University of Glasgow

School of Mathematics and Statistics

Level-5 Honours project

The Dold-Kan and Dwyer-Kan Correspondences

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Abstract

We begin by giving an introduction to basic homological algebra: chain complexes and homology are discussed with illustrative examples. Likewise, we introduce double complexes and the homotopy theory of chain complexes, followed by a short discussion of spectral sequences. We then proceed to the topic of simplicial objects, first discussing the simplex category Δ, then defining the notion of a simplicial object, with illustrative examples. This is followed by a discussion of the notion of the geometric realization of a simplicial set, as well as the notion of a combinatorial simplicial complex. We then define some basic operations on simplicial objects, and discuss the homotopy theory of simplicial objects. Then, assembling all the previous work, we state and prove the Dold-Kan Correspondence. We then define cyclic objects, and the cyclic homology associated to them, before defining duplicial objects and mixed complexes. We then state and prove the Dwyer-Kan Correspondence. In the concluding section of the main body of the paper, we give some directions for further study. Many categorical terms and results used in the main body of the text without comment are discussed in the Appendix.

Contents

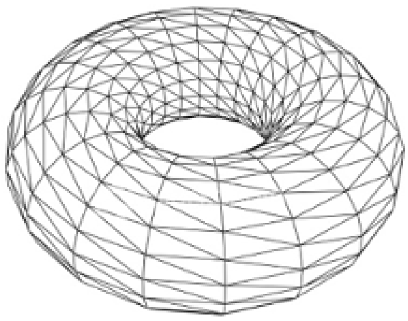
1 Introduction

All through mathematics, one encounters notions of correspondence: in other words, ways to recognise when mathematical objects are similar in some respect. This ranges from the very simplest cases, such as when two numbers are equal, through to more complex situations, such as when two groups are isomorphic. As the setting gets more complex, the number of ways in which we can recognise similarity tends to increase. For example, one can define a notion of isomorphism in the setting of category theory: two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. However, this turns out to be too strong a condition to be often considered or used. The central notion of correspondence for categories is the slightly weaker *equivalence of categories*, in which the composite functors above are only required to be naturally isomorphic to the appropriate identity functors. This extra subtletly is a consequence of the shift from set-theoretic to category-theoretic statements: we have no general notion of a morphism between functions, but the notion of *natural transformations* between functors is ubiquitous. With this in mind, we say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories if they are isomorphic up to a natural isomorphism.

The notion of equivalence of categories turns out to be strong enough to imply big resemblances between the structures of the categories involved, but weak enough so that the recognised similarities actually provide new insight. To give one example, the vast field of algebraic geometry is built around an equivalence of categories between sets of zeroes of polynomials and finitely-generated algebras over a field. Another example comes from functional analysis: the category of commutative C^*-algebras with unity is contravariantly equivalent to the category of compact Hausdorff spaces. This is known as the *Gelfand representation*.

The present paper is primarily concerned with two equivalances in particular: the *Dold-Kan Correspondence*, an equivalence between *simplicial objects* and *chain complexes*, and the *Dwyer-Kan Correspondence*, an equivalence between *duplicial objects* and *mixed complexes*. The Dold-Kan Correspondence can be thought of as a vast generalization of the notion of simplicial homology from algebraic topology, a fact we will illustrate with an example.

The figure below shows a possible *triangulation* of the torus.



Such pictures emerge when computing the simplicial homology of the torus, which is done as follows: first, we view the torus as a simplicial complex, consisting of 0-simplices (points), 1-simplices (lines), and 2-simplices (triangles). We then build a chain complex C of abelian groups: in degree 0, C has the free abelian group generated by the set of 0-simplices, and the other degrees are completely analogous. Now, let $v = [v_0, \dots, v_n]$ be an n-simplex. The differential $d_n : C_n \to C_{n-1}$ is given by

$$d_n(v) = \sum_{i=0}^{n} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_n),$$

where the addition of a hat to a vertex denotes deletion. So, for example, $(v_0, \dots, \hat{v}_k, \dots, v_n)$ is the kth face of v, which we obtain by deleting the kth vertex.

As the nomenclature may suggest, the collection of n-simplices (where $n \in \{0, 1, 2\}$) of the torus is a kind of simplicial object, and, in fact, the notion of a simplicial object is a generalization of the combinatorial structure of a simplicial complex. In this more abstract language, when we compute the simplicial homology of some simplicial complex X, we start with a *simplicial set*, then create a *simplicial abelian group* from it, then, finally, use this simplicial abelian group to build a chain complex: the chain complex whose homology is the simplicial homology of X.

The last step of the above example, where we construct a chain complex from a simplicial abelian group, is a kind of early premonition of the Dold-Kan Correspondence, which we will now state.

Theorem (Dold-Kan). *Let $\mathcal{A}$ be an abelian category, let $\mathcal{SA}$ be the category of simplicial objects in $\mathcal{A}$, and let $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ be the category of non-negatively graded chain complexes in $\mathcal{A}$. Then there exist functors $N : \mathcal{SA} \to \mathbf{Ch}_{\geq 0}(\mathcal{A})$ and $K : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{SA}$ such that $KN \cong \mathrm{id}_{\mathcal{SA}}$ and $NK \cong \mathrm{id}_{\mathbf{Ch}_{\geq 0}(\mathcal{A})}$.*

The bulk of the paper deals with the preliminary material required for a full understanding of the Dold-Kan Correspondence, as well as its proof. For this, we draw primarily on Chapter 8 of Charles A. Weibel's *An Introduction to Homological Algebra* [17]. For more information on the Dold-Kan Correspondence, the reader may wish to consult [5], [8], or [9].

The Dwyer-Kan Correspondence is an extension of the Dold-Kan Correspondence, in the sense that a duplicial object can be built from a simplicial object with the addition of extra structure, and a mixed complex can be built from a chain complex again by the addition of extra structure. The functors which form the Dwyer-Kan Correspondence are extensions of those which form the Dold-Kan Correspondence, and for this reason we also denote them by N and K:

Theorem (Dwyer-Kan). *Let $\mathcal{A}$ be an abelian category, let $\mathcal{DA}$ be the category of duplicial objects in $\mathcal{A}$, and let $\mathbf{Mix}_{\geq 0}(\mathcal{A})$ be the category of non-negatively graded mixed complexes in $\mathcal{A}$. Then there exist functors $N : \mathcal{DA} \to \mathbf{Mix}_{\geq 0}(\mathcal{A})$ and $K : \mathbf{Mix}_{\geq 0}(\mathcal{A}) \to \mathcal{DA}$ such that $KN \cong \mathrm{id}_{\mathcal{DA}}$ and $NK \cong \mathrm{id}_{\mathbf{Mix}_{\geq 0}(\mathcal{A})}$.*

Additionally, we discuss ways in which the correspondences are even stronger than simple equivalences of categories.

In Section 2 we discuss chain complexes, as well as the homological algebra necessary to prove the Dold-Kan Correspondence. In Section 3, we discuss simplicial objects. In Section 4, we state and prove the Dold-Kan Correspondence. In Section 5, we introduce duplicial objects and mixed complexes, then state and prove the Dwyer-Kan Correspondence. Finally, in Section 6, we briefly discuss some possible directions for further research. Various category-theoretic terms and results used in the main body of the paper are explained in the Appendix.

2 Basic homological algebra

To begin, it will be necessary to cover some of the basics of homological algebra, starting with the central notion of the subject: chain complexes.

2.1 Chain complexes

Definition 2.1.1. Let $\mathcal{A}$ be an abelian category. A *chain complex* (C, d) is a sequence of objects and morphisms in $\mathcal{A}$ with $d_n : C_n \to C_{n-1}$ and the property that $d_n \circ d_{n+1} = 0$ for all n. For convenience, we will often write C for (C, d) and d^2 for $d_n \circ d_{n+1}$.

Remark. One can dualize the previous definition, obtaining the notion of a *cochain complex*, which is exactly the same, except that the maps d_n go upward in degree, rather than downward.

This definition is valid in any abelian category, but the following theorem, known as the Freyd-Mitchell Embedding Theorem, implies that we can consider the objects in question to be modules over a ring R, without any loss of generality. As this viewpoint will be notationally convenient, as well as allowing us to use element-theoretic proofs, we will adopt it from this point on.

Theorem 2.1.2 (Freyd-Mitchell)**.** *Let $\mathcal{A}$ be a small abelian category. Then there exists a unital ring R, not necessarily commutative, and a functor $F : \mathcal{A} \to$ **R-mod** which is full, faithful, and exact, where **R-mod** denotes the category of left R-modules.*

Proof. See [15]. □

Each chain complex of R-modules gives rise to two further sequences of R-modules $Z_n = \ker d_n$ and $B_n = \operatorname{im} d_{n+1}$ which are called, respectively, the *cycles* and *boundaries*. Since $d_n \circ d_{n+1} = 0$, we have the inclusions $B_n \subseteq Z_n \subseteq C_n$. This motivates the following definition, which is valid since we can quotient by any submodule.

Definition 2.1.3. Let C be a chain complex of R-modules. The quotient module Z_n/B_n is known as the *nth homology of C*. It is denoted $H_n(C)$. A chain complex whose homology is trivial for all n is known as *acyclic*.

Example 2.1.4. Consider the diagram

$$\cdots \to \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 2} \mathbb{Z}_8 \to 0$$

The notation here means that each object of non-negative degree is $\mathbb{Z}_8$, that each object of negative degree is 0, and that d alternates between $\cdot 2$ and $\cdot 4$ over the whole complex. This is a chain complex; wherever the differential is not simply 0, it is a multiple of 8, which means that any argument of d^2 will be congruent to 0 mod 8. Computing the homology is simple:

$$\begin{aligned} H_0 &= \ker d_0/\operatorname{im} d_1 = \mathbb{Z}_8/\mathbb{Z}_4 \cong \mathbb{Z}_4 \\ H_1 &= \ker d_1/\operatorname{im} d_2 = \mathbb{Z}_2/\mathbb{Z}_2 \cong \{0\} \\ H_2 &= \ker d_2/\operatorname{im} d_3 = \mathbb{Z}_4/\mathbb{Z}_4 \cong \{0\} \end{aligned}$$

This pattern will repeat. So the chain complex has trivial homology except in degree 0.

We can form a category, denoted $\mathbf{Ch}(R)$, whose objects are chain complexes of R-modules. To do so, however, we must define the notion of a morphism of chain complexes.

Definition 2.1.5. A *chain map* $f : C \to D$ is a collection of morphisms (in **R-mod**) $f_n : C_n \to D_n$ such that $f_{n-1} \circ d_n = d_{n-1} \circ f_n$. The last condition can be summed up by stating that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \xrightarrow{d'} & D_{n+1} & \xrightarrow{d'} & D_n & \xrightarrow{d'} & D_{n-1} & \xrightarrow{d'} & \cdots \end{array}$$

We will often denote the chain map as a whole by $\{f_n\}$. The category $\mathbf{Ch}(R)$ has chain complexes of R-modules as objects and chain maps as morphisms.

A map of chain complexes $f : C \to D$ maps boundaries to boundaries and cycles to cycles, and hence induces maps $H_n(C) \to H_n(D)$. These induced maps make each H_n into a functor from $\mathbf{Ch}(R)$ to **R-mod**. This motivates the following definition.

Definition 2.1.6. Let $f : C \to D$ be a map of chain complexes. We refer to f as a *quasi-isomorphism* if each induced map $H_n(C) \to H_n(D)$ is an isomorphism.

The category $\mathbf{Ch}(R)$ inherits some of the structure of $\boldsymbol{R}$**-mod**, in that it is also an additive category. To see this, we must, per Definition A.4.1, show that the hom-sets of $\mathbf{Ch}(R)$ have an abelian group structure over which composition of morphisms distributes, that $\mathbf{Ch}(R)$ has a zero object, and that binary products and coproducts exist. The first condition is certainly true; if $\{f_n\}$ and $\{g_n\}$ are two chain maps, we can simply add their individual components degreewise, because each one is a morphism in $\boldsymbol{R}$**-mod**, which is itself an additive category. The sum of the two chain maps is the family $\{f_n + g_n\}$.
The zero object in $\mathbf{Ch}(R)$ is the trivial complex, which is $\{0\}$ in each degree and has the zero map for each differential. The product and coproduct are defined degreewise. The differentials are as follows:

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1}$$

$$\bigoplus d_\alpha : \bigoplus_\alpha A_{\alpha,n} \to \bigoplus_\alpha A_{\alpha,n-1}$$

Definition 2.1.7. Let B and C be chain complexes. We refer to B as a *subcomplex* of C if, for every n, B_n is a submodule of C_n and the differential on B is the restriction of the differential on C; that is to say, the family of inclusions $\{i_n\}$ of B_n in C_n is a chain map $B \to C$. If this is the case, we can build a chain complex using the quotient modules:

$$\cdots \to C_{n+1}/B_{n+1} \xrightarrow{d} C_n/B_n \xrightarrow{d} C_{n-1}/B_{n-1} \xrightarrow{d} \cdots$$

This complex has the same differential as C, is denoted C/B, and is called the *quotient complex.*

Definition 2.1.8. Let C be a complex, let n be an integer, and let $\tau_{\geq n}C$ denote the subcomplex of C defined as follows:

$$(\tau_{\geq n}C)_i = \begin{cases} 0 & \text{if } i < n \\ Z_n & \text{if } i = n \\ C_i & \text{if } i > n \end{cases}$$

By construction, $H_i(\tau_{\geq n}C) = 0$ for $i < n$ and $H_i(\tau_{\geq n}C) = H_i(C)$ for $i \geq n$. We refer to the subcomplex $\tau_{\geq n}C$ as the *good truncation* of C below n, and we refer to the complex $\tau_{<n}C = C/(\tau_{\geq n}C)$ as the good truncation of C above n; $H_i(\tau_{<n}C) = H_i(C)$ for $i < n$ and 0 for $i \geq n$.

Remark. Related to the definition above, there exist the less refined *brutal truncations*, denoted $\sigma_{<n}C$ and $\sigma_{\geq n}C = C/(\sigma_{<n}C)$. By construction, $\sigma_{<n}C$ is C_i if $i < n$ and 0 if $i \geq n$. These are easier to describe directly, but, disadvantageously, they are not acyclic, having a homology group $H_n(\sigma_{\geq n}C) = C_n/B_n$.

Another operation which can be performed on chain or cochain complexes is shifting indices, or *translation.*

Definition 2.1.9. Let C be a complex and let p be an integer. From this, we can form a new complex, denoted $C[p]$ and defined as follows:

$$C[p]_n = C_{n+p} \text{ if } C \text{ is a chain complex,} \quad C[p]^n = C^{n-p} \text{ if } C \text{ is a cochain complex.}$$

The differential is given by $(-1)^p d$. This sign convention is used to simplify notation in some situations. We refer to $C[p]$ as the p-th translate of C.

Note that translation shifts homology in the expected manner:

$$H_n(C[p]) = H_{n+p}(C) \quad H^n(C[p]) = H^{n-p}(C).$$

We can view translation as a functor from $\mathbf{Ch}(R)$ to $\mathbf{Ch}(R)$ by translating the degrees of chain maps; if $f : C \to D$ is a chain map, then $f[p]$ is the chain map defined as follows:

$$f[p]_n = f_{n+p}.$$

The situation is similar for maps of cochain complexes.

Definition 2.1.10. Let $f : B \to C$ be a chain map. The *mapping cone* of f is the chain complex, denoted $\mathrm{cone}(f)$, whose entry in degree n is $C_{n-1} \oplus C_n$. In the name of consistence with other sign conventions, the differential of $\mathrm{cone}(f)$ is given by the formula

$$d_{\mathrm{cone}}(b, c) = (-d(b), d(c) - f(b))$$

where $b \in B_{n-1}, c \in C_n$. Predictably, there is a dual notion for a map $B \to C$ of cochain complexes; the mapping cone is a cochain complex whose entry in degree n is $B^{n+1} \oplus C^n$. The differential is given by the same formula as for chain complexes.

2.2 Double complexes

We proceed now to the key notion of a *double complex* or *bicomplex*. These can be thought of as chain complexes whose objects are themselves chain complexes.

Definition 2.2.1. A *double complex* or *bicomplex* in ***R*-mod** is a family $\{C_{p,q}\}$ of objects of ***R*-mod**, together with maps

$$d : C_{p,q} \to C_{p-1,q} \text{ and } b : C_{p,q} \to C_{p,q-1}$$

which, for reasons which will become clear, are often referred to as the *horizontal* and *vertical* differentials. Further, these maps are subject to the conditions that $d^2 = b^2 = db + bd = 0$. This construction is best viewed as a lattice:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q+1} & \xleftarrow{d} & C_{p,q+1} & \xleftarrow{d} & C_{p+1,q+1} & \longleftarrow & \cdots \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \\
\cdots & \longleftarrow & C_{p-1,q} & \xleftarrow{d} & C_{p,q} & \xleftarrow{d} & C_{p+1,q} & \longleftarrow & \cdots \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \\
\cdots & \longleftarrow & C_{p-1,q-1} & \xleftarrow{d} & C_{p,q-1} & \xleftarrow{d} & C_{p+1,q-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

This viewpoint is the reason for the use of the terms 'horizontal' and 'vertical'. The requirement that $db + bd = 0$ means that each square in the lattice anticommutes. We call a double complex C *bounded* if it has only finitely many nonzero terms along each diagonal; one example of this is if C is zero everywhere except the first quadrant of the plane.

