b) = 1(a, b) = (a, b)1 \in S$ for $(a, b) \in A \times B$. Note that $0_S \neq (0_A, 0_B) = 1(0_A, 0_B) \in S$.

Definition 3.2.2. Firstly, let $a, a' \in A, b, b' \in B$, and $r \in R$ all be arbitrary. Furthermore, let H be the additive subgroup of S generated by all $\mathbb{Z}$-linear combinations of all possible elements of types y_1, y_2, and y_3, defined by the following relations:

1. $(a + a', b) - (a, b) - (a', b) = y_1$
2. $(a, b + b') - (a, b) - (a, b') = y_2$
3. $(ar, b) - (a, rb) = y_3$

Then the *tensor product over* R of A and B, denoted $A \otimes_R B$, is defined to be the abelian quotient group S/H.

Note. If R is fixed and understood, the abbreviation $A \otimes B$ is used. For $(a, b) \in S$, define $a \otimes b = (a, b) + H = 1(a, b) + H \in A \otimes B$. Now, let $\rho : S \to S/H$ be the natural quotient map $\rho(s) = s + H$, and let π be the restriction $\pi = \rho \; |A \times B$ of ρ to the subset $A \times B \subset S$. This means that $\pi(a, b) = a \otimes b$.

This definition is both complicated and abstract. We will now verify a few properties of the quotient group S/H, with the goal of gaining some familiarity and comfortability with it. Notice that the calculations all rely on using the coset absorption properties of y_1, y_2, and y_3 to write expressions in a manner which is desirable for the proof.

Remark. Recall the action of $\mathbb{Z}$ from Example 3.1.3 which causes an arbitrary abelian group G to become a $\mathbb{Z}$-module. We will make use of it here, allowing it to act on the abelian group $A \otimes B$. Now, for the whole of the following example, let a and b be fixed elements of A and B respectively.

Example 3.2.3. (i) If $1 \notin R$, then $0_S = (a, b) - (a, b) \in H$ is not of the form y_3 as defined above.

(ii) $(0_A, 0_B) \in H$.

Proof.

$$\begin{aligned}&(0_A, 0_B) + H \\ &= (0_A, 0_B) + \underbrace{[(0_A + 0_A, 0_B) - (0_A, 0_B) - (0_A, 0_B)]}_{\text{Of the form } y_1} + H \\ &= 2(0_A, 0_B) + (-2)(0_A, 0_B) + H = 0(0_A, 0_B) + H = H.\end{aligned}$$

□

(iii) $(0_A, b), (a, 0_B) \in H$.

Proof.

$$\begin{aligned}&(0_A, b) + H \\ &= (0_A + 0_A, b) - \underbrace{[(0_A + 0_A, b) - (0_A, b) - (0_A, b)]}_{\text{Of the form } y_1} + H \\ &= 2(0_A, b) + H.\end{aligned}$$

From this it follows that $(0_A, b) + H = H$. For $(a, 0_B)$ the proof is similar, except that we use the y_2 relation. □

(iv) For an integer $n \geq 1, (na) \otimes b = a \otimes (nb) = n(a \otimes b)$.

Proof. We proceed by induction. For $n = 1$, the statement is trivially true, so we take it as the base case. For the inductive step, assume that

$$((n-1)a) \otimes b = a \otimes ((n-1)b) = (n-1)(a \otimes b)$$

Then

$$\begin{aligned}(na) \otimes b &= (na, b) + H \\ &= (na, b) - \underbrace{[a + (n-1)a, b) - (a, b) - ((n-1)a, b)]}_{\text{Of the form } y_1} + H \\ &= -[(-a, -b) - (n-1)(a, b)] + H \\ &= -[(-na, -nb)] + H \\ &= n(a, b) + H\end{aligned}$$

To complete the proof, it must be shown that $a \otimes (nb) = n(a \otimes b)$, but, as with (iii), we can proceed in an analogous way, using the y_2 relation instead of y_1. □

(v) $(-a) \otimes b = -(a \otimes b) = a \otimes (-b)$.

Proof. By (iii) and (iv) above:

$$\begin{aligned}0_{S/H} &= (0_A, b) + H \\ &= (a + (-a), b) = H \\ &= a \otimes b + (-a) \otimes b\end{aligned}$$

Hence, $(-a) \otimes b = -(a \otimes b)$. Analogously, $a \otimes (-b) = -(a \otimes b)$. □

(vi) For $n \geq 1, (-na) \otimes b = -n(a \otimes b) = a \otimes (-b)$.

Proof. Simply apply (iv), except with $-a$ in the place of a. Then apply (v):

$$\begin{aligned}&(-na) \otimes b \\ &= (n(-a)) \otimes b \\ &= n((-a) \otimes b) \\ &= -n(a \otimes b)\end{aligned}$$

Analogously, $a \otimes (-nb) = -n(a \otimes b)$. □

Note. So far, we have not used the y_3 relation.

Definition 3.2.4. For an additive abelian group C, a function $\varphi : A \times B \to C$ is said to be *R-bilinear* if, for all $a, a' \in A, b, b' \in B$ and $r \in R$ the following hold:

1. $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$
2. $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$
3. $\varphi(ar, b) = \varphi(a, rb)$

Recall from Definition 3.1.6 the notion of the direct sum of a family of modules. Let A and B be as they have been so far in this subsection. The notation and construction of the direct sum often cause vague ideas regarding connections between $A \oplus B$ and S, between $A \oplus B$ and bilinear maps, or between $A \oplus B$ and $A \otimes B$, to be formed. The following remarks are intended to remove this confusion and demonstrate that there is no deep connection.

Remarks.

1. The set $A \times B$ can be given the group structure of $A \oplus B$, and is certainly a subset of S, but it is not a subgroup of S: we have that $(a + a', 0_B) - (a, 0_B) - (a', 0_B) = 0_{A \oplus B}$, but this can never be equal to 0_S.

2. Consider $\varphi : A \times B \to C$, a bilinear map. Then there is no way to construct φ as a group homomorphism from $A \oplus B$ to C. To see this, consider the element $(a, b) + (a', b') \in A \oplus B$. Define a congruence between $(a, b) + (a', b') \in A \oplus B$ and $(a + a', b + b') \in A \times B$ (this is suggested by the identification of $A \times B$ with $A \oplus B$.) Then $\varphi[(a, b) + (a', b')] \cong \varphi[(a, b) + (a', b')] = \varphi(a, b) + \varphi(a, b') + \varphi(a', b) + \varphi(a', b') \neq \varphi(a, b) + \varphi(a', b')$ in general. ($\varphi(a, b') + \varphi(a', b) \neq 0$ in general.)

3. Let π be the restriction $\pi = \rho \mid A \times B$ as previously. It can be the case that $\pi(a, b) = a \otimes b = 0$ for $a \neq 0_A$ and $b \neq 0_B$.

4. Consider the subset $\pi(A \times B) \subset A \otimes B$. We will show that this does not inherit the group structrue from $A \oplus B$. If $A \otimes B \neq 0$, then $\exists\ 0 \neq a \otimes b \in A \otimes B$. Then, in $A \oplus B$, $(a, b) = (a, 0) + (0, b)$, but modulo H, $0 \neq a \otimes b = (a, b) + H \neq (a, 0) + (b, 0) + H = H$.

For concrete computations, we consider the general forms of the elements of the modules in question and use them to determine the general form of an element of the tensor product.

Proposition 3.2.5. *For an integer $n \geq 2$, consider $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$, the tensor product of the ideal generated by n with $\mathbb{Q}$. It can be shown that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n) = 0$.*

Proof. Let v be an element of $\mathbb{Q}$. Then, for some $q \in \mathbb{Q}, v = nq$. The general form of a member of $\mathbb{Z}/(n)$ is $(x + (n))$, where $n \nmid x \in \mathbb{Z}$. So we have $v \otimes (x + (n)) = qn \otimes (x + (n))$, which is equal to $q \otimes (nx + (n)) = q \otimes (n)$ by Example 3.2.3.(iii). But this is simply $q \otimes 0_{\mathbb{Z}/(n)} = 0$.