Remark. The anticommutativity of the squares means that the maps b are not chain maps, but we can form chain maps $f_{*,q} : C_{*,q} \to C_{*,q-1}$ from them by introducing sign changes:

$$f_{p,q} = (-1)^p b_{p,q} : C_{p,q} \to C_{p,q-1}$$

Using this 'sign trick' allows us to identify double complexes in ***R*-mod** with chain complexes in $\mathbf{Ch}(R)$.

Definition 2.2.2. Given a double complex C_{**}, we can construct a pair of chain complexes, which we call the *total complexes* of C, denoted by $\mathrm{Tot}^{\Pi}(C)_n$ and $\mathrm{Tot}^{\oplus}(C)_n$, and defined as follows

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \text{ and } \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

Due to the anticommutativity of the squares, the formua $d_{\mathrm{Tot}} = d + b$ defines maps

$$d_{\mathrm{Tot}} : \mathrm{Tot}^{\Pi}(C)_n \to \mathrm{Tot}^{\Pi}(C)_{n-1} \text{ and } d_{\mathrm{Tot}} : \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d_{\mathrm{Tot}}^2 = 0$, which means that the two total complexes are chain complexes. Notice that, due to finite direct sums and products being the same in an additive category, $\mathrm{Tot}^{\Pi}(C) = \mathrm{Tot}^{\oplus}(C)$ if C is bounded.

2.3 Chain homotopies

We proceed now to discuss the notion of *chain homotopy*; this is an equivalence relation of chain maps which, as the nomenclature suggests, shares many properties with the classical notion of homotopy of continuous maps from topology.
We begin, by way of motivation, with a special case, of historical importance. Let C be a chain complex of vector spaces, that is to say, a chain complex of R-modules where the ring R is a field. In this situation, we can always choose the following vector space decompositions:

$$\begin{aligned}
&C_n = Z_n \oplus B'_n \\
&B'_n \cong C_n/Z_n = d(C_n) = B_{n-1} \\
&Z_n = B_n \oplus H'_n \\
&H'_n \cong Z_n/B_n = H_n(C).
\end{aligned}$$

We can therefore form the compositions

$$C_n \to Z_n \to B_n \cong B'_{n+1} \subseteq C_{n+1}$$

to obtain maps $s_n : C_n \to C_{n+1}$, such that for each s in the family, $d = dsd$.
The compositions ds and sd are, respectively, projections from C_n onto B_n and B'_n: therefore, the sum $ds + sd$ is a map from C_n to C_n whose kernel H'_n is isomorphic to the homology $H_n(C)$. Both the kernel and the cokernel of $ds + sd$ are the acyclic chain complex $H_*(C)$, and both chain maps $H_*(C) \to C$ and $C \to H_*(C)$ are quasi-isomorphisms. In addition, C is an exact sequence (a sequence where $\ker d_n = \operatorname{im} d_{n+1}$ if and only if $ds + sd$ is equal to the identity map.
If R is an arbitrary ring and hence not necessarily a field, it is not always possible to split chain complexes in this fashion, so we give a name to the situation in which we are able to do so.

Definition 2.3.1. A chain complex C is called *split* if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$ for all n. We call the maps s_n the *splitting maps*, and, if C is acyclic, we say that C is *split exact.*

Example 2.3.2. Let R be $\mathbb{Z}$ or $\mathbb{Z}_4$, and let C be the complex

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \cdots$$

It is obvious that this complex is acyclic, but it is not split exact; there is no family of maps $\{s_n\}$ such that $ds+sd$ is the identity for each member s of the family, nor is there any direct sum decomposition $C_n \cong Z_n \oplus B'_n$.

Now, suppose that we are given two chain complexes C and D, together with arbitrary maps $s_n : C_n \to D_{n+1}$. Let f_n be the map from C_n to D_n given by the formula $f_n = d_{n+1}s_n + s_{n-1}d_n$:

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} \\ & \swarrow^{s} & \downarrow f & \swarrow^{s} & \\ D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

Forgetting the subscripts for concision, we compute

$$df = d(ds + sd) = dsd + (ds + sd)d = fd$$

Therefore, f is a chain map from C to D. This motivates the following definition.

Definition 2.3.3. A chain map $f : C \to D$ is called *null homotopic* if there are maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$. The maps $\{s_n\}$ are called a *chain contraction* of f.

Definition 2.3.4. Let f and g be two chain maps mapping from C to D. We say that f and g are *chain homotopic*, and write $f \simeq g$, if their difference $f - g$ is null homotopic. That is to say, $f - g = ds + sd$. We refer to the maps $\{s_n\}$ as a *chain homotopy* from f to g. Finally, we refer to $f : C \to D$ as a *chain homotopy equivalence* if there is a map $g : D \to C$ such that $g \circ f \simeq \mathrm{id}_C$ and $f \circ g \simeq \mathrm{id}_D$.

Remark. This choice of terminology is inspired by topology, via the following observation. A map $f : X \to Y$ of topological spaces induces a map $f_* : S(X) \to S(Y)$ of chain complexes between the corresponding singular chain complexes. It can be shown that f being topologically null homotopic (or, respectively, a topological homotopy equivalence) implies that the chain map f_* is chain null homotopic (or, respectively, a chain homotopy equivalence.) Furthermore, if two maps f and g are topologically homotopic, then f_* and g_* are chain homotopic.

We shall now prove a lemma which will be required later; the result corresponds to the fact that if the identity on a topological space X is null homotopic, X is contractible.

Lemma 2.3.5. *Let $f : C \to D$ be null homotopic. Then every induced map $f_* : H_n(C) \to H_n(D)$ is zero. In particular, if* $\mathrm{id} : C \to C$ *is null homotopic, C is acyclic.*

Proof. Suppose that $f = ds + sd$. Every element of $H_n(C)$ is represented by an n-cycle x, but $f(x) = d(sx)$. That is to say, $f(x)$ is an n-bounday in D. Therefore, $f(x)$ represents 0 in $H_n(D)$. □

We proceed now to a topic of vast computational usefulness: the theory of spectral sequences. Because of the subject's complexity, we will cover only the very small amount necessary for a proof of the Dold-Kan Correspondence. For a full and proper treatment of spectral sequences, consult Chapter 5 in [17].

2.4 Spectral sequences

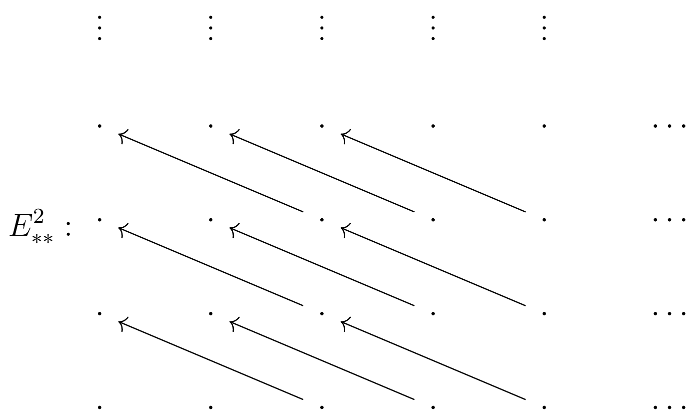
By way of motivation, first consider the problem of computing the homology of the total chain complex, denoted T_*, of a double complex E_{**} which is zero everywhere except for the first quadrant in the plane. (We will refer to such a double complex as a *first quadrant double complex* for concision.) As a first step, it will be useful to temporarily forget the existence of the horizontal differentials, denote the act of doing so by the addition of a superscript 0, and consider only the vertical differentials b along the columns E^0_{p*}:

$$E^0_{**}: \quad \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

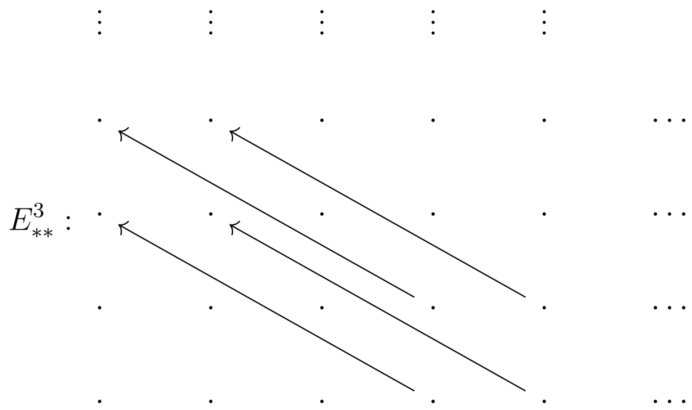
Now, if we write E^1_{pq} for the vertical homology $H_q(E^0_{p*})$ at the position (p, q), we can arrange this information in another lattice, this time using the horizontal differentials d:

$$E^1_{**}: \quad \begin{array}{ccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdots \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdots \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdots \\ \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdots \end{array}$$

We have just performed the first two steps of an algorithim for computing the homology $H_*(T)$, which is called a spectral sequence. To see how the algorithim progresses, we write E^2_{pq} for the hoizontal homology $H_p(E^1_{q*})$ at the position (p, q). Predictably, this data can once again be arranged in a lattice with the differentials going left by two positions and up by one position:



The pattern continues in this fashion; we form a lattice E^3_{**} using the "L-shaped" homology from the diagram above, with differentials as in the diagram below:



With this in mind, we now make the formal definition of a spectral sequence.

Definition 2.4.1. A *homology spectral sequence* (starting with E^a) in $\boldsymbol{R}$**-mod** consists of the following data:

1. A family $\{E^r_{pq}\}$ of R-modules which are defined for all integers p, q and $r \geq a$.

2. Maps $d_pq^r : E^r_{pq} \to E_{p-r\, q-r+1}$ which are differentials in the sense that $d^r d^r = 0$ so that the lines which we think of of having 'slope' $\frac{-(r-1)}{r}$ in the lattice E^r_{**} form chain complexes (compare the diagrams above).

3. Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the position E^r_{pq}:

$$E^{r+1}_{pq} \cong \ker(d^r_{pq})/\mathrm{im}(d^r_{p+r,q-r+1})$$

   The *total degree* of the term E^r_{pq} is $n = p + q$. The terms which have total degree n lie on a line of 'slope' -1, and each differential d^r_{pq} decreases the total degree by 1.

Remark. One can form a category whose objects are homology spectral sequences. A morphism $f : E \to E'$ is a family of maps $f^r_{pq} : E^r_{pq}{}' \to E^r_{pq}$ in $\boldsymbol{R}$**-mod**, for suitably large r, with $d^r f^r = f^r d^r$ such that f^{r+1}_{pq} is the map induced by f^r_{pq} on homology.

Example 2.4.2. A first quadrant homology spectral sequence is a homology spectral sequence E^r_{pq} such that $E^r_{pq} = 0$ for $p < 0$ or $q < 0$; that is to say, the spectral sequence is concentrated in the first quadrant of the plane. For a fixed p and q, $E^r_{pq} = E^{r+1}_{pq}$ for all sufficiently large r, because the differentials which map to the module in the position (p, q) will come from the fourth quadrant, and the differentials which map from the module in the position (p, q) will land in the second quadrant, which means all differentials eventually become 0. We denote this stable value of E^r_{pq} by E^∞_{pq}.

Remark. One can define the dual notion of a *cohomology spectral sequence*; the modules are reindexed via $E^r_{pq} = E^r_{-p\,-q}$ so that the differentials increase, rather than decrease, the total degree $p + q$ by one. As with homology spectral sequences, there is a category of cohomology spectral sequences.

A first quadrant spectral sequence is a special case of a *bounded* spectral sequence; before discussing this, we require a supporting definition.

Definition 2.4.3. Let C be an R-module. A *filtration* of C is a set $\{F_iC\}$ of submodules of C, indexed by $i \in I$. The index set I is totally ordered, and the filtration is subject to the condition that if $i \leq j$ in I, $F_iC \subseteq F_jC$.

Definition 2.4.4. Let E be a homology spectral sequence. We say that E is bounded if, for each n, there are only finitely many non-zero terms of total degree n in E^a_{**}. If this is the case, then, for each p and q, there exists some r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. As in the case of a first quadrant spectral sequence, of which this case is a generalization, we write E^∞_{pq} for this stable value of E^r_{pq}. We say that a bounded spectral sequence *converges* to H_* if we have a family of R-modules H_n, each having a finite filtration

$$0 = F_sH_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq F_tH_n = H_n$$

and if we have isomorphisms

$$E^\infty_{pq} \cong F_pH_{p+1}/F_{p-1}H_{p+q}.$$

This situation is denoted thus:

$$E^a_{pq} \implies H_{p+q}$$

This covers the basics of the theory of spectral sequences. For the proof of the Dold-Kan Correspondence, we now require a little material on the relationship between spectral sequences and filtrations of chain complexes. To begin, a filtration of a chain complex corresponds to the definition for modules above, except that that family F_iC are chain subcomplexes of a chain complex C. Our goal is to associate a spectral sequence to every such filtration, as well as establish conditions under which these spectral sequences converge; doing so will be crucial to the proof of a lemma which is itself crucial to the proof of the Dold-Kan Correspondence.

Definition 2.4.5. A filtration F_iC is called *exhaustive* if $C = \cup F_iC$. When we perform the construction of the spectral sequence associated to a given filtration, it will be clear that F_iC and $\cup F_iC$ both give rise to the same spectral sequence. We therefore, in practice, insist that filtrations are exhaustive; from the perspective of spectral sequences, we can do so without loss of generality.

Definition 2.4.6. A filtration on a chain complex C is called *bounded* if, for each n, there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. If this is the case, there are only finitely many nonzero terms of total degree n in E^0_{**}, so the spectral sequence is bounded.
A filtration on a chain complex C is called *bounded below* if, for each n, there is an integer s such that $F_sC_n = 0$, and it is called *bounded above* if, for each n, there exists a t such that $F_tC_n = C_n$. Bounded filtrations are both bounded above and bounded below. A bounded above filtration is trivially exhaustive. The Classical Convergence Theorem, which we will discuss shortly, says that the spectral sequence always converges to $H_*(C)$ when the filtration is bounded below and exhaustive; hence, bounded filtrations of C always have an associated spectral sequence which converges to $H_*(C)$.

Example 2.4.7. We call the filtration *canonically bounded* if $F_{-1}C = 0$ and $F_nC_n = C_n$ for each n. We will see that the construction of the associated spectral sequence begins with the identity $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$; every canonically bounded filtration thus gives rise to a first quadrant spectral sequence.

We will now formally construct the spectral sequence arising from a filtered chain complex.

Theorem 2.4.8. *A filtration F of a chain complex C naturally determines a spectral sequence starting with $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $E^1_{pq} = H_{p+q}(E^0_{p*})$.*

Proof. For concision, we will temporarily ignore the subscript q and write η_p for the surjection $F_pC \to F_pC/F_{p-1}C = E^0_p$. We next introduce

$$A^r_p = \{c \in F_pC \mid d(c) \in F_{p-r}C\},$$

which consists of the elements of F_pC that are cycles module $F_{p-r}C$. We also introduce their images $Z^r_p = \eta_p(A^r_p)$ in E^0_p and $B^{r+1}_{p-r} = \eta_{p-r}(d(A^r_p))$ in E^0_{p-r}. This indexing is chosen so that Z^r_p and $B^r_p = \eta_p(d(A^{r-1}_{p+r-1}))$ are subobjects of E^0_p. We now set $Z^\infty_p = \cap^\infty_{r=1} Z^r_p$ and $B^\infty_p = \cup^\infty_{r=1} B^r_p$. Assembling these definitions shows that we have defined a tower of subobjects of each E^0_p:

$$0 = B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^r_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots \subseteq Z^1_p \subseteq Z^0_p = E^0_p.$$

Note that $A^r_p \cap F_{p-1}C = A^{r-1}_{p-1}$, so that $Z^r_p \cong A^r_p/A^{r-1}_{p-1}$. Hence

$$E^r_p = \frac{Z^r_p}{B^r_p} \cong \frac{A^r_p + F_{p-1}(C)}{d(A^{r-1}_{p+r-1}) + F_{p-1}(C)} \cong \frac{A^r_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}}$$

Let $d^r_p : E^r_p \to E^r_{p-r}$ be the map induced by the differential of C. To define the spectral sequence, it is now necessary only to give the isomorphism between E^{r+1} and $H_*(E^r)$.
Towards this goal, we will show that the map d determines isomorphisms $Z^r_p/Z^{r+1}_p \xrightarrow{\cong} B^{r+1}_{p-r}/B^r_{p-r}$. First, note that $d(A^r_p) \cap F_{p-r-1}C = d(A^{r+1}_p)$, so that $B^{r+1}_{p-r} \cong d(A^r_p)/d(A^{r+1}_p)$ and hence B^{r+1}_{p-r}/B^r_{p-r} is isomorphic to $d(A^r_p)/d(A^{r+1}_p + A^{r-1}_{p-1})$. Now, the other term Z^r_p/Z^{r+1}_p is isomorphic to $A^r_p/(A^{r+1}_p + A^{r-1}_{p-1})$. As the kernel of $d : A^r_p \to F_{p-r}C$ is contained in A^{r+1}_p, the two sides are isomorphic.
In light of this, we can continue constructing the spectral sequence. The kernel of d^r_p is

$$\frac{\{z \in A^r_p \mid d(z) \in d(A^{r-1}_{p-1}) + A^{r-1}_{p-r-1}\}}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}} = \frac{A^{r-1}_{p-1} + A^{r+1}_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}} \cong \frac{Z^{r+1}_p}{B^r_p}.$$

Now, the map d^r_p factors as

$$E^r_p = Z^r_p/B^r_p \to Z^r_p/Z^{r+1}_p \xrightarrow{\cong} B^{r+1}_{p-r}/B^r_{p-r} \hookrightarrow Z^r_{p-r}/B^r_{p-r} = E^r_{p-r}.$$

This shows that the image of d^r_p is B^{r+1}_{p-r}/B^r_{p-r}; replacing p with $p+r$ shows that the image of d^r_{p+r} is B^{r+1}_p/B^r_p. This provides the isomorphism

$$E^{r+1}_p = Z^{r+1}_p/B^{r+1}_p \cong \ker(d^r_p)/\mathrm{im}(d^r_{p+r})$$

required to complete the construction of the spectral sequence. □

For our purposes, we now require only the following theorem, which gives a condition under which a spectral sequence associated to a filtration converges.