□

Proposition 3.2.6. *Let p and q be coprime. Then $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} \mathbb{Z}/(q) = 0$.*

Proof. The key fact here is that, due to the coprimality of p and q, $\exists s, t \in \mathbb{Z}$ such that $sp + tq = 1$. We have:

$$\begin{aligned}
&(x + (p)) \otimes (y + (q)) \\
&= xy(sp + tq + (p)) \otimes (1 + (q)) \\
&= xy(tq + (p)) \otimes (1 + (q)) \\
&= xy((t + (p))q) \otimes (1 + (q))
\end{aligned}$$

Once again, by 3.2.3(iii), this is equal to $xy(t + (p)) \otimes (q(1 + (q)) = xy(t + (p)) \otimes 0_{\mathbb{Z}/q} = 0$. □

We now proceed to *associative algebras*, which follow on conceptually from modules in a natural way, and can, in fact, be constructed from them.

3.3 Associative algebras

Note. The remainder of the paper is based on [4].

Definition 3.3.1. Let R be a commutative ring. An *associative R-algebra* is a ring B which is also an R-module such that the multiplication operation in the ring is *R-bilinear.* This means that, $\forall \alpha \in B$ and $\forall a, b \in B$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.
The attributes of B can vary as they can for rings without the additional structure: associative algebras can possess or lack a multiplicative identity, be commutative or non-commutative, and so on. We are primarily concerned with the case where R is a field.

Example 3.3.2. Given an R-module M, we can construct an associative R-algebra from it by furnishing it with an R-bilinear map from $M \times M$ to M such that, $\forall m_1, m_2, m_3 \in M$, $m_1(m_2m_3) = (m_1m_2)m_3$.

Example 3.3.3. Any ring can be viewed as a $\mathbb{Z}$-algebra. This corresponds to the fact that any abelian group can be viewed as a $\mathbb{Z}$-module.

Example 3.3.4. Any set of matrices with entries in a commutative ring R forms an R-algebra with matrix addition and multiplication.

Example 3.3.5. Let V be a vector space over a field $\mathbb{F}$. Consider the set of linear maps $\phi : V \to V$. (These are known as the *endomorphisms* of V.) For $\alpha \in \mathbb{F}$ and $\mathbf{v} \in V$, define $(\alpha\phi)(\mathbf{v}) = \alpha\phi(\mathbf{v})$. This construction, together with pointwise addition of functions as the ring addition and composition of functions as the ring multiplication, is an $\mathbb{F}$-algebra. It is denoted End V and is known as the *endomorphism algebra* of V.

Example 3.3.6. Just as the complex numbers are obtained from the real numbers by adjoining a new element i such that $i^2 = -1$, we can obtain an algebra D, which is called the *dual numbers*, by adjoining a new element x such that $x^2 = 0$. Note that $x \neq 0$. We can write D as $\mathbb{R}[x]/(x^2)$. It is a two-dimensional associative $\mathbb{R}$-algebra which is unital and commutative. This process works for a general field $\mathbb{F}$: as the notation for D suggests, the algebra of dual numbers over $\mathbb{F}$ is nothing more than the quotient of the polynomial ring by the ideal generated by x^2.

Example 3.3.7. Let Γ be a group. Then we can define the *group algebra* $\mathbb{F}[\Gamma]$, which is the $\mathbb{F}$-vector space with basis Γ, with multiplication defined by

$$(\alpha_1\gamma_1)(\alpha_2\gamma_2) = (\alpha_1\alpha_2)(\gamma_1\gamma_2),$$

for all $\alpha_1, \alpha_2 \in \mathbb{F}, \gamma_1, \gamma_2 \in \Gamma$.

Definition 3.3.8. Let B and C be two associative R-algebras. Then $\varphi : B \to C$ is an *algebra homomorphism* if the following conditions hold $\forall \alpha \in R, x, y \in B$:

- $\varphi(\alpha x) = \alpha\varphi(x)$
- $\varphi(x + y) = \varphi(x) + \varphi(y)$
- $\varphi(xy) = \varphi(x)\varphi(y)$

In other words, φ is a ring homomorphism which is also R-linear.

Note. In the case that B and C are unital, we also require that $\varphi(1_B) = 1_C$.

For the remainder of this section, let B be a unital associative $\mathbb{F}$-algebra.

Definition 3.3.9. A *representation* of B is a unital algebra homomorphism with domain B and codomain End V for some $\mathbb{F}$-vector space V.

Definition 3.3.10. A *left B-module* is a left B-module for the underlying ring of B. Let M be a left B-module. Then M is also a left $\mathbb{F}$-module, with the following action:

$$\alpha m = (\alpha 1_B)m, \text{ for all } \alpha \in \mathbb{F}, m \in M.$$

Right B-modules are defined analogously.
Let M be a simple B-module, containing an arbitrary nonzero element m. Then $Bm = \{bm \,|\, b \in B\}$ is a nontrivial submodule of M. Since we have defined M to be simple, it must be the case that $Bm = M$. Therefore, if M is a simple B-module, it is generated by any one of its nonzero elements. Consider the homomorphism $f : B \to M$, defined by $f(b) = bm$. Since $Bm = M$, f is surjective. Recall Example 2.1.4: by the First Isomorphism Theorem for modules, M and $B/\ker f$ are isomorphic as B-modules. Since $B/\ker f \cong M$ and M is simple, it must be the case that $\ker f$ is a maximal ideal of B. Therefore, because we imposed nothing other than the simplicity of M, *all* simple B-modules are isomorphic to quotients of B by its maximal ideals.

Example 3.3.11. (i) Up to isomorphism, the only simple $\mathbb{F}$-module is the one-dimensional vector space $\mathbb{F}$.
(ii) The only maximal ideal of D, as defined in Example 3.3.6, is the principal ideal generated by x. It follows that, up to isomorphism, the only simple D-module is $D/(x)$. This module is one dimensional: in it, x acts, multiplicatively, as 0 does. 1_D is the identity.

The concept of a representation of B and the concept of a left B-module are, in fact, equivalent. Hence, in an abuse of terminology, the two terms will sometimes be used interchangeably. Similarly, when "B-module" is written with no specification of whether the module in question has scalar multiplication on the left or right, "left B-module" is meant.

Definition 3.3.12. A left B-module M is *finitely generated* if it has a finite generating set $\{b_i \,| i \in I\}$.

In the following definition, the trivial module is denoted by 0.

Definition 3.3.13. A *short exact sequence* of modules is a sequence of B-module homomorphisms

$$0 \xrightarrow{\psi_0} M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3 \xrightarrow{\psi_3} 0$$

such that $\mathrm{Im}\psi_i = \mathrm{Ker}\psi_{i+1}$ for $i = 0, 1, 2$. Now, consider the following equivalent conditions:

1. There exists a B-module homomorphism $\varphi_1 : M_2 \to M_1$ such that $\varphi_1 \circ \psi_1 = id_{M_1}$
2. There exists a B-module homomorphism $\varphi_2 : M_3 \to M_2$ such that $\varphi_2 \circ \psi_2 = id_{M_3}$

If they hold, we say that the sequence is a *split exact sequence.*

Note. In the above case we have that $M_2 \cong M_1 \oplus M_3$.

Definition 3.3.14. Consider a short exact sequence of B-modules:

$$0 \to M \to N \to P \to 0$$

If every short exact sequence of this form is a split exact sequence, we say that P is a *projective B-module.* Also P is projective if and only if it is equal to $F \oplus M$ for some module M and free module F. In particular, all free modules are projective.

Definition 3.3.15. Let ***B*-mod** be the category whose objects are finitely generated B-modules and whose morphisms are B-module homomorphisms. Let ***B*-pmod** be the category with the same class of morphisms, but with finitely generated *projective* B-modules as objects. Note that ***B*-pmod** is a subcategory of ***B*-mod**.