Theorem 2.4.9. *Suppose that we have a bounded filtration on a chain complex C. Then the associated spectral sequence is bounded and converges to $H_*(C)$:*

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \implies H_{p+q}(C)$$

Proof. See [17], Chapter 5. □

Bibliographical Note. The preceding section is based primarily on Chapter 5 of [17]. [16] was also consulted.

3 Simplicial objects

We proceed now to *simplicial objects*. Intuitively, these can be thought of as a vast generalization of the notion of a simplex from topology. In fact, as we will see, one can construct a topological space from a simplicial object in a fairly natural way. There are similarities between this approach to modelling topological spaces and the approach of CW complexes: one crucial difference is that simplicial objects, considered alone, are not imbued with any topology, whereas each CW complex is a topological space.

3.1 The simplex category

Definition 3.1.1. Let Δ be the category whose objects are the finite ordered sets

$$[n] := \{0 < 1 < \cdots < n\} \text{ for integers } n \geq 0,$$

and whose morphisms are the nondecreasing functions between them. We call Δ the *simplex category.* Let $\mathcal{A}$ be any category. Then a *simplicial object* A in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$. Dually, a *cosimplicial object* A in $\mathcal{A}$ is a covariant functor $A : \Delta \to \mathcal{A}$. A morphism of simplicial objects is a natural transformation and the category $\mathcal{SA}$ of all simplicial objects in $\mathcal{A}$ is the functor category $\mathcal{A}^{\Delta^{\text{op}}}$.

Remark. We stipulate that morphisms of Δ must be nondecreasing to preserve the order structure on the sets. For concision, we will use the notational convention of writing A_n for the image of $[n]$ in $\mathcal{A}$, reflecting the standard notation for the objects of a chain complex. Our notational convention for the case of a cosimplicial object is to write A^n for the image of $[n]$, reflecting the standard notations for the objects of a cochain complex.

Example 3.1.2. Let A be some fixed object of a category $\mathcal{A}$. The functor $\Delta \to \mathcal{A}$ which sends each $[n]$ to A and each morphism in Δ to id_A is called the *constant simplicial object* at A.

It will be useful in some situations to take a view of simplicial and cosimplicial objects more directly rooted in combinatorics. To develop this viewpoint, it will be necessary to deal with Δ directly, starting with the notion of *face maps* and *degeneracy maps.* These can be thought of, in a sense which will become apparent, as "generating" the morphisms of Δ.

Definition 3.1.3. For each n and $i \in \{0, 1 \dots n\}$ the *i-th face map*, which maps from $[n-1]$ to $[n]$, is the unique nondecreasing injective map which does not contain i in its image. We will denote these by ε_i. Similarly, the *i-th degeneracy map*, mapping from $[n+1]$ to $[n]$, is the unique nondecreasing surjective map which "hits" i twice. We will denote these by η_i.

Due to the nature of the face and degeneracy maps, as well as the fact that they must be nondecreasing, we have the following formulae:

$$\varepsilon_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\eta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Proposition 3.1.4. *The following identities hold in Δ:*

$$\begin{aligned} \varepsilon_j \varepsilon_i &= \varepsilon_i \varepsilon_{j-1} && \textit{if} \quad i < j \\ \eta_j \eta_i &= \eta_i \eta_{j+1} && \textit{if} \quad i \leq j \end{aligned}$$

$$\eta_j\varepsilon_i = \begin{cases} \varepsilon_i\eta_{j-1} & \textit{if } i < j \\ \mathrm{id} & \textit{if } i = j \textit{ or } i = j+1 \\ \varepsilon_{j-1}\eta_j & \textit{if } i > j+1 \end{cases}$$

Proof. We will use the formulae above to verify the first identity directly. All the others may be verified in a similar fashion. Begin by letting $i < j$. We will compare the functions $\varepsilon_j\varepsilon_i(x)$ and $\varepsilon_i\varepsilon_{j-1}(x)$ for all possible values of x. Firstly, let $x < i$. Then $\varepsilon_j\varepsilon_i(x) = \varepsilon_j(x) = x$, and $\varepsilon_i\varepsilon_{j-1}(x) = \varepsilon_i(x) = x$.
Secondly, let $i \leq x < j$. Then $\varepsilon_j\varepsilon_i(x) = \varepsilon_j(x+1) = x+1$, and $\varepsilon_i\varepsilon_{j-1}(x) = \varepsilon_i(x+1) = x+1$.
Finally, let $x \geq j$. Then $\varepsilon_j\varepsilon_i(x) = \varepsilon_j(x+1) = x+2$, and $\varepsilon_i\varepsilon_{j-1}(x) = \varepsilon_i(x+1) = x+2$.
Since we have shown the two composite functions to be equal on all possible values of x, we have verified the first identity. □

Lemma 3.1.5. *Every morphism $\alpha : [n] \to [m]$ in Δ has a unique epi-monic factorization $\alpha = \varepsilon\eta$, where the monic ε has a unique factorization consisting of face maps*

$$\varepsilon = \varepsilon_{i_1} \cdots \varepsilon_{i_s} \quad \textit{with } 0 \leq i_s < \cdots < i_1 < m$$

and the epi η has a unique factorization consisting of degeneracy maps

$$\eta = \eta_{j_1} \cdots \eta_{j_t} \quad \textit{with } 0 \leq j_1 \cdots < j_t < n.$$

Proof. Let $i_s < \cdots < i_1$ be the elements of $[m]$ which are not in the image of α, and let $j_1 < \cdots < j_t$ be the elements of $[n]$ such that $\alpha(j) = \alpha(j+1)$. Now, let $p = n - t = m - s$. The morphism α factors thus:

$$[n] \xrightarrow{\eta} [p] \xrightarrow{\varepsilon} [m]$$

Let $\eta = \eta_{j_1} \cdots \eta_{j_t}$ and $\varepsilon = \varepsilon_{i_1} \cdots \varepsilon_{i_s}$. Then η is the unique map from $[n]$ to $[m]$ which "hits" each of $j_1 < \cdots < j_t$ twice, and ε is the unique map from $[p]$ to $[m]$ which 'misses' all of $i_s < \cdots < i_1$. This makes their composition the unique map from $[n]$ to $[m]$ which hits all of $j_1 < \cdots < j_t$ twice and misses all of $i_s < \cdots < i_1$, which is α. The uniqueness comes from the uniqueness of the face and degeneracy maps. □

Proposition 3.1.6. *To determine a simplicial object A in $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A_0, A_1, \ldots$ of $\mathcal{A}$ together with face operators $\partial_i : A_n \to A_{n-1}$ and degeneracy operators $\sigma_i : A_n \to A_{n+1}$, where i can range from 0 to n. Furthermore, these operators satisfy the following "simplicial" identities:*

$$\partial_i\partial_j = \partial_{j-1}\partial_i \qquad \textit{if } \; i < j$$
$$\sigma_i\sigma_j = \sigma_{j+1}\sigma_i \qquad \textit{if } \; i \leq j$$

$$\partial_i\sigma_j = \begin{cases} \sigma_{j-1}\partial_i & \textit{if } i < j \\ \mathrm{id} & \textit{if } i = j \textit{ or } i = j+1 \\ \sigma_j\partial_{i-1} & \textit{if } i > j+1 \end{cases}.$$

Considering A as a functor from Δ to A, $\partial_i = A(\varepsilon_i)$ and $\sigma_i = A(\eta_i)$.

Proof. For the first direction of the proof, let A be a simplicial object in $\mathcal{A}$. If this is the case, we obtain the data above naturally by setting $A_n = A([n])$. The identities follow from the identities of Proposition 3.1.4, using the assumption that A is a contravariant functor.
Conversely, assume we are given the data above. Let α be a map in Δ, and write it in the form of Lemma 3.1.5: $\alpha = \varepsilon_{i_1} \cdots \eta_{j_t}$. Set $A(\alpha) = \sigma_{j_t} \cdots \partial_{i_1}$, where A is a map from Δ to $\mathcal{A}$. The fact that the simplicial identities of Proposition 3.1.4 govern the behaviour of morphism composition in Δ means that A is a contravariant functor; in other words, a simplicial object of $\mathcal{A}$. □

Corollary 3.1.7. *To determine a cosimplicial object A in $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A^0, A^1, \ldots,$ together with coface operators $\partial^i : A^{n-1} \to A^n$ and codegeneracy operators $\sigma^i : A^{n+1} \to A^n$ $(i = 0, \ldots, n)$ which satisfy the following "cosimplicial" identities:*

$$\partial^j \partial^i = \partial^i \partial^{j-1} \qquad \text{if} \quad i < j$$
$$\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \qquad \text{if} \quad i \leq j$$

$$\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ \partial^{j-1} \sigma^j & \text{if } i > j+1 \end{cases}$$

Example 3.1.8. Recall the definition of the *standard n-simplex*, denoted by Δ^n:

$$\Delta^n = \{(v_0, \ldots, v_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \Sigma x_i = 1\}$$

So that the standard 1-simplex is a line, the standard 2-simplex is a triangle, the standard 3-simplex is a tetrahedron, and so on. If we identify the elements of the ordered set $[n]$ with the vertices $v_0 = (1, 0, \ldots, 0), v_2 = (0, 1, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$ of Δ^n, a morphism $\alpha : [n] \to [p]$ in Δ induces a morphism α_* which sends the vertices of Δ^n to the vertices of Δ^p, defined by the rule $\alpha_*(v_i) = v_{\alpha(i)}$. Extending linearly, so that α_* has the whole simplex in its image rather than just the vertices, gives a map $\alpha_* : \Delta^n \to \Delta^p$. This means that the sequence $\Delta^0, \Delta^1, \ldots, \Delta^n, \ldots,$ together with the face and degeneracy maps obtained by factoring α in the usual manner, defines a cosimplicial object in the category of topological spaces. (We will, for brevity, refer to these as cosimplicial topological spaces. A similar convention will be used for other categories.)
Geometrically speaking, the face map ε_i induces ∂^i, the inclusion of Δ^{n-1} in Δ^n as the ith face, which is the face opposite the vertex v_i. The degeneracy map η_i induces σ^i, which is the projection $\Delta^{n+1} \to \Delta^n$ onto the ith face, identifying v_i and v_{i+1}.

Remark. This is example is, in some ways, prototypical; as one may expect, its geometric interpretation gave the general notions of face and degeneracy maps their names.

3.2 Geometric realization and combinatorial simplicial complexes

We proceed now to discuss a variety of simplicial structures, as a way of gaining intuition; chief among them are the notion of the *geometric realization* of a simplicial set, and the notion of a *combinatorial simplicial complex.*

Definition 3.2.1. Let X be a simplicial set. Its *geometric realization*, which we will denote by $|X|$, is a topological space constructed in the following manner.
For each $n \geq 0$, make the product $X_n \times \Delta^n$ into a topological space by viewing it as the disjoint union of a number of copies of Δ^n. We index these copies by the elements x of X_n, so that the number of copies is equal to the cardinality of X_n. Now, consider the disjoint union of each $X_n \times \Delta^n$ as a topological space, and denote this larger space by $\bar{X}$. Define the equivalence relation $\sim$ on $\bar{X}$ by the rule that $(x, s) \in X_m \times \Delta^m$ and $(y, t) \in X_n \times \Delta^n$ are equivalent if and only if there exists a map $\alpha : [m] \to [n]$ in Δ such that $\alpha_*(y) = x$ and $\alpha^*(s) = t$. That is to say:

$$(\alpha_*(y), s) \sim (y, \alpha^*(s))$$

The space $\bar{X}/\sim$ is the geometric realization $|X|$.

Remark. Geometric realization is the way of constructing topological spaces from simplicial objects discussed at the beginning of the section.

In constructing $|X|$, for obvious reasons, we refer to elements of the form $\sigma(y)$ for some y as *degenerate*. Elements not of this form are called non-degenerate. Lemma 14.2 in [14] implies that, for the purposes of forming $|X|$, elements of the form $(\sigma(y), t)$ can be ignored. Another consequence of Lemma 14.2 in [14] is that for any X, $|X|$ is a CW complex, whose n-cells are indexed by the non-degenerate elements of X_n.

Remark. Geometric realization defines a functor from $\mathcal{S}$**Set** to **Top**.

Example 3.2.2. Let G be a group, and let BG be the simplicial set defined as follows.
$BG_0 = \{1\}, BG_1 = G, \ldots, BG_n = G^n$, and so on. The face and degeneracy maps are defined by insertion of identities, deletion, and multiplication:

$$\sigma_i(g_1, \ldots, g_n) = (g_1, \cdots, g_i, 1, g_{i+1}, \ldots, g_n)$$

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

A routine verification shows that BG is indeed a simplicial set. The geometric realization $|BG|$ is called the *classifying space* of G.

Definition 3.2.3. A *combinatorial simplicial complex* is a collection, denoted K, of non-empty finite subsets of some set V (called the vertex set) such that if $\tau \subset \sigma \subset V$ and $\sigma \in K$, $\tau \in K$. If the set V is ordered, we call K an *ordered* combinatorial simplicial complex.

Example 3.2.4. Let K be an ordered combinatorial simplicial complex. From K, we can construct a simplicial set $SS(K)$ in the following manner.
Let $SS_n(K)$ be the set consisting of all ordered $(n+1)$-tuples $(v_0, \ldots, v_n)$ of vertices, possibly including repetitions, such that the underlying set of *distinct* vertices $\{v_0, \ldots, v_n\}$ is in K.
Let $\alpha : [n] \to [p]$ be a map in Δ, and define $\alpha_* : SS_p(K) \to SS_n(K)$ by $\alpha_*(v_0, \ldots, v_p) = (v_{\alpha(0)}, \ldots, v_{\alpha(n)}$. Note that $v_0 \leq \cdots \leq v_n$, and that

$$\partial_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$
$$\sigma_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_i, \ldots, v_n)$$

Intuitively, there is an obvious correspondence between combinatorial simplicial complexes and geometric n-simplices; if Δ^n is the standard n-simplex, there is a combinatorial simplicial complex K whose elements are the faces of Δ^n, and whose vertex set V corresponds to the vertices of Δ^n. The following two propositions make this explicit.

Proposition 3.2.5. *Let K be an ordered combinatorial simplicial complex. Then $SS(K)$ determines K, because there exists a bijection between K and the subset of $SS(K)$ consisting of non-degenerate elements.*

Proof. Let a member of K be denoted by $\{v_0, \ldots, v_n\}$. This is a set, so it can have no repetitions in its elements. Let f be the function which maps a tuple in $SS(K)$ to its underlying set; that is to say, it maps a tuple to an ordered set whose elements are the vertices which appear in the tuple. For example, $f(v_0, v_1, \ldots, v_i, v_i, \ldots, v_j, v_j, \ldots, v_n) = \{v_0, v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n\}$. Notice that, if any vertex appears multiple times, all of the positions in which it appears must be directly next to each other, because the tuple is ordered.
This function is surjective. However, if its domain is the whole of $SS(K)$, it fails to be injective: to give one example, the elements $(v_0, \ldots, v_i, \ldots, v_n)$ and $(v_0, \ldots, v_i, v_i, \ldots, v_n)$, distinct in $SS(K)$, are both mapped to $\{v_0, \ldots, v_i, \ldots, v_n\}$ in K. If we restrict the domain of f to non-degenerate elements

of $SS(K)$, however, there can be no repetitions, because any tuple with repetitions can be viewed as a member of the image of some appropriate degeneracy map. Therefore, f, when restricted to non-degenerate elements of $SS(K)$, is surjective and injective. □

Proposition 3.2.6. *Let K be the collection of non-empty subsets of a vertex set V having $n+1$ elements. The geometric realization $|SS(K)|$ is homeomorphic to the standard n-simplex Δ^n.*

Proof. In forming the geometric realization, we can ignore the degenerate elements of $SS(K)$, and, from the above proposition, we have a bijection between the set of non-degenerate elements of $SS(K)$ and the members of K. In general, the number of members of K with cardinality k is $\binom{n+1}{k}$, which is the number of k-faces in Δ^n. We now must check that the information on how to "glue" the faces together, which is encoded by the face maps, gives us a topological space homeomorphic to Δ^n. We will consider the situation for the elements of $SS_0(K)$ and $SS_1(K)$: the rest of the result follows by similar arguments.
Now, elements of $SS_0(K)$ are of the form $(v_0), (v_1), (v_2)$, and so on. Likewise, elements of $SS_1(K)$ are of the form $(v_0, v_1), (v_0, v_2), (v_1, v_2)$, and so on. In forming the geometric realization, we have $n+1$ copies of the 0-simplex Δ^0, and $\binom{n+1}{1}$ copies of the 1-simplex Δ^1. Consider one identification made in forming $|SS(K)|$:

$$(\partial_0(v_0, v_1) \times \Delta_0^0) \sim ((v_0, v_1) \times \partial^0(\Delta_0^0))$$

That is to say, Δ_0^0 is identified with its inclusion in a copy of Δ^1. But this happens for each copy of Δ^0, and going further, each copy of Δ^1 is identified with its inclusion in a copy of Δ^2. This continues inductively until all is included in Δ^n; therefore, $|SS(K)| \cong \Delta^n$. □

Definition 3.2.7. Let Δ_S be the subcategory of Δ whose morphisms are the injective morphisms in Δ. A *semi-simplicial object* K in a category $\mathcal{A}$ is a contravariant functor from Δ_S to $\mathcal{A}$.