Definition 3.3.16. Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of ***B*-mod**. Let 0 denote the trivial object, and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor such that $F(0) = 0$. Further, let $0 \to M \to N \to P \to 0$ be a short exact sequence of objects in $\mathrm{Ob}(\mathcal{C})$. We say that F is an *exact functor* if it maps short exact sequences to short exact sequences, that is to say, F is exact if $0 \to F(M) \to F(N) \to F(P) \to 0$ is exact in $\mathcal{D}$.

Remark. The definition of an exact functor is valid for any category which has the property of being *abelian*; ***B*-mod** and ***B*-pmod** are specific examples. Since our approach to categorification is based on B-modules, we have, for conciseness, defined exact functors only in relation to these specific categories. For a proper treatment of abelian categories, see [3].

Definition 3.3.17. Let N be a submodule of a B-module M. Suppose that, for any other submodule H of M, $N + H = M \implies H = M$. Then N is said to be *superfluous.* A *superfluous epimorphism* of B-modules $M \to N$ whose kernel is superfluous in M.

Definition 3.3.18. Let M be a B-module. A *projective cover* of M is a projective module P, together with a superfluous epimorphism $P \to M$.

Example 3.3.19. Any projective module is its own projective cover, for example, the projective cover of $\mathbb{F}$ considered as a one-dimensional vector space over itself is $\mathbb{F}$.

Example 3.3.20. Recall D, the algebra of dual numbers. The projective cover of the simple D-module $D/(x)$ is D itself.

Note. Let M be an arbitrary B-module. Then its projective cover and associated superfluous epimorphism, if they exist, are unique up to isomorphism. However, in general, it is not necessary that they should exist.

Definition 3.3.21. Let A and B be unital associative algebras, let M be an A-module, and let N be a B-module. Then we can define an $(A \otimes_{\mathbb{F}} B)$-module, denoted $M \otimes_{\mathbb{F}} N$, by the following action:

$$(a \otimes b)(m \otimes n) = (am) \otimes (bn),$$

for all $a \in A, b \in B, m \in M$, and $n \in N$.

This construction is known as the *external tensor product* of M and N.

3.4 Finite-dimensional algebras

We are most concerned here with associative $\mathbb{F}$-algebras which are finite-dimensional as $\mathbb{F}$-vector spaces and which are unital. We will now state some of the properties of such algebras. For this subsection, we assume that B is an associative $\mathbb{F}$-algebra which is finite dimensional and unital. We assume also that all modules are finitely generated.

Proposition 3.4.1.

1. *Every left or right B-module has a projective cover. ([8], Theorem I.4.2)*
2. *The algebra B has a finite number of nonisomorphic simple modules. ([8], Proposition I.3.1)*
3. *The projective covers of the nonisomorphic simple modules form a complete list of nonisomorphic indecomposable projective B-modules. ([8], Corollary I.4.5)*

Note. Proofs of all three statements in the proposition above are given in [8].

Lemma 3.4.2. *Suppose V is a simple B-module with projective cover P. Then, for any simple B-module W, we have an isomorphism of $\mathbb{F}$-modules*

$$Hom_B(P, W) \cong Hom_B(V, W) = \begin{cases} 0 & \text{if } W \ncong V \\ End_B(V) & \text{if } W \cong V \end{cases}$$

Example 3.4.3. As seen in Examples 3.3.12 and 3.3.20 , the algebra D has one simple module $D/(x)$. Its projective cover D is, up to isomorphism, the only indecomposable projective D-module.

Definition 3.4.4. An associative algbra B is called *simple* if it has no nontrivial proper two-sided ideals and the set $B^2 = \{ab \,|\, a, b \in B\} \neq \{0\}$.

Note. If B is a nonzero unital associative algebra, the second condition is automatically satisfied.

Example 3.4.5. The algebra of square matrices with entries in $\mathbb{F}$ is a simple $\mathbb{F}$-algebra.

Definition 3.4.6. A finite dimensional unital associtive algebra is *semisimple* if it is isomorphic to a Cartesian product of simple subalgebras. A module over an associative algebra is semisimple if it is isomorphic to a direct sum of simple submodules.

Proposition 3.4.7. *Let B be a semisimple finite dimensional unital associative algebra. Then all B-modules are semisimple and projective. In particular, every B-module is its own projective cover.*

4 Weak categorification

We are now almost equipped with the necessary concepts to begin an explanation of weak categorification; we have presented the required basic category theory, as well as the required theory of modules over associative algebras. All that is left to deal with is the notion of the *Grothendieck group* of a category. For the first part of this culminating section, we present an overview of it.

4.1 Grothendieck groups

Once again, we fix a unital associative $\mathbb{F}$-algebra B. Recall the categories ***B*-mod** and ***B*-pmod**, and let $\mathcal{C}$ be a subcategory of ***B*-mod**. We are primarily interested in the case where $\mathcal{C}$ is either ***B*-mod** or ***B*-pmod**.

Definition 4.1.1. Let $\mathrm{F}(\mathcal{C})$ be the free abelian group (analogously to Definition 3.2.1, an abelian group with a basis) whose basis is the isomorphism classes of objects $M \in \mathrm{Ob}(\mathcal{C})$, which we denote by $[M]$. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a split exact sequence in $\mathcal{C}$, and, in addition, let $N^{split}(\mathcal{C})$ be the subgroup of $\mathrm{F}(\mathcal{C})$ generated by elements of the form $[M_1] - [M_2] + [M_3]$. The *split Grothendieck group* of $\mathcal{C}$, denoted $\mathcal{K}_0^{split}(\mathcal{C})$, is the quotient group $\mathrm{F}(\mathcal{C})/N^{split}(\mathcal{C})$. We will usually denote the image of $[M]$ in $\mathcal{K}_0^{split}(\mathcal{C})$ again by $[M]$.

Remark. Equivalently to the above, $N^{split}(\mathcal{C})$ is also generated by elements of the form $[M_3] - [M_1] - [M_2]$ for every $M_1, M_2, M_3 \in \mathrm{Ob}(\mathcal{C})$ with $M_3 = M_1 \oplus M_2$.)

Proposition 4.1.2. *Let $\mathbb{F}$-**mod** be the category whose objects are all finite dimensional $\mathbb{F}$-vector spaces and whose morphisms are the $\mathbb{F}$-linear maps between the spaces. Then $\mathcal{K}_0^{split}(\mathbb{F}\textbf{-mod}) \cong \mathbb{Z}$.*

Proof. Consider the surjective homomorphism

$$\begin{aligned} f : F(\mathbb{F}\textbf{-mod}) &\to \mathbb{Z} \\ [V] &\mapsto \dim(V) \\ -[V] &\mapsto -\dim(V) \end{aligned}$$

Since $\dim(V \oplus W) = \dim(V) + \dim(W)$, we have that $N^{split}(\mathbb{F}\textbf{-mod}) \subseteq \ker(f)$. Now, let $\sum_{i=1}^{n} c_i[V_i]$ be an arbitrary element of $\ker(f)$. We have $\sum_{i=1}^{n} c_i \dim(V_i) = f(\sum_{i=1}^{n} c_i[V_i]) = 0$. In $\mathcal{K}_0^{split}(\mathbb{F}\textbf{-mod})$,

since $[V_i] = \dim(V_i)[\mathbb{F}]$, we have $\sum_{i=1}^{n} c_i[V_i] = (\sum_{i=1}^{n} c_i \dim(V_i))[\mathbb{F}] = 0$, so $\sum_{i=1}^{n} c_i[V_i] \in N^{split}(\mathbb{F}\textbf{-mod})$, but, because $\sum_{i=1}^{n} c_i[V_i]$ was an arbitrary element of $\ker(f)$, $\ker(f) \in N^{split}(\mathbb{F}\textbf{-mod})$. So $\ker(f) = N^{split}(\mathbb{F}\textbf{-mod})$.
Therefore, by the First Isomorphism Theorem,

$$\mathcal{K}_0^{split}(\mathbb{F}\textbf{-mod}) \cong F(\mathbb{F}\textbf{-mod})/\ker(f) \cong \mathbb{Z}$$

□

Definition 4.1.3. Let $F(\mathcal{C})$ be the free abelian group with basis the isomorphism classses $[M]$ of objects M in $\mathcal{C}$. Let $N(\mathcal{C})$ be the subgroup of $F(\mathcal{C})$ generated by the elements $[M_1] - [M_2] + [M_3]$ for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{C}$. The *Grothendieck group* of $\mathcal{C}$, denoted $\mathcal{K}_0(\mathcal{C})$, is the quotient group $F(\mathcal{C})/N(\mathcal{C})$. Customarily, we will denote the image of $[M]$ in $\mathcal{K}_0(\mathcal{C})$ again by $[M]$.

Remark. Analogously to the notion of an exact functor, as discussed in 3.3.17, it is possible to define $\mathcal{K}_0(\mathcal{C})$ for any abelian category $\mathcal{C}$; but, once again, it is not necessary to deal with this in depth.

Example 4.1.4. Note that $\mathbb{F}$**-mod** is projective: every short exact sequence in it splits. Therefore, $\mathcal{K}_0^{split}(\mathbb{F}\textbf{-mod}) = \mathcal{K}_0(\mathbb{F}\textbf{-mod}) \cong \mathbb{Z}$.

Notation. From this point onward, let $\mathcal{K}_0(\boldsymbol{B}\textbf{-mod.}) = G_0(B)$ and $\mathcal{K}_0(\boldsymbol{B}\textbf{-pmod}) = K_0(B)$.

Lemma 4.1.5. *If B is semisimple, all short exact sequences in **B-mod** split. Therefore, all modules are projective. Hence, **B-mod** = **B-pmod** and $G_0(B) = K_0(B)$.*

Proof. This is a direct consequence of Proposition 3.4.7. □

For the rest of this section, we assume that B is finite dimensional. Let $V_1, \dots, V_s$ be a complete list of nonisomorphic simple B-modules. By Proposition 3.4.1, if P_i is the projective cover of V_i for $i = 1, \dots, s$, then $P_1, \dots, P_s$ is a complete list of nonisomorphic indecomposable projective B-modules. Recall Theorem 3.1.11 (the Jordan-Hölder theorem), from which it follows that

$$G_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[V_i].$$

The class $[M] \in G_0(B)$ of any $M \in \boldsymbol{B}\textbf{-mod}$ is the sum (with multiplicity) of the classes of the simple modules appearing in any composition series of M. Moreover, since any $P \in \boldsymbol{B}\textbf{-pmod}$ can be written uniquely as a sum of indecomposable projective modules, we also have

$$K_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[P_i].$$

Example 4.1.6. Recall Example 3.4.3. Due to it, for the algebra D of dual numbers, we have that $G_0(D) = \mathbb{Z}[D/(x)]$ and that $K_0(D) = \mathbb{Z}[D]$.

We can now define a natural bilinear form as follows:

$$\langle -, - \rangle : K_0(B) \otimes_{\mathbb{Z}} G_0(B) \to \mathbb{Z}$$

Given by

$$\langle [P], [M] \rangle = \dim_{\mathbb{F}} \mathrm{Hom}_B(P, M), \tag{4.1.1}$$

for all $P \in \boldsymbol{B}\textbf{-pmod}$, $M \in \boldsymbol{B}\textbf{-mod}$.
Here, $\mathrm{Hom}_B(P, M)$ denotes the F-vector space of all B-module homomorphisms from P to M (the

operation is function composition). Note that it is crucial that P is projective here; there is no analogous bilinear form mapping from $G_0(B) \otimes G_0(B)$ to $\mathbb{Z}$ except in the case that B is semisimple, which implies that $G_0(B) = K_0(B)$.
By Lemma 3.4.2, we have

$$\langle [P_i], [V_j] \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \dim_{\mathbb{F}} \mathrm{End}_B(V_i) \geq 1 & \text{if } i = j \end{cases}$$

Therefore, the form $\langle -, - \rangle$ is nondegenerate. For a commutative ring R and R-module V, let $V^\vee$ denote the dual space, the set of all linear maps from V to R. A bilinear form $\langle -, - \rangle : V \otimes_R W \to R$ of R-modules induces maps

$$V \to W^\vee,\ v \mapsto (w \mapsto \langle v, w \rangle)$$
$$W \to V^\vee,\ w \mapsto (w \mapsto \langle v, w \rangle)$$

If the form is nondegenerate, these maps are injective. If these maps are isomorphisms, then the form is known as a *perfect pairing.*

Example 4.1.7. Consider once again the algebra D of dual numbers. Let $f : D \to D/(x)$ be a homomorphism of D-modules. Since the codomain of f is a simple D-module, the kernel of f must be a maximal ideal of D. But it is known that the only maximal ideal of D is (x). So, f factors through a map $D/(x) \to D/(x)$. But $D/(x)$ is a one dimensional $\mathbb{F}$-vector space, so any such map is simply multiplication by some scalar in $\mathbb{F}$. So $\mathrm{Hom}_D(D, D/(x)) \cong \mathbb{F}$. $\langle D, D/(x) \rangle = 1$.

4.2 Some further properties of functors

We are now almost in a position to discuss categorification proper; we require only the definitions of two specific functors which were not accesible to us without the theory of modules over associative algebras. Now that the required theory has been given, we can begin to define the functors in question.

Definition 4.2.1. Let B_1 and B_2 both be unital associative $\mathbb{F}$-algebras, let $\mathcal{C}_1$ be a subcategory of the category of B_1-modules, and let $\mathcal{C}_2$ be a subcategory of the category of B_2-modules. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be *additive* if $F(M \oplus N) = F(M) \oplus F(N)$ for all $M, N \in \mathcal{C}_1$.

Now, if F is additive, then it induces a group homomorphism

$$[F] : \mathcal{K}_0^{split}(\mathcal{C}_1) \to \mathcal{K}_0^{split}(\mathcal{C}_2)$$
$$[F]([M]) \mapsto [F(M)]$$

for all $M \in \mathcal{C}_1$.

Analogously, if F is exact, then it induces a group homomorphism

$$[F] : \mathcal{K}_0(\mathcal{C}_1) \to \mathcal{K}_0(\mathcal{C}_2)$$
$$[F]([M]) \mapsto [F(M)]$$

for all $M \in \mathcal{C}_1$.

Definition 4.2.2. Let A and B be two finite-dimensional unital associative $\mathbb{F}$-algebras. Now, let M be a left A-module and a right B-module, where the A and B actions commute. Then M is known as an *(A, B)-bimodule.*

Example 4.2.3. A itself is an (A, A)-bimodule with left and right multiplication. More generally, if B is a subalgebra of A, we can consider A as an $(A, B), (B, A)$, or (B, B)-bimodule.

If B and C are subalgebras of A, we let ${}_BA_C$ denote A considered as a (B, C)-bimodule. If B or C is equal to A, the corresponding subscript is omitted. For example, A_B denotes A considered as an (A, B)-bimodule.
Suppose M is an (A, B)-bimodule. Then we have the functor

$$M \otimes_N - : \boldsymbol{B}\textbf{-mod} \to \boldsymbol{A}\textbf{-mod}, \quad N \mapsto M \otimes_B N.$$

Here we consider $M \otimes_B N$ as an A-module via the action

$$a(m \otimes n) = (am) \otimes n$$

Definition 4.2.4. Now, in the above scenario, suppose that B is a subalgebra of A. Then we have *induction* and *restriction* functors

$$\begin{aligned} \mathrm{Ind}_B^A &: \boldsymbol{B}\textbf{-mod} \to \boldsymbol{A}\textbf{-mod}, \quad N \mapsto A \otimes_B N \\ \mathrm{Res}_B^A &: \boldsymbol{A}\textbf{-mod} \to \boldsymbol{B}\textbf{-mod} \quad M \mapsto {}_BA \otimes_A M \end{aligned}$$

for all $M \in \boldsymbol{A}\textbf{-mod}$ and $N \in \boldsymbol{B}\textbf{-mod}$. If A is projective as a left and right B-module, both of these functors are exact and, in addition, induce functors on the corresponding categories of finitely-generated projective modules.