Example 3.2.8. Every simplicial object becomes a semi-simplicial object if we simply ignore the degeneracy maps.

We shall now discuss various operations on simplicial objects, beginning with a way to construct a chain complex from a simplicial object in ***R*-mod**.

3.3 Operations on simplicial objects

Definition 3.3.1. Let A be a simplicial or semi-simplicial R-module. The *unnormalized chain complex* associated to A, which we will denote by $C(A)$ or simply C, is defined as follows. In each degree, $C_n = A_n$, and the differential $d_n : C_n \to C_{n-1}$ is given by the alternating sum of the face operators $\partial_i : C_n \to C_{n-1}$:

$$d_n = \partial_0 - \partial_1 + \cdots + (-1)^n \partial_n.$$

Note that the unnormalized complex does meet the definition of a chain complex, because the simplicial identities ensure that d squares to 0. We will demonstrate this for the first two compositions. Firstly, consider $d_0 \circ d_1 = (\partial_0)(\partial_0 - \partial_1)$. This is equal to $\partial_0\partial_0 - \partial_0\partial_1$, but, by the simplicial identities, $\partial_0\partial_1 = \partial_0\partial_0$. So the composition is 0.
Secondly, consider $d_1 \circ d_2$:

$$\begin{aligned} d_1 \circ d_2 &= (\partial_0 - \partial_1)(\partial_0 - \partial_1 + \partial_2) \\ &= \partial_0\partial_0 - \partial_0\partial_1 + \partial_0\partial_2 - \partial_1\partial_0 + \partial_1\partial_1 - \partial_1\partial_2 \end{aligned}$$

By the result for $d_0 \circ d_1$ and the simplicial identities, this expression is equal to $\partial_1\partial_0 - \partial_1\partial_0 + \partial_1\partial_1 - \partial_1\partial_1 = 0$. The case for any other composition of differentials may be verified similarly.

Remark. Recall Theorem 2.1.2: although we discussed simplicial R-modules in the previous definition, and will continue to do so, everything we write applies, via the Freyd-Mitchell Embedding Theorem, to abelian categories in general.

A useful observation is that applying a functor $F : \mathcal{A} \to \mathcal{B}$ to a simplicial object A in $\mathcal{A}$ returns a simplicial object in $\mathcal{B}$. This is the case because composing F with the contravariant functor $A : \Delta \to \mathcal{A}$ gives another contravariant functor $\Delta \to \mathcal{B}$, which is the definition of a simplicial object in $\mathcal{B}$. This observation is also true of cosimplicial and semi-simplicial objects, by similar reasoning.

Example 3.3.2. Let R be a ring. Then the functor which sends a set X to the free R-module with basis X, denoted $R[X]$, is a functor mapping from **Set** to ***R*-mod**. (This is another instance of the situation discussed in Example A.3.2.) By the above observation, whenever $X = \{X_n\}$ is a simplicial set, $R[X] = \{R[X_n]\}$ is a simplicial R-module. In light of this, we define the *simplicial homology* $H(X; R)$ of a simplicial set X as the homology of the unnormalized chain complex of the simplicial module $R[X]$.

Example 3.3.3. Let X be a topological space, and consider the contravariant functor $\mathrm{Hom}_{\mathbf{Top}}(-, X)$, which maps from **Top** to **Set**. If we apply this functor to the cosimplicial topological space $\{\Delta^n\}$, the result is a simplicial set $S(X)$ with $S_n(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$. We call $S(X)$ the *singular simplicial set* of X. The singular chain complex associated to X, used to calcuate the singular homology of X is, by definition, the unnormalized chain complex of $R[S(X)]$.

Remark. There exists a natural continuous map $|S(X)| \to X$ which is a homotopy equivalence if and only if X is homotopy equivalent to a CW complex. It is induced by maps from $S_n(X) \times \Delta^n$ to X which map (f, t) to $f(t)$. In fact, the functor S and the geometric realization functor form an adjoint pair: for every simplicial set Y, $\mathrm{Hom}_{\mathbf{Top}}(|Y|, X) \cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(Y, S(X))$. These assertions are proven in Chapter III of [14].

Example 3.3.4. For every $n \geq 0$, a simplicial set $\Delta[n]$ is determined by the functor $\mathrm{Hom}_\Delta(-, [n])$. These are universal in the following sense. For every simplicial set A, the Yoneda Lemma (discussed in Section A.5) gives a one-to-one correspondence between elements $a \in A_n$ and natural transformations from $\Delta[n]$ to A. The morphism f determines the element $a_f = f(\mathrm{id}_{[n]})$. Conversely, f_a is defined on $\lambda \in \mathrm{Hom}_\Delta([m], [n])$ by $f_a(\lambda) = \lambda^*(a) \in A_m$.

Definition 3.3.5. Let A and B be simplicial objects in a category with products. Their *product* $A \times B$ is defined as follows. In each degree n, $(A \times B)_n = A_n \times B_n$. The face and degeneracy operators are defined componentwise; for $(a, b) \in A \times B, \partial_i(a, b) = (\partial_i a, \partial_i b)$ and $\sigma_i(a, b) = (\sigma_i a, \sigma_i b)$.

Remark. We can also define $A \times B$ if B is a simplicial set and A is a simplicial object in a category with products. We do this by setting $(A \times B)_n$ equal to the product of m copies of A_n, where m is the cardinality of the set B_n. This is most useful when each B_n is a finite set, in which case the category containing A only has to have finite products.

From a homotopy-theoretic point of view, it is useful to restrict attention to a certain class of simplicial sets, which we will now define.

Definition 3.3.6. Let X be a simplicial set satisfying the following criterion, which is known as the *Kan condition*: for every n and k with $0 \leq k \leq n+1$, we have that if $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in X_n$ are such that $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ $(i, j \neq 0)$, then there exists some $y \in X_{n+1}$ such that $\partial_i(y) = x_i$ for all $i \neq k$. If X satisfies the Kan condition, we call X a *fibrant simplicial set.*

Lemma 3.3.7. *Let G be a simplicial object in **Grp**, that is to say, a simplicial group. Then G is fibrant when considered as a simplicial set. In particular, this means that simplicial abelian groups and simplicial R-modules are all fibrant when considered as simplicial sets.*

Proof. Suppose we are given $x_i \in G_n$ $(i \neq k)$ such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$. We will use induction on r to find an element $g_r \in G_{n+1}$ such that $\partial_i(g_r) = x_i$ for all $i \leq r, i \neq k$. We begin the induction by setting $g_{-1} = 1 \in G_{n+1}$ and suppose, for the inductive step, that the case where $g = g_{r-1}$ is given. If $r = k$, set $g_r = g$. If $r \neq k$, consider $u = x_r^{-1}(\partial_r(g))$. If $i < r$ and $i \neq k$, then $\partial_i(u) = 1$ and hence, $\partial_i(\sigma_r u) = 1$. Hence $g_r = g(\sigma_r u)^{-1}$ satisfies the inductive hypothesis. The element $y = g_n$ therefore, has $\partial_i(y) = x_i$ for all $i \neq k$, so the Kan condition is satisfied. □

Definition 3.3.8. A map $\pi : E \to B$ of simplicial sets is called a *fibration* if, for every $n, b \in B_{n+1}$ and $k \leq n+1$, if $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in E_n$ are such that $\partial_i b = \pi(x_i)$ and $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j (i, j \neq k)$, then there exists an element $y \in E_{n+1}$ such that $\pi(y) = b$ and $\partial_i(y) = x_i$ for all $i \neq k$.

Remark. The notion of a fibration generalizes the notion of a fibrant simplicial set X; another way of saying X is fibrant is to say that X is a simplicial set such that $X \to *$ is a fibration, where $*$ means the constant simplicial object which sends every object in Δ to a singleton set and every morphism in Δ to the identity on that set. To see this, let $X \to *$ be a fibration. This means that for every $n, b \in *$ and $k \leq n+1$, if $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in X_n$ are such that $\partial_i * = *$ and $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j (i, j \neq k)$ then there exists an element $y \in X_{n+1}$ such that $\pi(y) = *$ and $\partial_i(y) = x_i$. But these are exactly the conditions for X to be fibrant.

Consider the involution (in this context, 'involution' means a functor which is its own inverse) denoted by $\check{}$ on Δ, and defined as follows. For every $[n]$ in Δ, $[\check{n}] = [n]$. Its effect on the morphisms is as follows:

$$\check{\partial}_i = \partial_{n-i} : [n-1] \to [n]$$
$$\check{\sigma}_i = \sigma_{n-i} : [n+1] \to [n]$$

Intuitively, we may think of the involution as reversing the ordering of $[n] = (0 < 1 < \cdots < n)$, which results in the ordering$(n < \cdots < 1 < 0)$. For a map $\alpha : [m] \to [n], \check{\alpha}(i) = n - \alpha(m-i)$. This motivates the following definition.

Definition 3.3.9. Let A be a simplicial object in a category $\mathcal{A}$. Then the *front-to-back* dual $\check{A}$ is the composition of A with the involution $\check{}$ defined above.

3.4 Simplicial homotopy groups and simplicially homotopic maps

Above, before we defined the notion of a fibrant simplicial set, it was stated that such simplicial sets are useful from a 'homotopy-theoretic' point of view, implying that we have some idea of homotopy theory for the category $\mathcal{S}$**Set**, just as we have for **Top.** We will now discuss some of this theory for $\mathcal{S}$**Set**, beginning with the simplicial analogue of the homotopy groups π_n of a based space $(X, x_0.)$

Definition 3.4.1. Let X be a fibrant simplicial set and choose a basepoint $* \in X_0$, forming the based fibrant simplicial set $(X, *)$. In what follows we will, in a useful abuse of notation, write $*$ for the element $\sigma_0^n(*)$ of X_n. Form the family of subsets $Z_n = \{x \in X_n \mid \partial_i(x) = * \text{ for all } i = 0, \ldots, n\}$. Now, let x and x' be two elements of Z_n. We say that x and x' are *homotopic*, and write $x \sim x'$, if there exists an element $y \in X_{n+1}$ (which we call a *homotopy* from x to x') such that the following is true:

$$\partial_i(y) = \begin{cases} * & \text{if } i < n \\ x & \text{if } i = n \\ x' & \text{if } i = n+1 \end{cases}$$

Remark. Notice the connection with the notion of homotopy of paths from topology. With this in mind, we would expect to be able to quotient by the relation $\sim$, obtaining a group. This is indeed the case, but we must first verify that $\sim$ is an equivalence relation.

Lemma 3.4.2. *Let X be a fibrant simplicial set. Then the relation $\sim$ is an equivalence relation.*

Proof. We must first verify that $\sim$ is reflexive. This is true since $y = \sigma_n x$ is a homotopy from x to itself. To see that $\sim$ is symmetric and transitive, suppose we are given homotopies y' and y'' from x to x' and from x to x''. Because X is fibrant, we can apply the Kan condition to the elements $*, \ldots, *, y', y''$ of X_{n+1} with $k = n+2$ yields an element $z \in X_{n+2}$ with $\partial_n z = y', \partial_{n+1} z = y''$, and $\partial_i z = *$ for $i < n$. Then the element $y = \partial_{n+2} z$ is a homotopy from x' to x''. Therefore, $x' \sim x''$ and hence, $\sim$ is both symmetric and transitive. □

Definition 3.4.3. For a fibrant simplicial set X, we set $\pi_n(X) = Z_n / \sim$.

Remark. If X is fibrant, then, for all $n \geq 0$, $\pi_n(X)$ is isomorphic to the topological homotopy group of the geometric realization of X. (See [14], Theorem 16.1.) Since $\pi_n(|X|) \cong \pi_n(S(X))$, where $S(X)$ is the singular simplicial set associated to the space $|X|$, we can define $\pi_n(X)$ to be $\pi_n(S(X))$ when X is not fibrant. Therefore, $\pi_1(X)$ is a group, and $\pi_n(X)$ is an abelian group for $n \geq 2$. For an explicit description of the group structure of $\pi_n(X)$, see Chapter 1 of [9].

Definition 3.4.4. For X a simplicial set, we define $\pi_0(X)$ to be $X_0/\sim$, where, for each $y \in X_1$, we define $\sim$ by $\partial_0(y) \sim \partial_1(y)$. As is the case for topological spaces, $\pi_0(X)$ need only be a set and is not necessarily a group.

Example 3.4.5. Recall the notion of the classifying space $|BG|$ of a group G. Our goal is to compute the simplicial homotopy groups of the simplicial set BG, which means we will also have computed the topological homotopy groups of the space $|BG|$. Firstly, let $n = 0$. Then $Z_0 = BG_0 = \{1\}$, so $\pi_0(BG)$ is the trivial group. Now let $n > 1$. Then $Z_n = \{g \in G^n \mid \partial_i(g) = (1, 1, \ldots, 1) = \mathrm{id} \in G^{n-1}\}$. This is because our only choice of basepoint in BG_0 is the identity on G, and degeneracies simply insert a copy of the identity. By inspection, the only element which is sent to the identity in G^{n-1} by every face map is the identity on G^n. So, for $n > 1$, $Z_n = \pi_n(X) = \{1\}$. Finally, consider the case where $n = 1$. Here, $Z_1 = \{g \in G \mid \partial_i(g) = 1\}$, but this is clearly the whole of G since $G^0 = \{1\}$. From this, we deduce that:

$$\pi_n(|BG|) = \pi_n(BG) = \begin{cases} G & \text{if } n = 1 \\ \{1\} & \text{if } n \neq 1 \end{cases}$$

Definition 3.4.6. Let G be a group. An *Eilenberg-MacLane space* of type $K(G, n)$ is a topological space X such that $\pi_n(X) = G$ and $\pi_i(X) = 0$ for $i \neq n$. Due to the isomorphisms between simplicial and topological homotopy groups, we will sometimes also refer to a simplicial set with geometric realization X using the same terminology. The immediately preceding example shows that $|BG|$ (or BG) is an Eilenberg-MacLane space of type $K(G, 1)$.

If G is a simplicial group or a simplicial module, considered as a fibrant simplicial set (by Lemma 3.3.7) with basepoint $* = 1$, it is helpful to consider the subgroups

$$N_n(G) = \{x \in G_n \mid \partial_i x = 1 \text{ for all } i \neq n\}$$

In light of this, $Z_n = \ker(\partial_n : N_n \to N_{n-1})$ and we define the image of the homomorphism $\partial_{n+1} : N_{n+1} \to N_n$ to be the subgroup $B_n = \{x \mid x \sim 1\}$. Thus, $\pi_n(G)$ is the same as the homology group Z_n/B_n of the corresponding chain complex:

$$\{1\} \leftarrow N_0 \xleftarrow{\partial_1} N_1 \xleftarrow{\partial_2} \cdots$$

With this observation, we can generalize the definition of the homotopy groups of a simplicial object to any abelian category $\mathcal{A}$, even if the objects of $\mathcal{A}$ have no underlying set structure. (If this is the case, our previous definition is not valid, since it relies on identifying and comparing specific elements of sets.)

Definition 3.4.7. Let A be a simplicial object in an abelian category $\mathcal{A}$. The *normalized* chain complex $N(A)$ is defined as follows. In each degree, $N_n(A) = \bigcap_{i=0}^{n-1} \ker(\partial_i : A_n \to A_{n-1})$. The differential d is given by $(-1)^n \partial_n$. By construction, $N(A)$ is a chain subcomplex of the unnormalized complex $C(A)$, and we define the simplicial homotopy groups thus:

$$\pi_n(A) = H_n(N(A)).$$

If $\mathcal{A}$ is **Ab** or ***R*-mod**, this definition is the same as Definition 3.4.3, which we obtained by regarding A as a fibrant simplicial set.

Definition 3.4.8. Consider $C(A)$, the unnormalized chain complex of a simplicial object A. Let $D(A)$ denote the 'degenerate' chain subcomplex of $C(A)$ which is generated by the images of the degeneracies σ_i so that $D_n(A) = \sum \sigma_i(C_{n-1}(A))$.

Lemma 3.4.9. *Let A be a simplicial object, let $C(A)$ be the associated unnormalized chain complex, let $D(A)$ be the associated degenerate chain complex, and let $N(A)$ be the normalized chain complex. Then we have the following identity:*

$$C(A) = N(A) \oplus D(A)$$

Hence $N(A) \cong C(A)/D(A)$.