4.3 Weak categorification

We can now describe the process of weak categorification. In what follows, let R be a commutative ring. Let B be an R-algebra which is unital and associative, and let $\{b_i\}_{i \in I}$ be a fixed generating set for B. If M is a B-module, then the action of each b_i in the generating set defines an R-linear endomorphism $b_i^M(x) = b_i x$ of M.

Definition 4.3.1. A *naive categorification* of $(B, \{b_i\}_{i \in I}, M)$ is a tuple $(\mathcal{M}, \varphi, \{F_i\}_{i \in I})$, where $\mathcal{M}$ is an abelian category, $\varphi : \mathcal{K}_0 \otimes_{\mathbb{Z}} R \to M$ is an isomorphism, and, for each $i \in I, F_i : \mathcal{M} \to \mathcal{M}$ is an exact endofunctor of $\mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_0(\mathcal{M}) \otimes_{\mathbb{Z}} R & \xrightarrow{[F_i]} & \mathcal{K}_0(\mathcal{M}) \otimes_{\mathbb{Z}} R \\ \varphi \downarrow & & \downarrow \varphi \\ M & \xrightarrow{b_i^M} & M \end{array}$$

Note. We say that a diagram such as this is *commutative* or *commutes* if any two paths with the same starting and ending points, where we define the traverse of a path to be the composition of the functions that comprise it, are equal.

Above, we have written $[F_i]$ in place of $[F_i] \otimes \mathrm{id}$. To say that this diagram commutes is, intuitively speaking, to say that the action of the functor $[F_i]$ lifts the action of b_i to the realm of categories.
This is indeed a form of categorification, but it is an extremely weak notion; it is often the case that we want the functors $[F_i]$ to preserve something more than just the action. For example, we could categorify the relations *between* the generators b_i or the induced maps b_i^M. This motivates the following definition.

Definition 4.3.2. Given a set of relations of B which generate all of the relations in B, a set of isomorphisms of functors which lift these relations to the Grothendieck group, along with a naive categorification of $(B, \{b_i\}_{i\in I}, M)$, is called a *weak categorification.*

Remark. Note that these definitions depend on a generating set for the algebra B. Because of this, it is more accurate to speak of the categorification of the *presentation* of a module. We will, in a slight abuse of terminology, proceed using the language of the definitions.

Example 4.3.3. Let $B = \mathbb{C}[b]/(b^2 - 2b)$, with the generating set $\{b\}$. Obviously, the relation satisifed by this generator is $b^2 = 2b$. Consider $\mathbb{C}$ as the B-module M with action given by $b \cdot 1 = 0$, and also as the B-module N with action given by $b \cdot 1 = 2$. Let $\mathcal{M}$ be $\mathbb{C}$-mod, the category of finite-dimensional $\mathbb{C}$-modules, and define the functors $F, G : \mathcal{M} \to \mathcal{M}$ by $F(V) = 0$ and $G(V) = V \oplus V$ for all $V \in \mathcal{M}$. Define $\varphi : \mathcal{K}_0 \otimes_{\mathbb{Z}} \mathbb{C} \to M$ and $\psi : \mathcal{K}_0 \otimes_{\mathbb{Z}} \mathbb{C} \to N$ both by $z[\mathbb{C}] \mapsto z$ (here, $[\mathbb{C}]$ denotes the isomorphism class of the simple one-dimensional $\mathbb{C}$-module).
Now, we have that

$$\varphi \circ [F](z[\mathbb{C}]) = 0 = b \cdot \varphi(z[\mathbb{C}])$$

for all $z \in \mathbb{C}$, so $(\mathcal{M}, \varphi, F)$ is a naive categorification of $(B, \{b\}, M)$. It is also the case that

$$\psi \circ [G](z[\mathbb{C}]) = \psi(z[G(\mathbb{C})]) = \psi(z[\mathbb{C} \oplus \mathbb{C}]) = \psi(2z[\mathbb{C}]) = 2z = b \cdot z = b \cdot \psi(z[\mathbb{C}]),$$

so $(\mathcal{M}, \psi, G)$ is a naive categorification of $(B, \{b\}, N)$. For the weak categorification, we begin by noting the following:

$$[F \circ F](z[\mathbb{C}]) = F(0) = 0 = [F](z[\mathbb{C}]) \oplus [F](z[\mathbb{C}])$$
$$[G \circ G](z[\mathbb{C}]) = [G](z[\mathbb{C} \oplus \mathbb{C}]) = z[(\mathbb{C} \oplus \mathbb{C}) \oplus (\mathbb{C} \oplus \mathbb{C})] = [G](z[\mathbb{C}]) \oplus [G](z[\mathbb{C}])$$

These equations are true for all $z \in \mathbb{C}$. This means that we have isomorphisms of functors $F \circ F \cong F \oplus F$ and $G \circ G \cong G \oplus G$. Therefore, considered as operators on $\mathcal{K}_0(\mathcal{M})$ we have $[F]^2 = 2[F]$ and $[G]^2 = 2[G]$, so the isomorphisms lift the generator relation $b^2 = 2b$. This means that $(\mathcal{M}, \varphi, F)$ and $(\mathcal{M}, \psi, G)$ are weak categorifications of $(B, \{b\}, M)$ and $(B, \{b\}, N)$, respectively.

4.4 Categorification of the polynomial representation of the Weyl algebra

The example immediately previous is very simple. We will now proceed to a more sophisticated example: the categorification of the polynomial representation of the Weyl algebra, which utilizes categories of modules over nilcoxeter algebras. We begin by defining the Weyl algebra. For the whole of this subsection, as before, we fix an arbitrary field $\mathbb{F}$ and assume all modules are finitely generated.

Definition 4.4.1. The *Weyl algebra*, denoted W, is the unital associative algebra over $\mathbb{Z}$ with generators x, ∂ and defining relation $\partial x = x\partial + 1$.

Now, let $R_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector space spanned by $x^0, x^1, x^2, \ldots$ There is a natural and familiar action of W on $R_{\mathbb{Q}}$ defined as follows:

$$x \cdot x^n = x^{n+1}, \ \ \partial \cdot x^n = nx^{n-1}$$

for all $n = 0, 1, 2, \ldots$ The polynomial representation of the Weyl algebra is this representation of W on $R_{\mathbb{Q}}$.

Proposition 4.4.2. *The abelian subgroups*

$$R = Span_{\mathbb{Z}}\{x^n/n!\}_{n=0}^{\infty} \ \text{and} \ R' = Span_{\mathbb{Z}}\{x^n\}_{n=0}^{\infty}$$

of $R_{\mathbb{Q}}$ are also submodules of $R_{\mathbb{Q}}$ considered as a W-module.

Proof. Firstly, we must show that $xR \subseteq R$ and $\partial R \subseteq R$. The action of W on R is linear with respect to $\mathbb{Z}$, so we only need to show the following, which are true for all $n \in \mathbb{N}$:

$$x\frac{x^n}{n!} = \frac{x^{n+1}}{n!} = (n+1)\frac{x^{n+1}}{(n+1)!} \in R$$
$$\partial\frac{x^n}{n!} = \frac{x^{n-1}}{(n-1)!} \in R$$

so R is a W-submodule of $R_{\mathbb{Q}}$. Similarly, the action of W on R' is linear, so we simply show that $xR' \subseteq R'$ and $\partial R' \subseteq R$:

$$xx^n = x^{n+1} \in R'$$
$$\partial x^n = nx^{n-1} \in R'$$

So both R and R' are W-submodules of $R_{\mathbb{Q}}$. □

Now, we define a bilinear form $\langle -, - \rangle : R_{\mathbb{Q}} \times R_{\mathbb{Q}} \to \mathbb{Q}$ by

$$\langle x^n, x^m \rangle = \delta_{m,n} n!$$

This form restricts to a perfect pairing $\langle -, - \rangle : R \times R' \to \mathbb{Z}$.
We will now proceed to the nilcoxeter algebra and its modules. We begin by recalling some basic facts about the symmetric group. $\mathbb{F}[S_n]$, the group algebra of the symmetric group on n symbols is generated by the simple transpositions $s_i = (i, i+1)$. The relations for these generators are the same as the relations for the transpositions in the symmetric group, and are as follows:

$$s_i^2 = 1 \text{ for } i = 1, 2, \dots, n-1,$$
$$s_i s_j = s_j s_i \text{ for } i, j = 1, \dots n-1 \text{ such that } |i-j| > 1,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i = 1, 2, \dots, n-2$$

The last two relations are called the *braid relations.* Now, any element $\sigma \in S_n$ can be written as a product

$$\sigma = s_{i_1} \cdots s_{i_k}.$$

If k is minimal, we call it the *length* of σ, and denote this value $\ell(\sigma)$. Any expression where $k = \ell(\sigma)$ is called a *reduced expression* for σ. Any reduced expression for σ can be obtained from any other reduced expression for σ by a sequence of braid relations. If the product in question is not a reduced expression, braid relations can be used to replace it with an expression in which two copies of s_j are directly next to each other. The relation $s_j^2 = 1$ can then be used to reduce the number of transpositions by two. In this way a reduced expression can be obtained from any expression.

Definition 4.4.3. Let n be a non-negative integer. Then the *nilcoxeter algebra* N_n is the unital $\mathbb{F}$-algebra generated by $u_1, \dots, u_{n-1}$ with defining relations

$$u_i^2 = 0 \text{ for } i = 1, 2, \dots, n-1,$$
$$u_i u_j = u_j u_i \text{ for } i, j = 1, \dots n-1 \text{ such that } |i-j| > 1,$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \text{ for } i = 1, 2, \dots, n-2$$

As a convention, we set $N_0 = N_1 = \mathbb{F}$.

Note. The only difference between $\mathbb{F}[S_n]$ and the nilcoxeter algebra is that the generators square to 0.

The unique maximal length element of S_n has length $n(n-1)/2$, which implies, for the Nilcoxeter algebra, that if $k > n(n-1)/2$, then $u_{i_1} u_{i_2} \cdots u_{i_k} = 0$ for all $i_1, i_2, \ldots, i_k \in \{1, \ldots, n-1\}$.

Definition 4.4.4. A *graded algebra* is an algebra which is also a direct sum of abelian groups M_i, such that $M_i M_j \subset M_{i+j}$. The decomposition of the sums is referred to as *grading*. An element of a member of the decomposition, say M_i, is referred to as *homogenous* of degree i.

Now, because the relations defining N_n are homogenous in the u_i, we can define a grading $N_n = \bigoplus_{m \in \mathbb{N}} N_n^{(m)}$ on N_n by setting the degree of u_i to be one for $i = 1, \ldots, n-1$. Let $I = \bigoplus_{m \geq 1} N_n^{(m)}$ be the sum of the positively graded pieces of N_n; that is to say, I is the ideal of N_n generated by the u_i for $i = 1, \ldots, n-1$. Since N_n/I is a one-dimensional N_n-module (spanned by the image of the unit of N_n), I is a maximal ideal of N_n and hence simple. It follows from the above result that $I^k = 0$ for $k > n(n-1)/2$.

Proposition 4.4.5. *The nilcoxeter algebra N_n has a unique simple module, denoted L_n. This is the one-dimensional module on which all u_i such that $i = 1, \ldots, n-1$ act by zero. The projective cover of L_n is N_n.*

Proof. Let V be a simple N_n-module. Then IV is a submodule of V. So it must be the case that $IV = 0$ or $IV = V$. if $IV = V$, for $k > n(n-1)/2$, we have

$$V = IV = I^2 V = \cdots = I^k V = 0$$

However, since V is simple, it must be nonzero. So we have $IV = 0$ and $V = L_n$.
Now, since N_n is a free, and hence projective, N_n-module, to show that it is the projective cover of L_n, it is enough to show the kernel I of the canonical map $N_n \to N_n/I$ is a superfluous submodule of N_n. If $I + H = N_n$ for some submodule (or, equivalently, ideal) H of N_n, then H contains an element of the form $1 - a$ with $a \in I$. This element is invertible; the inverse is given by $1 + a + a^2 + \cdots + a^k$, where k is some integer greater than $n(n-1)/2$. Hence, $H = N_n$. So, as desired, I is superfluous. □

Recall from Section 4.1 the results stemming from the Jordan-Hölder Theorem for modules; from these, and Proposition 4.4.5, we have

$$G_0(N_n) = \mathbb{Z}[L_n], \quad K_0(N_n) = \mathbb{Z}[L_n].$$

Now, let

$$\mathcal{G}_N = \mathcal{K}_0\left(\bigoplus_{n=0}^{\infty} \boldsymbol{N_n}\textbf{-mod}\right) = \bigoplus_{n=0}^{\infty} G_0(N_n) = \bigoplus_{n=0}^{\infty} \mathbb{Z}[L_n], \text{ and}$$

$$\mathcal{K}_N = \mathcal{K}_0\left(\bigoplus_{n=0}^{\infty} \boldsymbol{N_n}\textbf{-pmod}\right) = \bigoplus_{n=0}^{\infty} K_0(N_n) = \bigoplus_{n=0}^{\infty} \mathbb{Z}[N_n].$$

We define a bilinear form $\langle -, - \rangle : \mathcal{G}_N \otimes_{\mathbb{Z}} \mathcal{K}_N \to \mathbb{Z}$ by setting $\langle G_0(N_n), K_0(N_m) \rangle = 0$ for $n \neq m$ and using the form (4.1.1) when $n = m$.
Define isomorphisms of $\mathbb{Z}$-modules

$$\varphi_{\mathcal{G}_N} : \mathcal{G}_N \to R = \text{Span}_{\mathbb{Z}}\{x^n/n!\}_{n=0}^{\infty}, \ [L_n] \mapsto x^n/n!$$

$$\varphi_{\mathcal{K}_N} : \mathcal{K}_N \to R' = \text{Span}_{\mathbb{Z}}\{x^n\}_{n=0}^{\infty}, \ [N_n] \mapsto x^n$$

Since

$$\langle x^m, x^n/n! \rangle = \delta_{m,n} = \langle [N_m], [L_n] \rangle,$$

we see that the above maps respect the bilinear forms on the involved spaces (the form on the left hand side is the $\mathbb{Q}$-valued bilinear form discussed above.)
Thus we have

$$\langle a, b\rangle = \langle \varphi_{\mathcal{K}_N}(a), \varphi_{\mathcal{G}_N}(b)\rangle, \quad \text{for all } a \in \mathcal{K}_N, b \in \mathcal{G}_N.$$

We now proceed to categorify the action of the Weyl algebra on the modules R and R'.