Proof. We will use an element-theoretic proof, which is valid by the Freyd-Mitchell Embedding Theorem 2.1.2. A general element of $D_n(A)$, which we will denote by y, is a sum $y = \Sigma \sigma_j(x_j)$, with $x_i \in C_{n-1}(A)$. If $y \in N_n(A)$ and i is the smallest integer such that $\sigma_i(x_i) \neq 0$, then $y = y - \sigma_i \partial_i(y) = \Sigma_{j>i} \sigma_j(x'_j)$. By induction, $y = 0$. Hence, $D_n \cap N_n = 0$.
To see that $D_n \oplus N_n = C_n$, we choose some $y \in C_n$ and use downward induction on the smallest integer j such that $\partial_j(y) \neq 0$. The element y is the same, modulo D_n, as $y' = y - \sigma_j \partial_j(y)$, and for $i < j$ the simplicial identities yield

$$\partial_i(y') = \partial_i(y) - \sigma_{j-1} \partial_{j-1} \partial_i(y) = 0$$

Since $\partial_i(y') = 0$, y' is equal, modulo D_n, to an element of N_n by induction; thus, $D_n \oplus N_n = C_n$. □

Theorem 3.4.10. *The homotopy $\pi_*(A)$ of a simplicial object A in **R-mod** is naturally isomorphic to the homology $H_*(C)$ of the unnormalized chain complex $C = C(A)$:*

$$\pi_*(A) = H_*(N(A)) \cong H_*(C(A)).$$

Proof. By the lemma above, it suffices to show that $D(A)$ is acyclic, as homology is preserved by taking direct sums. To see this, filter $D(A)$ by setting $F_0 D_n = 0, F_p D_n = D_n$ if $n \geq p$, and $F_p D_n = \sigma_0(C_{n-1}) + \cdots + \sigma_p(C_{n-1})$ otherwise. The simplicial identities show that each $F_p D$ is a chain subcomplex of $D(A)$. Recall from Section 2.4 that this filtration meets the criteria to be canonically bounded, which means that we have a convergent first quadrant spectral sequence

$$E^1_{pq} = H_{p+q}(F_p D / F_{p-1} D) \implies H_{p+q}(D).$$

It shall therefore suffice to show that each subcomplex $F_p D / F_{p-1} D$ is acyclic.
First, note that $(F_p D / F_{p-1} D)_n$ is a quotient of $\sigma_p(C_{n-1})$ and is zero for $n < p$. In element-theoretic language, if $x \in C_{n-1}(A)$, the simplicial identities yield the following equalities in $F_p D / F_{p-1} D$:

$$d\sigma_p(x) = \sum_{i=p+2}^{n} (-1)^i \sigma_p \partial_{i-1}(x)$$

$$\begin{aligned} d\sigma_p^2(x) + \sigma_p d\sigma_p(x) &= \sum_{i=p+2}^{n+1} (-1)^i \sigma_p \partial_{i-1} \sigma_p(x) + \sum_{i=p+2}^{n} (-1)^i \sigma_p^2 \partial_{i-1}(x) \\ &= (-1)^p \sigma_p(x) \end{aligned}$$

Hence, $\{s_n = (-1)^p \sigma_p\}$ is a chain contraction of the identity map on $F_pD/F_{p-1}D$ which is null homotopic. Hence, by Lemma 2.3.5, $F_pD/F_{p-1}D$ is acyclic. □

Example 3.4.11. Let X be a fibrant simplicial set, and let $\mathbb{Z}[X]$ be the simplicial abelian group which, in degree n, is the free abelian group with basis X_n. The map of simplicial sets $h : X \to \mathbb{Z}[X]$ which sends $x \in X$ to the corresponding basis element in $\mathbb{Z}[X]$ is called the *Hurewicz homomorphism* since, on homotopy groups, it is the map

$$\pi_*(X) \to \pi_*(\mathbb{Z}[X]) \cong H_*(\mathbb{Z}[X]) = H_*(X; \mathbb{Z})$$

which corresponds to the topological Hurewicz homomorphism $\pi_*(|X|) \to H_*(|X|; \mathbb{Z})$. To see this, represent an element φ of $\pi_n(|X|)$ by a map $f : \Delta^n \to |X|$ and view f as an element of $S_n(|X|)$. The class of $h(f)$ in $H_n(\mathbb{Z}[S(|X|)]) = H_n(|X|; \mathbb{Z})$ is the topological Hurewicz element $h(\varphi)$.

Proposition 3.4.12. *Let A be a simplicial abelian group. Then the Hurewicz map $h_* : \pi_*(A) \to H_*(A; \mathbb{Z}) = H_*(|A|; \mathbb{Z})$ is a split monomorphism; that is to say, there exists a map $r : H_*(A; \mathbb{Z}) \to \pi_*(A)$ such that $r \circ h_* = \mathrm{id}_{\pi_*(A)}$.*

Proof. For every abelian group G, there is a natural surjection from $\mathbb{Z}[G]$, the free abelian group generated by the elements of G, to G, defined to be the identity on the basis elements. There therefore exists a natural surjection of simplicial abelian groups $j : \mathbb{Z}[A] \to A$. Thus, the composite simplicial set map $j \circ h : A \to \mathbb{Z}[A] \to A$ is equal to the identity on A, so, passing to maps of homotopy groups, $j_* h_* : \pi_*(A) \to \pi_*(\mathbb{Z}[A]) \to \pi_*(A)$ is the identity, by functoriality. □

Having defined the notion of chain homotopic maps in the previous section, and, bearing in mind the classical notion of homotopy of maps from topology, we proceed now to define the equivalent notion for simplicial maps.

Definition 3.4.13. Let A and B be simplicial objects in a category $\mathcal{A}$. Two simplicial maps f and g, both mapping from A to B, are said to be *homotopic* if there are morphisms $h_i : A_n \to B_{n+1}$ in $\mathcal{A}$ $(i = 0, \ldots, n)$ such that $\partial_0 h_0 = f$ and $\partial_{n+1} h_n = g$. In addition, the following should hold:

$$\partial_i h_j = \begin{cases} h_{j-1}\partial_i & \text{if } i < j \\ \partial_i h_{i-1} & \text{if } i = j \neq 0 \\ h_j \partial_{i-1} & \text{if } i > j+1 \end{cases},$$

$$\sigma_i h_j = \begin{cases} h_{j+1}\sigma_i & \text{if } i \leq j \\ h_j \sigma_{i-1} & \text{if } i > j \end{cases}$$

We refer to the family of maps $\{h_j\}$ as a *simplicial homotopy* from f to g and write $f \simeq g$.

Proposition 3.4.14. *Simplicial homotopy is an additive equivalence relation on maps between simplicial objects in **R**-**mod**. That is to say, if f, f', g, and g' are simplicial maps from A to B the following conditions hold:*

1. $f \simeq f$,
2. *If $f \simeq g$ and $f' \simeq g'$, then $(f + f') \simeq (g + g')$,*

3. If $f \simeq g$, then $(-f) \simeq (-g), (f-g) \simeq 0$ and $g \simeq f$,

4. If $f \simeq g$ and $g \simeq h$, $f \simeq h$.

Proof. 1. Let $h_j = \sigma_j f$. Now, we simply check that each condition for $\{h_j\}$ to be a homotopy is true. Firstly, we have, by the simplicial identities,

$$\partial_0 h_0 = \partial_0 \sigma_0 = \text{id} \circ f = f,$$

amd, again by the simplicial identities:

$$\partial_{n+1} h_n = \partial_{n+1} \sigma_n f = \text{id} \circ f.$$

Next, we check the equations for $\partial_i h_j$ above. Firstly, let $i < j$. Then we have, by the simplicial identities,

$$\partial_i h_j = \partial_i \sigma_j f = \sigma_{j-1} \partial_i f.$$

However, because f is a simplicial map, it commutes with all face and degeneracy maps, so the above is equal to $\sigma_{j-1} f \partial_i$ which is itself equal to $h_{j-1}\partial_i$, as required.
Secondly, let $i = j \neq 0$. Then $\partial_i h_j = \partial_i \sigma_j f = f$, by the simplicial identities. We also have $\partial_i h_{i-1} = \partial_i \sigma_{i-1} f = f$, again by the simplicial identities. So $\partial_i h_j = \partial_i h_{i-1}$, as required.
Thirdly, let $i > j+1$. By a combination of the simplicial identities and the assumption that f is a simplicial map, we have

$$\partial_i h_j = \partial_i \sigma_j f = \sigma_j \partial_{i-1} f = \sigma_j f \partial_{i-1} = h_j \partial_{i-1}$$

as required.
To finish the proof that simplicial homotopy is a reflexive relation, we must check that the equations above for $\sigma_i h_j$ hold. First, let $i \leq j$. Then, again by the simplicial identities and the assumption that f is a simplicial map, we have

$$\sigma_i h_j = \sigma_i \sigma_j f = \sigma_{j+1} \sigma_i f = \sigma_{j+1} f \sigma_i = h_{j+1} \sigma_i$$

as required.
Finally, let $i > j$. By the definition of h_j, we have $\sigma_i h_j = \sigma_i \sigma_j f$. This expression cannot be rewritten directly by using the simplicial identities; however, we require it to be equal to $h_j \sigma_{i-1}$. Once again, by the simplicial identities and the assumption that f is a simplicial map, we have

$$H_j = \sigma_{i-1} = \sigma_j f \sigma_{-1} = \sigma_j \sigma_{i-1} f = \sigma_i \sigma_j f$$

as required. Therefore, $\{h_j\} = \{\sigma_j f\}$ fulfils every condition to be a homotopy from f to f: $f \simeq f$.

2. We have that $f \simeq g$ and $f' \simeq g'$, so we have a homotopy $\{h_j\}$ from f to g and a homotopy $\{h'_j\}$ from f' to g'. We wish to construct a homotopy $\{H_j\}$ from $f + f'$ to $g + g'$. Simply let $\{H_j\} = \{h_j + h'_j\}$. Then, by the assumption that $\{h_j\}$ and $\{h'_j\}$ are homotopies and by the fact that function composition distributes over addition in ***R*-mod**, we have $\partial_0(h_0 + h'_0) = f + f'$ and $\partial_{n+1}(h_n + h'_n) = g + g'$. The rest of the conditions for $\{H_j\}$ to be a homotopy can be verified in an exactly analogous fashion.

3. Assuming that $f \simeq g$, we must verify three things. The first is that $(-f) \simeq (-g)$. To see this, let $\{h_j\}$, be a homotopy from f to g, and consider $\{-h_j\}$. Because $\{h_j\}$ is a homotopy, we have $\sigma_0(-h_0) = -\sigma_0 h_0 = -f$ and $\partial_{n+1}(-h_n) = -\partial_{n+1} h_n = -g$. The rest of the conditions for $\{-h_j\}$ to be a homotopy from $(-f)$ to $(-g)$ can be verified similarly.
The second thing to be verified is that $f - g \simeq 0$. To see this, let $h'_j = (h_j - \sigma_j g)$, where h_j

is a homotopy from f to g. We claim that $\{h'_j\}$ is a homotopy from $f-g$ to 0. This can be verified by checking each condition. For example, $\partial_0 h'_0 = \partial_0(h_0 - \sigma_0 g) = \partial_0\sigma_0 - g = f - g$, by the simplicial identities and the conditions for $\{h_j\}$ to be a homotopy from f to g. By a similar argument, $\partial_{n+1}h'_n = g - g = 0$. As another example, we shall check the first equation for $\partial_i h_j$. First, let $i < j$. Then $\partial_i h'_j = \partial_i(h_j - \sigma_j g) = \partial_i h_j - \partial_i \sigma_j g = h_{j-1}\partial_i - \sigma_{j-1} g \partial_i = h'_{j-1}\partial_i$, by the simplicial identities, the assumption that g is a simplicial map, and by the assumption that $\{h_j\}$ is a homotopy. The rest of the conditions for $\{h'_j\}$ to be a homotopy can be verified similarly.
The final thing to be verified is that $g \simeq f$. This can be done using the results we have proved so far. Firstly, we know that $f \simeq g \implies (f-g) \simeq 0$. We also know that this implies that $(g-f) \simeq 0$. By the first part of this proof, we know that $f \simeq f$, so, by the second part, we can add f to both sides of $(g-f) \simeq 0$, which gives $g \simeq f$, as required.

4. This can be verified easily using results already proven. If $f \simeq g$ and $g \simeq h$, $(g-f) \simeq 0$ and $(h-g) \simeq 0$. By part two above, we can add these two equivalances to obtain $h - f \simeq 0$ and, adding f to both sides of this, we obtain $h \simeq f$. So $f \simeq h$ as required.

□

We now proceed to a key lemma, forming part of the Dold-Kan Correspondence, which establishes a connection between simplicial homotopy and chain homotopy.

Lemma 3.4.15. *Let f and g be two simplicially homotopic maps $A \to B$, where A and B are simplicial R-modules. Then $f_*, g_* : N(A) \to N(B)$, the induced maps between the corresponding normalized chain complexes, are chain homotopic.*

Proof. By Proposition 3.4.14 above, we can assume that $f = 0$, by replacing g with $g - f$. Let $\{h_j\}$ be a simplicial homotopy from 0 to g, and define $s_n = \Sigma(-1)^j h_j$, which is a map from A_n to B_{n+1}. Now, the image of the restriction of s_n to $Z_n(A)$ is contained in $Z_n(B)$, so it is enough to show that $ds + sd = -g$. To see this, first consider the expression $\partial_{n+1}s_n - s_{n-1}\partial_n$. We have the expansion

$$\partial_{n+1}(h_0 - h_1 + h_2 - h_3 + \cdots (-1)^n h_n) - (h_0 - h_1 + h_2 - h_3 + \cdots + (-1)^{n-1}h_{n-1})\partial_n$$

by the definition of s_n. Expanding the brackets yields

$$\partial_{n+1}h_0 - \partial_{n+1}h_1 + \partial_{n+1}h_2 - \cdots + (-1)^n\partial_{n+1}h_n - h_0\partial_n + h_1\partial_n - h_2\partial_n + \cdots - (-1)^n h_{n-1}\partial_n.$$

The identities of Definition 3.4.13 cause cancellations such that the only remaining term is $(-1)^n\partial_{n+1}h_n$, but, again by 3.4.13, this is simply $(-1)^n g$. So we have the identity

$$\partial_{n+1} - s_{n-1}\partial_n = (-1)^n g \tag{3.1}$$

We will use this identity to show that $ds + sd = -g$. First, recall the definition of d: $d_n = (-1)_n\partial_n$. Therefore,

$$ds + sd = (-1)^{n+1}\partial_n s_n + s_{n-1}(-1)^n\partial_n.$$

Taking out a common factor of $(-1)^n$, we have

$$(-1)^n(-\partial_n s_n + s_{n-1}\partial_n) = (-1)^n(-1)^{n+1}g$$

by (3.1) above. But this is equal to $(-1)^{2n+1}g = -g$. Therefore, $\{s_n\}$ is a chain homotopy from 0_* to g_*. □

We will shortly define a notion which will be key to the proof of the Dold-Kan Correspondence. We require a preliminary discussion first: consider the functor $P : \Delta \to \Delta$ such that $P[n] = [n+1]$, and such that the map $\varepsilon_0 : [n] \to [n+1] = P[n]$ is a natural transformation from the identity functor on Δ to P. At the level of the elements of the objects of Δ, we obtain $P[n]$ from $[n]$ by formally adding an initial element, which will be denoted by $0'$, to each set $[n]$ and identifying the new set $(0' < 0 < \cdots < n)$ with $[n+1]$. The effect that this has on the face and degeneracy maps is to translate everything one position to the right; $P(\varepsilon_i) = \varepsilon_{i+1}$ and $P(\eta_i) = \eta_{i+1}$. Having defined the functor P, we can proceed.

Definition 3.4.16. Let A be a simplicial object in a category $\mathcal{A}$. The *path space*, denoted PA, is the simplicial object obtained by composing A with P. Thus $(PA)_n = A_{n+1}$, ∂_i in A is ∂_{i+1} in PA, and likewise for the degeneracy maps. Moreover, in accordance with ε_0 being a natural transformation $\mathrm{id}_\Delta \to P$, the maps $\partial_0 : A_{n+1} \to A_n$ form a simplicial map from PA to A.

The notion of the path space is important to the proof of the Dold-Kan Correspondence via the following lemma.

Lemma 3.4.17. *Let A be a simplicial R-module, and let ΛA denote the simplicial R-module which is the kernel of the simplicial map $\partial_0 : PA \to A$. Furthermore, let $A_0[1]$ denote the chain complex which has A_0 in degree -1 and 0 elsewhere. Then $N_n(\Lambda A) \cong N_{n+1}(A)$ for all $n \geq 0$, and we have the exact sequence:*

$$0 \to A_0[1] \to NA[1] \to N(\Lambda A) \to 0,$$

where, as in Definition 2.1.9, $NA[1]$, is NA translated by one degree.

Proof. Let $n \geq 0$. The simplicial R-module $\Lambda A = \ker \partial_0$, where ∂_0 is the simplicial map from $PA \to A$ formed by each $\partial_0 : A_{n+1} \to A_n$. Therefore

$$N_n(\Lambda A) = \bigcap_{i=1}^{n} \ker(\partial_i : A_{n+1} \to A_n),$$

with differential $d = (-1)^{n+1}\partial_{n+1}$. Because the simplicial object to which N is being applied is ΛA, everything must be in the kernel of ∂_0. So we can include $\ker \partial_0$ in the intersection, which gives

$$\bigcap_{i=0}^{n} \ker(\partial_i : A_{n+1} \to A_n)$$

which is the definition of $N_{n+1}(A)$.

Now, to prove the existence of the exact sequence, we will compare $NA[1]$ and $N(\Lambda A)$. By juxtaposing NA and $NA[1]$ like so

$$\begin{aligned} &\cdots \to NA_3 \to NA_2 \to NA_1 \to NA_0 \to 0 \\ &\cdots \to NA_4 \to NA_3 \to NA_2 \to NA_1 \to NA_0 \to 0, \end{aligned}$$

we see that $NA[1]$ has NA_0, which is isomorphic to A_0 by the definition of N, in degree -1. The simplicial object $N(\Lambda A)$, however, has 0 in degree -1, because it is a subobject of the path space; for the path space to have a non-zero entry in degree -1, it would be necessary for a set $[-1]$ to exist in Δ. As this is clearly not the case, we can conclude that $N(\Lambda A)$ has 0 in degree -1. As $NA[1]$ and $N(\Lambda A)$ are isomorphic except for degree -1, we can obtain $N(\Lambda A)$ from $NA[1]$ by quotienting by $A_0[1]$. Therefore, we can form the exact sequence

$$0 \to A_0[1] \to NA[1] \to N(\Lambda A) \to 0,$$

which completes the proof. □

Bibliographical Note. The preceding section is based primarily on Chapter 8 of [17]. Some exercises in Weibel become propositions or lemmas in the present paper, with the proofs given being solutions of the exercises: specifically, Proposition 3.4.14 and Lemma 3.4.17. Chapter 1 of [14] was also consulted.

4 The Dold-Kan Correspondence

In light of all the preliminary work done so far, we are now ready to tackle the Dold-Kan Correspondence itself. The theorem contains enough information to make attempts to prove it all at once difficult to follow; we will therefore state it first, then prove it in a piecewise manner. Before the statement, recall that, given a simplicial R-module, the normalized chain complex $N(A)$ gives rise to a functor N which maps from the category of simplicial R-modules to the category of chain complexes of R-modules.

4.1 Statement of the theorem

Theorem 4.1.1 (Dold-Kan). *Let $\mathcal{S}\boldsymbol{R}$-**mod** be the category of simplicial R-modules, and let $\mathbf{Ch}_{\geq 0}(R)$ be the category of chain complexes in $\boldsymbol{R}$-**mod**, with $C_n = 0$ for all $n < 0$. The functor N which sends a simplicial R-module to its normalized chain complex is an equivalence of categories between $\mathcal{S}\boldsymbol{R}$-**mod** and $\mathbf{Ch}_{\geq 0}(R)$. Moreover, under this equivalence, simplicial homotopy corresponds to the homology of the normalized chain complex, and simplicially homotopic maps correspond to chain homotopic maps.*

We can immediately dualize the theorem, to obtain the following.

Corollary 4.1.2. *For the category $\boldsymbol{R}$-**mod**, there is an equivalence between the category of cosimplicial objects in $\boldsymbol{R}$-**mod** and the category of non-negatively graded cochain complexes in $\boldsymbol{R}$-**mod**. This is realized by a functor N^*, which is a summand of the unnormalized cochain complex CA of A. Maps which are homotopic as cosimplicial maps correspond to maps which are homotopic as cochain maps.*

4.2 Proof of the theorem

The Dold-Kan Correspondence is an equivalence of categories. This means that we require a functor, which will be denoted by K, which maps from $\mathbf{Ch}_{\geq 0}(R)$ to $\mathcal{S}\boldsymbol{R}$-**mod**, such that KN is naturally isomorphic to the identity on $\mathcal{S}\boldsymbol{R}$-**mod** and such that NK is naturally isomorphic to the identity on $\mathbf{Ch}_{\geq 0}(R)$. The functor K can be explicity defined as follows. Given a chain complex C, we define $K_n(C)$ to be the finite direct sum $\bigoplus_{p\leq n} \bigoplus_{\eta} C_p[\eta]$, where, for a fixed $p \leq n$, the index η ranges over the surjections $[n] \to [p]$ in Δ, and $C_p[n]$ denotes a copy of C_p. For a given chain complex C, the first few parts of $K_n(C)$ are as follows:

$$
\begin{aligned}
&K_0(C) = C_0 \\
&K_1(C) = C_0 \oplus C_1 \\
&K_2(C) = C_0 \oplus C_1 \oplus C_1 \oplus C_2 \\
&K_3(C) = C_0 \oplus (C_1 \oplus C_1 \oplus C_1) \oplus (C_2 \oplus C_2 \oplus C_2) \oplus C_3
\end{aligned}
$$

To make K into a simplicial object, we must define how it induces maps in $\mathcal{S}\boldsymbol{R}$-**mod**. To see this, first let $\alpha : [m] \to [n]$ be a morphism in Δ. We will define $K(\alpha) : K_n(C) \to K_m(C)$ by defining its restrictions $K(\alpha, \eta) : C_p[\eta] \to K_m(C)$. Recall from Lemma 3.1.5 that each map in Δ has an epi-monic factorization. For each surjection $\eta : [n] \to [p]$, find the epi-monic factorization $\varepsilon\eta'$ of $\eta\alpha$, as indicated by the following diagram:

$$
\begin{array}{ccc}
[m] & \xrightarrow{\alpha} & [n] \\
\downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} \\
[q] & \xhookrightarrow{\varepsilon} & [p]
\end{array}
$$

If $p = q$, in which case $\eta\alpha = \eta'$, we define $K(\alpha, \eta)$ to be the natural identification of $C_p[\eta]$ with the summand $C_p[\eta']$ of $K_m(C)$. If $p = q + 1$ and $\varepsilon = \varepsilon_p$, in which case the image of $\eta\alpha$ is the subset $(0 < 1 \cdots < p - 1)$ of $[p]$, we define $K(\alpha, \eta)$ to be the map

$$C_p \xrightarrow{d} C_{p-1} = C_q[\eta'] \subseteq K_m(C).$$

In every other case, we define $K(\alpha, \eta)$ to be zero.

Lemma 4.2.1. *For a given C in $\mathbf{Ch}_{\geq 0}(R)$, $K(C)$ is a simplicial R-module; furthermore, because K is natural in C, K is a functor from $\mathbf{Ch}_{\geq 0}(R)$ to $\mathcal{S}\boldsymbol{R}$-mod.*

Proof. We will show that $K(C)$ is a contravariant functor from Δ to $\boldsymbol{R}$**-mod**. We have already seen how $K(C)$ associates a simplicial R-module to each object in Δ: $K(C)([n]) = K_n(C)$, which is defined above. So we must check the rest of the conditions for $K(C)$ to be a contravariant functor. First, let the map α be $\mathrm{id}_{[n]}$, and let η be a surjection in Δ from $[n]$ to $[p]$. In this situation, in terms of the diagram above, $\varepsilon = \mathrm{id}, \alpha = \mathrm{id}$, and $\eta = \eta'$. Therefore, by definition, $K(\alpha, \eta)$ is the identification of $C_n[\eta]$ in $K_n(C)$ with $C_n[\eta]$ in $K_n(C)$, which is the identity on $K_n(C)$. So K preserves identity morphisms.
We must also verify that $K(\beta\alpha, \eta) = K(\alpha, \eta) \circ K(\beta, \eta)$. To see this, we begin with the diagram

$$\begin{array}{ccccc} [m] & \xrightarrow{\alpha} & [n] & \xrightarrow{\beta} & [t] \\ \downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle \eta_\beta} & & \downarrow{\scriptstyle \eta} \\ [q] & \xrightarrow{\varepsilon_\alpha} & [q'] & \xrightarrow{\varepsilon_\beta} & [p] \end{array}$$

where $\varepsilon_\alpha\eta_\alpha$ is the epi-monic factorization of $\eta_\beta\alpha$, and $\varepsilon_\beta\eta_\beta$ is the epi-monic factorization of $\eta\beta$. Using this, we construct $\varepsilon_\beta\varepsilon_\alpha\eta_\alpha$, which is the epi-monic factorization of $\eta\beta\alpha$. We will use this factorization to show that, for each possible situation, $K(\beta\alpha, \eta) = K(\alpha, \eta) \circ K(\beta, \eta)$.
We will consider first the situations where $K(\beta\alpha, \eta)$ is nonzero. For the first case, let $p = q$. Then, as monics in Δ must either be the identity or have a codomain with a greater cardinality than their domain, $\varepsilon_\alpha = \varepsilon_\beta = \mathrm{id}$. Therefore, $\eta\beta\alpha = \eta_\alpha$, so we take $K(\beta\alpha, \eta)$ to be the identification of $C_p[n]$ with the summand $C_p[\eta_\alpha]$ of $K_m(C)$. Now consider the right square in the diagram; this gives us that $K(\beta, \eta)$ is the identification of $C_p[\eta]$ in $K_t(C)$ with $C_p[\eta_\beta]$ in $K_n(C)$. Similarly, the left square gives us that $K(\alpha, \eta_\beta)$ identifies $C_p[\eta_\beta]$ in $K_n(C)$ with $C_p[\eta_\alpha]$ in $K_m(C)$. Therefore, composing the two gives the identification of $C_p[n]$ in $K_t(C)$ with the summand $C_p[\eta_\alpha]$ of $K_m(C)$, which is the same as $K(\beta\alpha, \eta)$.
For the second case, let $p = q + 1$, and let $\varepsilon_\beta\varepsilon_\alpha = \varepsilon_p$. In this situation, either $\varepsilon_\alpha = \mathrm{id}$ and $\varepsilon_\beta = \varepsilon_p$, or vice versa. As the two cases are almost exactly similar, we will write only the proof for the case where $\varepsilon_\alpha = \mathrm{id}$. This means that $q = q'$. We know, by definition, that $K(\beta\alpha, \eta)$ is the map

$$C_p \xrightarrow{d} C_{p-1} = C_q[\eta_\alpha] \subseteq K_m(C),$$

and that $K(\beta, \eta)$ is the map

$$C_p \xrightarrow{d} C_{p-1} = C_{q'}[\eta_\beta] \subseteq K_n(C).$$

Furthermore, we know that $K(\alpha, \eta)$ is the identification of $C_{q'}[\eta_\beta]$ with $C_q[\eta_\alpha] \subseteq K_m(C)$. Therefore, $K(\beta\alpha, \eta) = K(\alpha, \eta) \circ K(\beta, \eta)$.
We will now turn our attention to the situations where $K(\beta\alpha, \eta) = 0$. The first of these is when $p = q + 1$, but $\varepsilon_\beta\varepsilon_\alpha \neq \varepsilon_p$. This means that, because either ε_α or ε_β is equal to the identity, either ε_α or ε_β is equal to some face map which is not ε_p. This means that either $K(\alpha, \eta)$ or $K(\beta, \eta)$ is equal to 0, which means that $K(\beta\alpha, \eta) = K(\alpha, \eta) \circ K(\beta, \eta)$.
Next, consider the case where $p = q + 2$. If $q = q'$, $K(\beta, \eta) = 0$ because the difference between q'

and p is greater than 1. So $K(\beta\alpha, \eta) = K(\alpha, \eta) \circ K(\beta, \eta)$. If $q' = p$, the situation is the same, except that $K(\alpha, \eta) = 0$. If $q' = q + 1 = p - 1$, assume that $\varepsilon_\alpha = \varepsilon_{q'}$ and $\varepsilon_\beta = \varepsilon_p$. (Otherwise, $K(\alpha, \eta_\beta)$ of $K(\beta, \eta)$ will be 0 by definition.) We have

$$K(\alpha, \eta_\beta) \circ K(\beta, \eta) = d \circ d : C_p \to C_{p-2}$$

which is equal to 0 because C is a chain complex.
Lastly, consider the case where $p = q + i$, where $i \geq 3$. The condition for $K(C)$ to be a contravariant functor holds here too, because either $K(\alpha, \eta_\beta)$ or $K(\beta, \eta)$ must be 0; this is because either $q' > q+1$, or $p > q'+1$. So, in every possible case, all the conditions for $K(C)$ to be a contravariant functor from Δ to $\mathcal{A}$, that is, a simplicial R-module, hold. As this is natural in C, K is a functor from $\mathbf{Ch}_{\geq 0}(R)$ to $\boldsymbol{\mathcal{S}R}$**-mod**: a chain map $f : C \to D$ is sent to a natural transformation $K(C) \to K(D)$. □

Having shown that K is a functor, we will prove one half of the statement that N and K form an equivalence of categories.

Lemma 4.2.2. *Let C be a non-negatively graded chain complex in $\boldsymbol{R}$-**mod**, and let the functors N and K be as defined above. Then $NK(C) \cong C$.*

Proof. Let $\eta : [n] \to [p]$ be a surjection, with $n \neq p$. Then there is a factorization of degeneracy maps $\eta = \eta_{i_1} \cdots \eta_{i_t}$. We can use this to extend the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} \\ [q] & \xhookrightarrow{\varepsilon} & [p] \end{array}$$

downwards, forming t similar diagrams, with η replaced by one of the degeneracy maps in the factorization. From this, we see that $C_p[\eta] = \sigma_{i_t} \cdots \sigma_{i_1}(C_p[\mathrm{id}_p])$, which lies in the degenerate subcomplex $D(K(C))$. If η is the identity on $[n]$, then ∂_0 restricted to $C_n[\mathrm{id}_{[n]}]$ is $K(\varepsilon_i, \mathrm{id}_{[n]})$, which is 0 if $i \neq n$ and the differential d if $i = n$. Thus, $N_n(KC) = C_n[\mathrm{id}_{[n]}]$ and the differential is d. Therefore $NK(C) \cong C$. □

In light of this, in order to prove the Dold-Kan Correspondence, we must show that $KN(A)$ is naturally isomorphic to A, for every simplicial R-module A. First, we shall construct a natural simplicial map $\psi_A : KN(A) \to A$. If, as above, $\eta : [n] \to [p]$ is a surjection in Δ, the corresponding summand of $KN_n(A)$ is $N_p(A)$, and the restriction of ψ_A to this summand is $N_p(A) \subset A_p \xrightarrow{\eta} A_n$.

Let the situation be as at the beginning of the subsection, with a map $\alpha : [m] \to [n]$ in Δ and the epi-monic factorization $\varepsilon\eta'$ of $\eta\alpha$ in Δ. In this case, the diagram

$$\begin{array}{ccccccc} KN_n(A) & \longrightarrow & N_p(A) & \longrightarrow & A_p & \xrightarrow{\eta_*} & A_n \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \alpha_*} \\ KN_m(A) & \longrightarrow & N_q(A) & \longrightarrow & A_q & \xrightarrow{\eta'_*} & A_m \end{array}$$

commutes, because, from the definition of the normalized chain complex, $\varepsilon_* : N_p(A) \to N_q(A)$ is equal to 0 unless $\varepsilon_* = \partial_p$. Therefore, ψ_A is a simplicial map from $KN(A)$ to A which is natural in A. To prove the equivalence of categories, we must show that ψ_A is an isomorphism for all A. It follows from the definition of ψ_A that $N_{\psi_A} : NKN(A) \to N(A)$ is the above isomorphism $NK(NA) \cong NA$. The following lemma states that ψ_A is an isomorphism; proving it, therefore, is equivalent to proving that N and K form two halves of an equivalence of categories.

Lemma 4.2.3. *Let $f : B \to A$ be a simplicial map such that $N(f) : N(B) \to N(A)$ is an isomorphism. Then f is an isomorphism.*

Proof. We shall prove that each $f_n : B_n \to A_n$ is an isomorphism by induction on n. Begin with $n = 0$; by the definition of N, $B_0 = N_0B \cong N_0A = A_0$. Recall from Lemma 3.4.17 the object ΛA, the kernel of $\partial_0 : PA \to A$, where $(PA)_n = A_{n+1}$. Recall also that $N(\Lambda A)$ is equal to $NA[1]/A_0[1]$. By our initial assumption, $N\Lambda f : N(\Lambda B) \to N(\Lambda A)$ is an isomorphism, and, by the inductive hypothesis, both f_n and $(\Lambda f)_n$ are isomorphisms. An application of the 5-lemma to the following diagram shows that f_{n+1} is an isomorphism, proving the lemma, and hence, that N and K form an equivalence of categories.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\Lambda B)_n & \longrightarrow & B_{n+1} & \xrightarrow{\partial_0} & B_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Lambda f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \\ 0 & \longrightarrow & (\Lambda A)_n & \longrightarrow & A_{n+1} & \xrightarrow{\partial_0} & A_n & \longrightarrow & 0 \end{array}$$

□

To complete the proof of the Dold-Kan Correspondence, we must show that simplicially homotopic maps correspond to chain homotopic maps. We are already halfway towards this; in Lemma 3.4.15, we showed that if f and g are simplicially homotopic, $N(f)$ and $N(g)$ are chain homotopic. On the other hand, suppose that we have two chain maps f and g mapping from C to D, and that we have a chain homotopy $\{s_n\}$ between them. Our aim is to define a family of maps $h_i : K(C)_n \to K(D)_{n+1}$ which form a simplicial homotopy from $K(f)$ to $K(g)$. We shall do so as follows.

On the summand C_n of $K(C)_n$ which corresponds to $\eta = \text{id}$, define

$$h_i = \begin{cases} \sigma_i f & \text{if } i < n-1 \\ \sigma_{n-1} f - \sigma_n s_{n-1} d & \text{if } i = n-1 \\ \sigma_n (f - s_{n-1} d) - s_n & \text{if } i = n \end{cases}$$

On the summand $C_p[\eta]$ of $K(C)_n$ which corresponds to $\eta : [n] \to [p]$ with $n \neq p$, we will define h_i by induction on $n - p$. To begin, let j be the largest element of $[n]$ such that $\eta(j) = \eta(j+1)$, and write $\eta = \eta' \eta_j$. Then σ_j maps $C_p[\eta']$ isomorphically onto $C_p[\eta]$, and we have already defined the maps h_i on $C_p[\eta']$. Writing h'_i for the composite of $C_p[\eta] \cong C_p[\eta']$ with h_i restricted to $C_p[\eta']$, we define

$$h_i \text{ on } C_p[\eta] = \begin{cases} \sigma_j h'_{i-1} & \text{if } j < i \\ \sigma_{j+1} h'_i & \text{if } j \geq i. \end{cases}$$

Performing calculations similar to those in Proposition 3.4.14 shows that the family $\{h_j\}$ forms a simplicial homotopy from $K(f)$ to $K(g)$. This completes the proof of the Dold-Kan Correspondence.