We can view the nilcoxeter algebra N_n as the subalgebra of N_{n+1} generated by $u_1, \ldots, u_{n-1}$. For each $n \in \mathbb{N}$, define

$$X_n = (N_{n+1})_{N_n}, \;\; D_n = {}_{N_n}(N_{n+1}).$$

That is to say, X_n is N_{n+1} considered as an (N_{n+1}, N_n)-bimodule, and D_n is N_{n+1} considered as an (N_n, N_{n+1})-bimodule. Thus

$$(X_n \otimes_{N_n} -) = \mathrm{Ind}_{N_n}^{N_{n+1}} : \boldsymbol{N_n}\textbf{-mod} \to \boldsymbol{N_{n+1}}\textbf{-mod}$$

$$(D_n \otimes_{N_{n+1}} -) = \mathrm{Res}_{N_n}^{N_{n+1}} : \boldsymbol{N_{n+1}}\textbf{-mod} \to \boldsymbol{N_n}\textbf{-mod}$$

Now, for $\sigma \in S_n$, let $\sigma = s_{i_1} \cdots s_{i_k}$ be a reduced expression and define $u_\sigma = u_{i_1} \cdots u_{i_k}$. It follows from the earlier-discussed facts about symmetric groups that u_σ is independent of the reduced expression of σ. It follows also that $\{u_\sigma\}_{\sigma \in S_n}$ is a basis for N_n and that the multiplication in this basis is defined thus

$$u_\sigma u_\tau = \begin{cases} u_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau), \\ 0 & \text{otherwise} \end{cases}$$

We require that the functors defined above are exact, and that they map projective modules to projective modules. As was mentioned in Section 4.2, the following lemma is required.

Lemma 4.4.6. *The bimodules X_n and D_n are projective as both left and right modules for all $n \in \mathbb{N}$.*

Proof. Recall that all free modules are projective. As a left N_{n+1}-module X_n is free, generated by one element, and hence, projective. We now must show that it is projective as a right module. Let $\sigma \in S_{n+1}$ and let $i = \sigma(n+1)$. Then σ can be written uniquely in the form

$$\sigma = s_i s_{i+1} \cdots s_n \sigma', \;\; \sigma' \in S_n$$

($\sigma' = s_n \cdots s_{i+1} s_i \sigma$). Therefore, X_n is a free right N_n-module with basis

$$1, u_n, u_{n-1}u_n, \ldots, u_1 u_2 \cdots u_n.$$

and is hence projective as a right module. By analogous arguments, it can be shown that D_n is left and right projective. □

Corollary 4.4.7. *The functors $X_n \otimes_{N_n} -$ and $D_n \otimes_{N_{n+1}} -$ are exact and induce functors*

$$(X_n \otimes_{N_n} -) = \mathit{Ind}_{N_n}^{N_{n+1}} : \boldsymbol{N_n}\textbf{-}\boldsymbol{pmod} \to \boldsymbol{N_{n+1}}\textbf{-}\boldsymbol{pmod}$$

$$(D_n \otimes_{N_{n+1}} -) = \mathit{Res}_{N_n}^{N_{n+1}} : \boldsymbol{N_{n+1}}\textbf{-}\boldsymbol{pmod} \to \boldsymbol{N_n}\textbf{-}\boldsymbol{pmod}$$

Let N be the direct sum $\bigoplus_{n=0}^{\infty} N_n$. Then N is an associative algebra. However, it is no longer unital. It instead has an infinite family $1_{N_n}, n \in \mathbb{N}$, of pairwise orthogonal idempotents. Let M be an N_n-module. Then M is naturally an N-module when we set $aM = 0$ for all $a \in N_m$ with $m \neq n$. Similarly, any (N_n, N_m)-bimodule can be viewed as an (N, N)-bimodule. Define the (N, N)-bimodules

$$X = \bigoplus_{n=0}^{\infty} X_n \;\text{ and }\; D = \bigoplus_{n=0}^{\infty} D_n.$$

Also let

$$\mathcal{N} = \bigoplus_{n=0}^{\infty} \boldsymbol{N_n}\textbf{-mod} \text{ and } \mathcal{N}_{\text{proj}} = \bigoplus_{n=0}^{\infty} \boldsymbol{N_n}\textbf{-pmod}$$

$\mathcal{N}$ and $\mathcal{N}_{\text{proj}}$ can be viewed as subcategories of the category of finite-dimensional N-modules. Define

$$\text{Ind} = \bigoplus_{n=0}^{\infty} \text{Ind}_{N_n}^{N_{n+1}} \text{ and } \text{Res} = \bigoplus_{n=0}^{\infty} \text{Res}_{N_n}^{N_{n+1}}$$

These are endofunctors of $\mathcal{N}$ or $\mathcal{N}_{\text{proj}}$. We now have isomorphisms of functors

$$(X \otimes_N -) \cong \text{Ind} \text{ and } (D \otimes_N -) \cong \text{Res}.$$

Recall that $\mathcal{G}_N = \mathcal{K}_0(\mathcal{N})$ and that $\mathcal{K}_N = \mathcal{K}_0(\mathcal{N}_{\text{proj}})$.

Proposition 4.4.8. *The tuples* $(\mathcal{N}, \varphi_{\mathcal{G}_N}, \{\mathit{Ind}, \mathit{Res}\})$ *and* $(\mathcal{N}_{\mathit{proj}}, \varphi_{\mathcal{K}_N}, \{\mathit{Ind}, \mathit{Res}\})$ *are naive categorifications of* $(W, \{x, \partial\}, R)$ *and* $(W, \{x, \partial\}, R')$*, respectively.*

Proof. We have already seen that $\varphi_{\mathcal{G}_N} : \mathcal{G}_n \to R$ and $\varphi_{\mathcal{K}_N} : \mathcal{K}_N \to R'$ are isomorphisms of $\mathbb{Z}$-modules. It was shown in the proof of Lemma 4.4.6 that X_n is a free right N_n-module generated by $n+1$ elements (we say that X_n has *rank* $n+1$). Recall the unique simple module of N_n, L_n. Since dim $L_n = 1$, we have

$$\dim_{\mathbb{F}}\text{Ind}(L_n) = \dim_{\mathbb{F}}(X_n \otimes_{N_n} L_n) = n+1$$

Therefore, the composition series of $X_n \otimes L_n$ consists of the unique simple N_{n+1}-module L_{n+1} occuring with multiplicity $n+1$. So we have the following equation of isomorphism classes:

$$[\text{Ind}(L_n)] = (n+1)[L_{n+1}] \in \mathcal{G}_N$$

We also have (considering the involved objects as left N_{n+1}-modules)

$$\text{Ind}(N_n) = X_n \otimes_{N_n} N_n = N_{n+1} \otimes_{N_n} N_n \cong N_{n+1},$$

thus

$$[\text{Ind}(N_n)] = [N_{n+1}] \in \mathcal{K}_N.$$

We now consider the Res endofunctors. We have

$$\dim_{\mathbb{F}}\text{Res}(L_{n+1}) = \dim_{\mathbb{F}}(D_n \otimes_{N_{n+1}} L_{n+1}) = \dim_{\mathbb{F}}(N_{n+1} \otimes_{N_{n+1}} L_{n+1}) = 1$$

So $\text{Res}(L_{n+1}) = L_n$ and therefore $[\text{Res}(L_{n+1})] = [L_n] \in \mathcal{G}_N$.
Finally, we have (considered as left N_n-modules)

$$\text{Res}(N_{n+1}) = D_n \otimes_{N_{n+1}} N_{n+1} = {}_{N_n}N_{n+1} \otimes_{N_{n+1}} N_{n+1} = {}_{N_n}N_{n+1} \cong N_n^{\oplus(n+1)}$$

where the last isomorphism stems from the fact that, as a left N_n-module, N_{n+1} is free and of rank $n+1$. Therefore, we have

$$[\text{Res}(N_{n+1})] = (n+1)[N_n] \in \mathcal{K}_N.$$

From these computations, it follows that we have, for all $n \in \mathbb{N}$,

$$\varphi_{\mathcal{G}_N} \circ \text{Res}([L_{n+1}]) = \varphi_{\mathcal{G}_N}([L_n]) = x^n/n! = \partial \cdot x^{n+1}/(n+1)! = \partial \cdot \varphi_{\mathcal{G}_N}([L_{n+1}]),$$

$$\varphi_{\mathcal{G}_N} \circ \text{Ind}([L_n]) = \varphi_{\mathcal{G}_N}((n+1)[L_{n+1}]) = (n+1)x^{n+1}/(n+1)! = x \cdot x^n/n! = x \cdot \varphi_{\mathcal{G}_N}([L_n]),$$

$$\varphi_{\mathcal{K}_N} \circ \text{Res}([N_{n+1}]) = \varphi_{\mathcal{K}_N}((n+1)[N_n]) = (n+1)x^n = \partial \cdot x^{n+1} = \partial \cdot \varphi_{\mathcal{K}_N}([N_{n+1}]),$$

$$\varphi_{\mathcal{K}_N} \circ \operatorname{Ind}([N_n]) = \varphi_{\mathcal{K}_N}([N_{n+1}]) = x^{n+1} = x \cdot x^n = x \cdot \varphi_{\mathcal{K}_N}([N_n]).$$