Proposition 4.2.4. *Let G be an abelian group and denote by $G[-n]$ the chain complex which is G concentrated in degree n. Then the simplicial abelian group $K(G[-n])$ is an Eilenberg-MacLane space of type $K(G, n)$ in the sense of Definition 3.4.6.*

Proof. We must show that $\pi_n(K(G[-n]))$ is isomorphic to G, and that $\pi_m(K(G[-n]))$ is trivial for all $n \neq m$. By Definition 3.4.7, this amounts to showing that $H_n(K(G[-n])) = G$ and that $H_m(K(G[-n])) = 0$ for $m \neq n$. However, this follows from the definition of $K(G[-n])$). □

Remark. The preceding proposition, an immediate consequence of the Dold-Kan Correspondence, is quite powerful; it allows one to construct an Eilenberg-Maclane space of type $K(G, n)$ for any abelian group G and $n \geq 1$.

Bibliographical Note. The preceding section is based primarily on Chapter 8 of [17]. Various details of the proof left as exercises by Weibel have been filled in.

5 Cyclic homology and the Dwyer-Kan Correspondence

So far, we have dealt with simplicial objects, chain complexes, and the Dold-Kan Correspondence, which links them. However, it is possible for a simplicial object A to have extra structure, in the form of an action of the cyclic group $\mathbb{Z}_{n+1}$ on each A_n, which interacts with the face and degenercy maps in a certain way. Simplicial objects of this kind are known as *cyclic objects*. Additionally, cyclic objects in the category ***R*-mod** have an associated *cyclic homology*. Cyclic objects form part of a larger class of objects called *duplicial objects*: there is an equivalence of categories between duplicial objects in ***R*-mod** and *mixed complexes* in ***R*-mod**, which will both be defined in the sequel. This equivalence is called the *Dwyer-Kan Correspondence*, and is the main subject of this section.

5.1 Cyclic objects and cyclic homology

We begin with a motivating example: a simplicial module which can be viewed as a cyclic module in a natural way.

Example 5.1.1. Fix a commutative ring k, let R be a k-algebra, let M be an $R-R$ bimodule, and let all tensor products in what follows be over the ring k. We can build a simplicial k-module $M \otimes R^{\otimes *}$ (where $R^{\otimes n}$ is the n-fold tensor product $R \otimes \cdots \otimes R$) with $[n] \mapsto M \otimes R^{\otimes n}$ by defining face and degeneracy maps as follows:

$$\partial_i(m \otimes r_1 \otimes \cdots \otimes r_n) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{if } i = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{if } 0 < i < n \\ r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{if } i = n \end{cases}$$

$$\sigma_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n$$

These formulas are k-multilinear, which means that the ∂_i and σ_i are well-defined k-module homomorphisms, and the simplicial identities can be verified directly. Our specific interest is in the case $R = M$. If this is the case, a generator of $R \otimes R^{\otimes n}$ has the form $r_0 \otimes r_1 \otimes \cdots \otimes r_n$. We can now define an action of the cyclic group $\mathbb{Z}_{n+1}$ on $R \otimes R^{\otimes n}$: to do so, denote the generator of $\mathbb{Z}_{n+1}$ by t and define the action of t by $t(r_0 \otimes \cdots \otimes r_n) = r_n \otimes r_0 \otimes r_1 \cdots \otimes r_{n-1}$. When the action is defined in this way, we have the identities $\partial_i t = t\partial_{i-1}$ and $\sigma_i t = t\sigma_{i-1}$ for $i > 0$. When $i = 0$, we have $\partial_0 t = \partial_n$ and $\sigma_o t = t^2 \sigma_n$: the operator t interacts with the face and degeneracy maps in a manner reminiscent of the simplicial identities. This motivates the following definition.

Definition 5.1.2. Let $\mathcal{A}$ be a category. A *cyclic object* A in $\mathcal{A}$ is a simplicial object with an automorphism t_n of order $n+1$ on each A_n such that $\partial_i t = t\partial_{i-1}$ and $\sigma_i t = t\sigma_{i-1}$ when $i > 0$. Also, $\partial_0 t_n = \partial_n$ and $\sigma_o t_{n+1}^2 = \sigma_n$.

Remark. Paralleling the terminology for simplicial objects, we will, for example, write "cyclic module" for a cyclic object in the category ***R*-mod**. This should not be confused with the identically-named notion of a module with one generator. We will follow a similar naming convention for cyclic objects in other categories.

Example 5.1.3. Recall the simplicial set BG associated to a group G, as defined in Example 3.2.2. We can make BG into a cyclic set by defining the action of t_n on $BG_n = G^n$ to be $t(g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_{n-1})$, where $g_0 = (g_1 \ldots g_n)^{-1}$.

In another parallel with simplicial objects, we will now construct a category, denoted ΔC and containing the simplicial category Δ, such that a cyclic object in a category $\mathcal{A}$ is the same thing as a contravariant functor from ΔC to $\mathcal{A}$. We call ΔC, which was first defined by Connes in [2], is called the *cyclic category*. In what follows, we will denote the objects and morphisms of Δ as in Section 3.

In addition, we will denote by t_n the "cyclic" automorphism of the set $[n]$ given by $t_n(0) = n$ and $t_n(j) = j - 1$ for $j \neq 0$.

To begin the construction, let $\mathrm{Hom}_{\Delta C}([n],[p])$ be the family of formal pairs (α, t^i), where $0 \leq i \leq n$ and $\alpha : [n] \to [p]$ is a morphism in Δ. Let $\mathrm{Hom}_{\mathcal{C}}([n],[p])$ denote the family of all functions $\varphi : [n] \to [p]$ which factor as $\varphi = \alpha t_n^i$ for some pair (α, t^i) in $\mathrm{Hom}_{\Delta C}([n],[p])$. Note that, if this is the case, $\varphi(i) \leq \varphi(i+1) \leq \cdots \leq \varphi(i-1)$, due to the cyclic action of the automorphism t. This means that the obvious surjection from $\mathrm{Hom}_{\Delta C}([n],[p])$ to $\mathrm{Hom}_{\mathcal{C}}([n],[p])$ is a bijection, except when the map φ is constant: φ uniquely determines (α, t^i) whenever φ is not constant. This means that the map between the hom-sets is an injection onto the subset consisting of non-constant functions φ. In the case where φ is constant, $\alpha = \varphi$ and all $n+1$ of the pairs (φ, t^i) yield the set map φ. In this setting, $\mathrm{Hom}_{\Delta}([n],[p])$ is the subset consisting of pairs of the form $(\alpha, 1)$ in $\mathrm{Hom}_{\Delta C}([n],[p])$.

As suggested by our choice of notation above, there exists a subcategory $\mathcal{C}$ of **Set**, containing Δ, whose objects are the finite ordered sets $[n]$ and whose morphisms are the functions in $\mathrm{Hom}_{\mathcal{C}}([n],[p])$. To see this, we must verify that the composition of $\phi = \beta t_m^j$ and $\varphi = \alpha t_n^i$ is in $\mathcal{C}$. This can be verified using the following identities, which are obtained from the formulas for the face and degeneracy maps in Δ (see Definition 3.1.3):

$$t_n \varepsilon_i = \begin{cases} \varepsilon_n & i = 0 \\ \varepsilon_{i-1} t_{n-1} & i > 0 \end{cases} \quad \text{and} \quad t_n \eta_i = \begin{cases} \eta_n t_{n+1}^2 & i = 0 \\ \eta_{i-1} t_{n+1} & i > 0 \end{cases}$$

The following proposition verifies that the category ΔC relates to cyclic objects in the same way that the category Δ relates to simplicial objects.

Proposition 5.1.4. *The sets $\mathrm{Hom}_{\Delta C}([n],[p])$ form the morphisms of a category ΔC which contains Δ. The objects are the finite ordered sets $[n]$ for $n \geq 0$. Furthermore, a cyclic object in a category $\mathcal{A}$ is the same as a contravariant functor from ΔC to $\mathcal{A}$.*

Proof. We wish to define the composition (γ, t^k) of $(\beta, t^j) \in \mathrm{Hom}_{\Delta C}([m],[n])$ and $(\alpha, t^i) \in \mathrm{Hom}_{\Delta C}([n],[p])$ such that $i = j = 0$ implies that $(\gamma, t^k) = (\alpha\beta, 1)$. If β is a non-constant map of sets, the composition $t^i \beta t^j$ in $\mathcal{C}$ is not constant, so there exists a unique (β', t^k) such that $t^i \beta t^j = \beta' t^k$: we set $(\gamma, t^k) = (\alpha\beta', t^k)$. In the case where β is constant, we set $(\gamma, t^k) = (\alpha\beta, t^j)$. By construction, the projections from $\mathrm{Hom}_{\Delta C}$ to $\mathrm{Hom}_{\mathcal{C}}$ are compatible with composition: because $\mathcal{C}$ is a category, the maps of the form $(\mathrm{id}, 1)$ are two-sided identity maps and composition in ΔC is associative. Therefore, ΔC is a category and the maps $\Delta \to \Delta C \to \mathcal{C}$ are functors. The final assertion is easily verified using the identities for $t\varepsilon_i$ and $t\eta_j$ above. □

Definition 5.1.5. Let A be a cyclic object in ***R*-mod**. The unnormalized chain complex $C_*^h(A)$ of the underlying simplicial object of A is called the *Hochschild complex* of A. We will denote the differential of $C_*^h(A)$ by b, so that $b = \partial_0 - \partial_1 + \cdots \pm \partial_n$ is a map from $C_n^h(A) = A_n$ to $C_{n-1}^h(A) = A_{n-1}$. The *Hochschild homology* $HH_*(A)$ of A is the homology of $C_*^H(A)$. The *acyclic complex* of A, $C_*^a(A)$ is the complex obtained from $C_*^h(A)$ by omitting the final face operator. Thus $C_n^a(A) = A_n$, and we denote the resulting differential $\partial_0 - \partial_1 + \cdots \mp \partial_{n-1}$ by b'.

Remark. The acyclic complex is so named because it is indeed acyclic: it is chain homotopic to the zero chain complex with contracting homotopy given by σ_{n+1}.

Drawing on Section 2.2, we can associate a double complex to each cyclic object in ***R*-mod**.

Definition 5.1.6. Let A be a cyclic object in ***R*-mod**. There exists an associated first quadrant double complex $CC_*(A)$, which is commonly referred to as *Tsygan's double complex*, after its first discoverer. The columns of $CC_*(A)$ are periodic of order 2: if p is even, the p-th column is the Hochschild complex C_*^h of A. If p is odd, the p-th column is the acyclic complex C_*^a with differential $-b'$, where the multipication of the differential by -1 stems from the "sign trick" discussed in

Section 2.2. Thus, $CC_{pq}(A)$ is A_q and hence independent of p. The q-th row of $CC_{**}(A)$ is the periodic complex associated with the action of the cyclic group C_{q+1} on A_q, where the generator acts as multiplication by $(-1)^q t$. Therefore, the differential $A_q \to A_q$ is multipliciation by $1-(-1)^q t$ when p is odd; when p is even it is multiplication by the norm operator

$$N := 1 + (-1)^q t + \cdots + (-1)^i t^i + \cdots + (-1)^q t^q$$

Combining all of this information, the following diagram depicts Tsygan's double complex:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A_2 & \xleftarrow[1-t]{} & A_2 & \xleftarrow[N]{} & A_2 & \xleftarrow[1-t]{} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A_1 & \xleftarrow[1-t]{} & A_1 & \xleftarrow[N]{} & A_1 & \xleftarrow[1-t]{} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A_0 & \xleftarrow[1-t]{} & A_0 & \xleftarrow[N]{} & A_0 & \xleftarrow[1-t]{} & \cdots
\end{array}$$

Definition 5.1.7. Recalling the notion of the total complex of a double complex from Definition 2.2.2, the *cyclic homology* $HC_*(A)$ of a cyclic object A is the homology of $\mathrm{Tot}^{\oplus} CC_{**}(A)$. The cyclic homology $HC_*(R)$ of a k-algebra R is the cyclic homology of the cyclic object $R \otimes R^{\otimes *}$.

Proposition 5.1.8. *$CC_{**}(A)$ is a double complex.*

Proof. To begin, set $\eta = (-1)^q$. We must show that $b(1-\eta t) = (1+\eta t)b'$ and $Nb = b'N$ considered as maps from A_q to A_{q-1}. Now $b - b' = \eta \partial_q$ and the cyclic relations impy that $bt = \partial_q - tb'$, which yields the first relation, $B(1-\eta t) = (1+\eta t)b'$. In addition, the cyclic relations imply that

$$b' = \sum_{i=0}^{q-1} (-t)^i \partial_q t^{q-i} \text{ and } b = \sum_{i=0}^{q} (-t)^{q-i} \partial_q t^i$$

Now, since $(1-\eta t)N = 0$, we have $T^i N = \eta^i N$ on A_q. Since $N(1+\eta t) = 0$, we have $Nt^i = (-\eta)^i N$ on A_{q-1}. Therefore

$$\eta N b = \eta \sum_{i=0}^{q} N(\eta)^{q-i} \partial_q t^i = \eta^{q+1} N \partial_q \sum (\eta t)^i = N \partial_q N,$$

$$\eta b' N = \eta \sum_{i=0}^{q-1} (-t)^i \partial_q \eta^{q-i} N = \eta^{q+1} \sum (\eta t)^i \partial_q N = N \partial_q N.$$

This yields the relation $Nb = b'N$. □

5.2 Duplicial objects and mixed complexes

We proceed now to duplicial objects, a more general notion than cyclic objects. In fact, a cyclic module is nothing other than a duplicial module which is subject to certain additional identities.

Bibliographical Note. From this point on, we draw primarily on the work of Dwyer and Kan in [6] and [7].

Definition 5.2.1. Let A be a simplicial object in a category $\mathcal{A}$. We refer to A as a *duplicial object* if, in each degree $n \geq 0$, there exists an extra degeneracy map σ_{n+1}. Moreover, the maps are subject to the following relations:

$$\partial_i \partial_j = \partial_{j-1}\partial_i \qquad \text{if} \quad i < j$$
$$\sigma_i \sigma_j = \sigma_{j+1}\sigma_i \qquad \text{if} \quad i \leq j$$

$$\partial_i \sigma_j = \begin{cases} \sigma_{j-1}\partial_i & \text{if } i < j \leq n \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

Remark. The identities defining a duplicial object are exactly those defining a simplicial object, except for one subtle difference: in general, contrary to what one may expect, $\partial_0\sigma_{n+1} \neq \sigma_n\partial_0$. The name "duplicial" was invented by Dwyer and Kan, based on the fact that removing the face maps $\partial_0 : A_{n+1} \to A_n$ leaves a cosimplicial object, with the degeneracy maps as coface maps and the remaining face maps as codegeneracy maps.

Paralleling our procedures for dealing with simplicial and cyclic objects, we can characterize duplicial objects as functors from a certain category into our category of interest.

Definition 5.2.2. The *duplicial category*, which we will denote by ΔD, is the category with objects $[n]$ for $n \geq 0$, and generating maps

$$\varepsilon_i : [n] \to [n-1] \qquad 0 \leq i \leq n$$
$$\eta_i : [n] \to [n+1] \qquad 0 \leq i \leq n+1$$

subject to the relations

$$\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_{j-1} \qquad \text{if} \quad i < j$$
$$\eta_j \eta_i = \eta_i \eta_{j+1} \qquad \text{if} \quad i \leq j$$

$$\eta_j \varepsilon_i = \begin{cases} \varepsilon_i \eta_{j-1} & \text{if } i < j \leq n \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{j-1}\eta_j & \text{if } i > j+1 \end{cases}$$

so that a duplicial object in a category $\mathcal{A}$ is a contravariant functor from ΔD to $\mathcal{A}$.

Analogously to the way we considered Δ to be the category consisting of finite ordered sets and the non-decreasing maps between them, we can view ΔD as the category whose objects are copies of the ordered set of non-negative integers $\mathbb{N}^+$, with morphisms being functions which are both nondecreasing and "periodic". To describe this with full precision, let P be the category whose objects consist of one copy p_n of $\mathbb{N}^+$ for each integer $n \geq 0$. The morphisms $p_n \to p_{n'}$ consist of the non-decreasing functions f which are periodic in the sense that $f(j+n+1) = f(j) + n' + 1$ for all $j \in \mathbb{N}^+$. With this information in place, it is easily verified that P is isomorphic to ΔD: agreeing with the formulas for the face and degeneracy maps in Section 3, the map $\varepsilon_i : [n] \to [n+1] \in \Delta D$ corresponds to the function $p_{n-1} \to p_n$ given by $j \mapsto j$ for $j < i$ and $j \mapsto j+1$ for $j \geq i$. There is a completely analagous correspondence between $\eta_i : [n] \to [n-1] \in \Delta D$ and the map $p_n \to p_{n-1}$ given by $j \to j$ when $j \leq i$ and $j \mapsto j-1$ if $j > i$. Taken together, these correspondences give a correspondence between the map $\eta_{n+1}\varepsilon_0 : [n] \to [n]$ and the map $p_n \to p_n$ given by $j \mapsto j+1$ for all j. This motivates the following proposition, which establishes the aforementioned connection between cyclic and duplicial objects.