These relations can be expressed in the form of commutative diagrams, thus:

$$\begin{array}{ccc} \mathcal{G}_N & \xrightarrow{\text{Ind}} & \mathcal{G}_N \\ {\scriptstyle \varphi_{\mathcal{G}_N}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathcal{G}_N}} \\ R & \xrightarrow{x} & R \end{array} \qquad \begin{array}{ccc} \mathcal{G}_N & \xrightarrow{\text{Res}} & \mathcal{G}_N \\ {\scriptstyle \varphi_{\mathcal{G}_N}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathcal{G}_N}} \\ R & \xrightarrow{\partial} & R \end{array} \qquad \begin{array}{ccc} \mathcal{K}_N & \xrightarrow{\text{Ind}} & \mathcal{K}_N \\ {\scriptstyle \varphi_{\mathcal{K}_N}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathcal{K}_N}} \\ R' & \xrightarrow{x} & R' \end{array} \qquad \begin{array}{ccc} \mathcal{K}_N & \xrightarrow{\text{Res}} & \mathcal{K}_N \\ {\scriptstyle \varphi_{\mathcal{K}_N}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathcal{K}_N}} \\ R' & \xrightarrow{\partial} & R' \end{array}$$

The result follows: the specified tuples are indeed the naive categorifications required. □

As with Example 4.3.3. this naive categorification, with the addition of an isomorphism of functors lifting the Weyl algebra's defining relation, can be strengthened to a weak categorification.

Proposition 4.4.9. *For each $n \in \mathbb{N}$, there exists an isomorphism of (N_n, N_n)-bimodules*

$$D_{n+1} \otimes_{N_{n+1}} X_n \cong (X_{n-1} \otimes_{N_{n-1}} D_n) \oplus N_n,$$

where N_n is considered as an (N_n, N_n)-bimodule in the natural way, with actions given by left and right multiplication. We thus have an isomorphism of (N, N)-bimodules

$$D \otimes_N X \cong (X \otimes_N D) \oplus N.$$

Proof. We have isomorphisms of (N_n, N_n)-bimodules

$$D_{n+1} \otimes_{N_{n+1}} X_n \cong {}_{N_n}(N_{n+1})_{N_n}, \quad X_{n-1} \otimes_{N_{n-1}} D_n \cong N_n \otimes_{N_{n-1}} N_n.$$

Let

$$m_1 : N_n \hookrightarrow N_{n+1}$$

be the natural inclusion of (N_n, N_n)-bimodules (uniquely determined by $1 \mapsto 1$). We also have an injective homomorphism of (N_n, N_n)-bimodules

$$m_2 : N_n \otimes_{N_{n-1}} N_n \hookrightarrow N_{n+1}, \quad m_2(a \otimes b) = a u_n b, \ a, b \in N_n.$$

For $\sigma \in S_{n+1}$, we have $u_\sigma \in m_1(N_n)$ if and only if $\sigma(n+1) = n+1$. If $\sigma(n+1) \neq n+1$, then we can wrote $\sigma = \tau_1 s_n \tau_2$ for $\tau_1, \tau_2 \in S_n$. Hence $u_\sigma \in m_2(N_n \otimes_{N_{n-1}} N_n)$. Therefore, m_1 and m_2 define an $N_n, N_n)$-bimodule homomorphism

$$(N_n \otimes_{N_{n-1}} N_n) \oplus N_n \cong N_{n+1}$$

as desired. □

Corollary 4.4.10. *There exist isomorphisms of endofunctors of* $\boldsymbol{N_n}$***-mod*** *(hence also of* $\boldsymbol{N_n}$***-pmod****)*

$$Res^{N_{n+1}}_{N_n} \circ Ind^{N_{n+1}}_{N_n} \cong (Ind^{N_n}_{N_{n-1}} \circ Res^{N_n}_{N_{n-1}}) \oplus id,$$

and hence isomorphisms of endofunctors of $\mathcal{G}_N$ (thus also of $\mathcal{K}_N$)

$$Res \circ Ind \cong (Ind \circ Res) \oplus id.$$

Proof. This follows from the fact that

$$(D_{n+1} \otimes_{N_{n+1}} X_n) \otimes_{N_n} - \cong \operatorname{Res}^{N_{n+1}}_{N_n} \circ \operatorname{Ind}^{N_{n+1}}_{N_n},$$
$$(X_{n-1} \otimes_{N_{n-1}} D_n) \otimes_{N_n} - \cong \operatorname{Ind}^{N_n}_{N_{n-1}} \circ \operatorname{Res}^{N_n}_{N_{n-1}},$$
$$N_n \otimes_{N_n} - \cong \operatorname{id}.$$

□

These isomorphisms categorify the relation $\partial x = x\partial + 1$. This, together with the naive categorification developed earlier, shows that we have a weak categorification of the modules R and R' of the Weyl algebra W.

4.5 Further directions

We have now developed notions of naive and weak categorification, with examples. This nomenclature is suggestive: why are such categorifications "naive" or "weak"? Are there "strong" categorifications? As a conclusion, we will discuss this, informally and briefly.

Recall from Section 2 that a monoid $(M, \circ)$ can be viewed as a category with a single object. This can be taken further; for example, we can view a group as a category with a single object in which all morphisms are isomorphisms. This corresponds to the fact that every element in a group must be invertible. We can, in fact, go even further and treat unital associative R-algebras as one-object categories with certain properties. Roughly speaking, a *strong categorification* of a unital associative R-algebra is a categorification of the algebra considered as a one-object category. Now, the categorifications that we have seen thus far all "lift", in some sense, mathematical objects to the "higher" realm of categories. However, we are now considering a categorification of a category; this suggests, intuitively, that it is possible to go "higher" still. This is indeed the case: for strong categorification, we require the notion of a so-called *2-category.* We will omit the precise definition here, but the notion is fairly intuitive: for example, rather than having only objects and morphisms, a 2-category has *1-morphisms* and *2-morphisms.* For any two objects X and Y in a 2-category, the morphisms $\mathrm{Mor}(X, Y)$ form a category; the objects in this category are the 1-morphisms and the morphisms are the 2-morphisms. Also fairly intuitive is the analogue of the notion of the Grothendieck group for a 2-category; just as taking the Grothendieck group of a category is moving "down" to the realm of single objects, the analogous operation for a 2-category also moves "down": the result is itself a category.

An example of a 2-category is the category of small categories discussed in Section 2: the objects are small categories, the 1-morphisms are functors, and the 2-morphisms are maps between functors known as *natural transformations.* With this in mind, we can define strong categorification in a slightly more precise manner than above: let R be a commutative ring and let $\mathcal{C}$ be a unital associative R-algebra considered as a one-object category. Then a strong categorification of $\mathcal{C}$ is a 2-category $\mathfrak{C}$ with certain properties, together with an isomorphism φ which has domain $\mathcal{K}_0^{\text{split}}(\mathfrak{C}) \otimes_{\mathbb{Z}} R$ or $\mathcal{K}_0(\mathfrak{C}) \otimes_{\mathbb{Z}} R$ and codomain $\mathcal{C}$.

This is, very roughly speaking, the idea behind strong categorification. For brevity, a full definition of it has been omitted. For the same reason, interesting examples of naive and weak categorification have been omitted. For more examples of weak and naive categorification, as well as a full treatment of strong categorification, see [1] and [4].

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