Proposition 5.2.3. *The cyclic category ΔC can be obtained from the duplicial category ΔD by adding the "cyclic" relations*

$$(\eta_{n+1}\varepsilon_0)^{n+1} = \mathrm{id} : [n] \to [n]$$

Proof. We will verify this directly by letting $t_n = (\eta_{n+1}\varepsilon_0)^n$. Recall from Definition 5.1.2 that a cyclic object is a simplicial object with an order $n+1$ automorphism t_n in each degree such that $\partial_i t = t\partial_{i-1}$ and $\sigma_i t = t\sigma_{i-1}$ when $i > 0$, and such that $\partial_0 t = \partial_n$ and $\sigma_0 t = t^2\sigma_n$ when $i = 0$. We also verified in Proposition 5.1.4 that a cyclic object in a category $\mathcal{A}$ is the same as a contravariant functor from ΔC to $\mathcal{A}$. Therefore, it is enough to show that $t_n = (\eta_{n+1}\varepsilon_0)^n$ is an order $n+1$ automorphism such that $t\varepsilon_i = \varepsilon_{i-1}t$ and $t\eta_i = \eta_{i-1}t$ when $i > 0$ and such that $t\varepsilon_0 = \varepsilon_n$ and $t\eta_0 = \eta_n t^2$ when $i = 0$.

By construction, $(\eta_{n+1}\varepsilon_0)^n$ is of order $n+1$. We will verify that $(\eta_{n+1}\varepsilon_0)^n\varepsilon_0 = \varepsilon_n$: the other identities can be verified similarly. First, write $(\eta_{n+1}\varepsilon_0)^n\varepsilon_0$ as $(\eta_{n+1})^n(\varepsilon_0)^{n+1}$. Then, because $\varepsilon_i(j) = j+1$ if $j \geq i, \varepsilon_0^{n+1}(j) = j+n+1$. Similarly, because $\eta_i(j) = j-1$ if $j > i, \eta_{n+1}^n(j+1+n) = n+1$ if $j \geq n$ and n otherwise. That is to say, $(\eta_{n+1}\varepsilon_0)^n\varepsilon_0 = \varepsilon_n$. □

We have now defined two ways to build an object with extra structure from a simplicial object, one of which is a special case of the other. We can, in fact, perform a similar procedure for chain complexes.

Definition 5.2.4. A *mixed complex* (M, b, d) is a sequence of R-modules and R-module homomorphisms, with $b_n : M_n \to M_{n-1}$ and $d_n : M_n \to M_{n+1}$. Also, $b^2 = d^2 = 0$ in each degree n. In other words, M is simultaneously a chain complex and a cochain complex. Consequently, it has both homology $H_n(M)$ and cohomology $H^n(M)$.

Example 5.2.5. Every chain complex (C, b) or cochain complex (C, d) can be made into a mixed complex in a trivial way: for the chain complex, simply define $d_n : C_n \to C_{n+1}$ to be 0, and similarly for $b_n : C_n \to C_{n-1}$ in the second case.

Example 5.2.6. Consider the diagram

$$\cdots \underset{d_2}{\overset{b_3}{\rightleftarrows}} \mathbb{Z}_{i_2} \underset{d_1}{\overset{b_2}{\rightleftarrows}} \mathbb{Z}_{i_1} \underset{d_0}{\overset{b_1}{\rightleftarrows}} \mathbb{Z}_{i_0} \rightleftarrows 0$$

where i_n is an integer for all n, and the maps b_n and d_n represent multiplication by some integer. The diagram depicts a mixed complex if $i_{n-1}|b_n b_{n+1}$ and $i_{n+2}|d_{n+1}d_n$ for all n.

Each mixed complex M has an associated homology and cohomology, obtained by considering b and d separately. It is natural at this stage to wonder whether one can, in some way or other, define a notion of homology by considering both differentials. This, with a little thought, turns out to be the case, and we will now proceed to define this notion. Beforehand, we will define the operator $\xi_n : M_n \to M_n$ by $\xi_n = b_{n+1}d_n + d_{n-1}b_n$ in each degree n.

Definition 5.2.7. Let (M, b, d) be a non-negatively graded mixed complex, and define $\hat{M} := M/\mathrm{im}\,\xi$. From the mixed complex $(\hat{M}, b, d)$, form the chain complex

$$CM_n := \hat{M}_n \oplus \hat{M}_{n-2} \oplus \hat{M}_{n-4} \cdots$$

whose differential ∂ is given by

$$\partial(m, m_{n-2}, m_{n-4}, \dots) = (bm_n + dm_{n-2}, bm_{n-2} + bm_{n-4}, \dots)$$

We refer to the homology of this chain complex, denoted $HM(M)$, as the *mixed homology* of M.

When constructing the mixed homology, we work with $\hat{M}$ rather than M. The reason for this is that the chain complex from which we calculate the mixed homology is the total complex of a certain double complex associated to M: this is called the *triangle complex* and is depicted by the following diagram

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
M_2 & \xleftarrow[d]{} & M_1 & \xleftarrow[d]{} & M_0 \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
M_1 & \xleftarrow[d]{} & M_0 & & \\
\downarrow{\scriptstyle b} & & & & \\
M_0 & & & &
\end{array}$$

Quotienting by $\operatorname{im}\xi$ ensures that the squares of this double complex anticommute as required. In fact, some authors have the condition $\xi = bd + db = 0$ as part of the definition of a mixed complex, for this reason.

The usual convention is to call the mixed homology the "cyclic homology", so that there is a notion of cyclic homology both for mixed complexes and for cyclic objects. The reason for this naming convention is that, as well as Tsygan's double complex $CC(A)$, one can associate a triangle complex to a cyclic object A. Per Theorem 2.5.11 in [12], the total complexes of the triangle complex and Tsygan's double complex have isomorphic homology. We deviate from convention here to avoid confusion in Section 6, where we will discuss two homology theories both of which would be called "cyclic" if we stuck rigidly to the usual terms.

5.3 The Dwyer-Kan Correspondence

We proceed now to discuss the main result of the section: the Dwyer-Kan Correspondence. This result, first proved by Dwyer and Kan in [6], generalizes the normalization of simplicial objects to duplicial objects. We fix some notation before proceeding: let $\mathcal{D}\boldsymbol{R}$**-mod** be the category of duplicial modules over a ring R, and let $\mathbf{Mix}_{n\geq 0}(R)$ be the category of non-negatively graded mixed complexes of R-modules.

Definition 5.3.1. Let A be a duplicial R-module. The associated *unnormalized mixed complex* $C(A)$ is given by $C(A)_n = A_n$ for all $n \geq 0$. The differentials are as follows:

$$b_n x = \sum_{i=0}^{n} (-1)^i \partial_i x \quad \text{and} \quad d_n x = \sum_{i=0}^{n+1} (-1)^i \sigma_{n+1-i} x \quad \text{for } x \in A_n.$$

Analogously to the simplicial case, we also have two subcomplexes of $C(A)$, called the *normalized mixed complex*, which we will again denote by $N(A)$, and the *degenerate mixed complex*, which we will again denote by $D(A)$. The normalized and degenerate complexes are defined exactly as in the chain complex case.

Proposition 5.3.2. *The mixed complex $C(A)$ is isomorphic to $D(A) \oplus N(A)$.*

Proof. This is completely analogous to the chain complex case. □

Generalizing the functor K from the Dold-Kan Correspondence, we now define its duplicial analogue.

Definition 5.3.3. Let M be a non-negatively graded mixed complex of R-modules. Then the functor K which sends M to the duplicial object $K(M)$ is defined as follows. In degree n, $K(M)$ is exactly the same as in the simplicial case. The maps are once again determined by the procedure used in the proof of Dold-Kan, except for the final degeneracy operator σ_{n+1}. This is determined by the requirement that, for all $x \in M_n$,

$$\sigma_{n+1}x = dx - \sum_{i=1}^{n+1}(-1)^i \sigma_{n+1-i}x,$$

which comes from the definition of the unnormalized mixed complex. The fact that K is a functor can be verified in a manner analogous to the simplicial Dold-Kan case, except for the extra degeneracy map σ_{n+1}: this is a matter of straightforward calculation.

With K defined, we can now state the Dwyer-Kan Correspondence.

Theorem 5.3.4 (Dwyer-Kan)**.** *Let $\mathcal{D}\boldsymbol{R}$-**mod** be the category of duplicial R-modules, and let $\mathbf{Mix}_{\geq 0}(R)$ be the category of mixed complexes in $\boldsymbol{R}$-**mod**, with $M_n = 0$ for all $n < 0$. The functor N which sends a duplicial R-module to its normalized mixed complex is an equivalence of categories between $\mathcal{D}\boldsymbol{R}$-**mod** and $\mathbf{Mix}_{\geq 0}(R)$.*

Proof. As above, this is simply a combination of the Dold-Kan Correspondence and its cosimplicial version in Corollary 4.1.2: we only need to check the behaviour of the extra degeneracy map, and this is, again, a matter of straightforward calculation. □

6 Further directions

We have now, after introducing the necessary preliminary concepts, stated and proved the Dold-Kan and Dwyer-Kan Correspondences. In this concluding section, we discuss, briefly and informally, some possible directions for further study and research.

6.1 Model categories

One notion underlying the paper is that of a *model category*. The rigorous definition can be found in Chapter 7.1 of [10], but it is rather technical. Informally, a model category is a category with some extra structure in the form of three distinguished classes of morphisms, which are all subject to certain conditions, and are known as weak equivalences, fibrations, and cofibrations. Even more informally, a model category can be thought of as a setting in which it "makes sense" to do homotopy theory of some kind. It will be instructive at this point to consider some examples.

Example 6.1.1. The classic example of a category which admits a model category structure is the category of topological spaces and continuous maps, **Top**. With the most common structure, we stipulate that the weak equivalences are maps which induce isomorphisms on all homotopy groups, the fibrations are maps which have the homotopy lifting property with respect to all spaces, and the cofibrations are retracts of relative CW complexes.

Example 6.1.2. Another category which admits a model category structure is $\mathbf{Ch}_{\geq 0}(R)$. Here, the weak equivalences are the quasi-isomorphisms, the fibrations are maps which are epimorphisms in each degree with injective kernel, and the cofibrations are maps which are monomorphisms in each non-zero degree.

Remark. Generally, there are multiple possible model category structures on a given category. The previous two examples merely represent two possibilities on their respective categories.

With this information about model categories in mind, it will be instructive to consider the Dold-Kan and Dwyer-Kan Correspondences again. Recall that, given a simplicial object A, the simplicial homotopy modules $\pi_n(A)$ are isomorphic to the homology of the normalized chain complex $N(A)$. In addition, consider the model category structure on $\mathbf{Ch}_{\geq 0}(R)$ which we have just discussed. It is natural, given the equivalence of categories, to attempt to describe a model category structure on $\boldsymbol{\mathcal{S}R}$**-mod** which corresponds with the one on $\mathbf{Ch}_{\geq 0}(R)$. This is done in a natural way: one can define a model category structure on $\boldsymbol{\mathcal{S}R}$**-mod** in which the weak equivalences are those maps which induce isomorphisms on all simplicial homotopy modules, and this model category structure is respected by the Dold-Kan Correspondence. The terminology for this situation is that there is a *Quillen equivalence* between the two model category structures.

The situation becomes more complicated when we deal with duplicial objects and mixed complexes. In [7], Dwyer and Kan detail three candidates for weak equivalences of duplicial R-modules: firstly, maps of duplicial modules which induce isomorphisms on all of the homotopy modules of the underlying simplicial modules; secondly, maps which induce isomorphisms on all of the cohomotopy modules of the underlying cosimplicial modules, and, finally, maps which induce isomorphisms on both the homotopy and the cohomotopy modules. The main concern of the paper of Dwyer and Kan was to show that each of these classes of weak equivalences corresponds to a class of weak equivalences of differential graded R-modules, and, hence, of mixed complexes. Predictably, the maps which induce isomorphisms on homotopy modules correspond to homological quasi-isomorphisms of mixed complexes, with the other two cases being completely analogous. This correspondence between possible weak equivalences is also discussed in [6].

Another class of interesting possible candidates for weak equivalences of mixed complexes incorporating both homology and cohomology, which Dwyer and Kan did not consider, is the class of maps which induce isomorphisms on mixed homology. As the following simple example shows, this is certainly a different class from the third of the three previously mentioned.

Example 6.1.3. Let M be the mixed complex

$$\cdots \underset{0}{\overset{\text{id}}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{\text{id}}{\rightleftarrows}} \mathbb{C} \rightleftarrows 0$$

Clearly, $H_n(M) = 0$ and $H^n(M) = \mathbb{C}$ for all n. Also, $\xi_n = 0$ for all n. Then, because $d_n = 0$ for all n, the following diagram depicts the chain complex CM,

$$\cdots \xrightarrow{\text{id}} \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \xrightarrow{0} \mathbb{C} \oplus \mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \oplus \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \to 0.$$

Therefore, the mixed homology $HM_n(M)$ is 0 for all n. This means that the map $0 : M \to M$ induces isomorphisms on homology and mixed homology, but not on cohomology. This suggests that there may be a fourth model category structure, not considered by Dwyer and Kan, in which the weak equivalences are those maps which induce isomorphisms on mixed homology.

6.2 Cyclic homology and mixed homology

Mixed homology, as well as possibly giving an interesting class of weak equivalences, is an active area of research in its own right. Parallel to the present paper, the author and his supervisor have undertaken work on a research paper entitled Cyclic VS Mixed Homology [11]. Broadly speaking, the paper concerns the mixed homology of mixed complexes. By way of a conclusion, we will now give an overview of this paper's motivation and results.

To begin, it will be necessary to make a definition.

Definition 6.2.1. Let (M, b, d) be a mixed complex. We refer to (M, b, d) as a *homological (resp. cohomologial) skyscraper* if the canonical map

$$M \to \hat{M} = M/\operatorname{im}\xi$$

is a quasi-isomorphism of chain (resp. cochain) complexes.

A common class of motivating examples for mixed complexes, of which a full treatment can be found in Section 2.6 of [12], are the noncommutative differential forms over an associative algebra, with the De Rham differential d and the Hochschild boundary map b. These do not form a skyscraper with respect to d, but they do so with respect to the coboundary operator B, first defined by Connes in [3] (Part II, Section 3), which defines cyclic homology.

The program of the paper, inspired by some results of Cuntz and Quillen in [4], is to examine the analogous situation for a general mixed complex of R-modules. To this end, we view the mixed complex M as a $k[x]$-module, where k is the centre of the ring R and x acts by the operator ξ. We also, by making the following definition, view B as a kind of deformation of d.

Definition 6.2.2. Given a mixed complex of R-modules (M, b, d) and a sequence of polynomials $c_n \in k[x]$, we define a new map

$$B_n = c_n d_n$$

and we, in line with the convention of Connes, call the mixed homology of (M, b, B) the cyclic homology, and denote it by $HC(M)$.

The main theorem of the paper describes the interactions between $HM(M)$ and $HC(M)$ when (M, b, B) is a homological skyscraper.

Theorem 6.2.3. *Let (M, b, d) be a mixed complex of R-modules, and assume that the polynomials $c_n \in k[x]$ defined above are invertible in $k[\![x]\!]$. Assume also that (M, b, B) is a homological skyscraper. Then there exists a graded R-module $X \subset HC(\hat{M})$ and short exact sequences*

$$0 \to X_n \to HM_n(M) \to HC_n(\hat{M})/X_n \to 0,$$

$$0 \to HC_n(M) \to HC_n(\hat{M}) \to HC_{n-1}(\operatorname{im}\xi) \to 0.$$

Thus, if the two short exact sequences split, then choosing a split for both yields an isomorphism

$$HM_n(M) \cong HC_n(M) \oplus HC_{n-1}(\operatorname{im}\xi)$$

Work on the paper is ongoing: at present, we are working towards building interesting examples which exhibit the behaviour described in the main theorem.

A Categorical language

In this section we give definitions and examples of various category-theoretic terms used in the main body of the paper. Basic familiarity with categories and functors is assumed. We begin with a central notion: that of a natural transformation. Intuitively, just as a functor is a kind of structure-preserving map between categories, we can think of natural transformations as structure preserving maps between functors.

A.1 Natural transformations and functor categories

Definition A.1.1. Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$, either both covariant or both contravariant. A *natural transformation* $\eta : F \to G$ is a family of morphisms, with a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$ for every object X in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X'}} \\ G(X) & \xrightarrow{G(f)} & G(X') \end{array}$$

We call the map η_X the *component* of η at X. If F and G are both contravariant, the directions of the horizontal arrows are reversed. If each component of η is an isomorphism, we call η a *natural isomorphism* and write $\eta : F \cong G$.

Example A.1.2. Consider the category **Grp**, consisting of all groups and group homomorphisms, and let $(G, \cdot)$ be a particular group. We now define the notion of the *opposite group*, denoted $(G^{\text{op}}, \cdot^{\text{op}}) : G^{\text{op}} = G$, and $a \cdot^{\text{op}} b = b \cdot a$. The process of forming the opposite group, if we set $f^{\text{op}} = f$ for a homomorphism $f : G \to H$, defines an endofunctor of **Grp.** To see this, we must verify that f^{op} is a group homomorphism from G^{op} to H^{op} :

$$f^{\text{op}}(a \cdot^{\text{op}} b) = f(b \cdot a) = f(b) \cdot f(a) = f^{\text{op}}(a) \cdot^{\text{op}} f^{\text{op}}(b).$$

We claim that the opposite functor $\textbf{Grp} \to \textbf{Grp}$ is naturally isomorphic to the identity functor on **Grp**. To show this, we require isomorphisms $\eta_G : G \to G^{\text{op}}$ for every group G such that the following diagram commutes:

$$\begin{array}{ccc} G^{\text{op}} & \xrightarrow{f^{\text{op}}} & H^{\text{op}} \\ {\scriptstyle \eta_G}\downarrow & & \downarrow{\scriptstyle \eta_H} \\ G & \xrightarrow{f} & H \end{array}$$

Set $\eta_G(a) = a^{-1}$ for all G. The identities $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$ show that η_G is a group homomorphism which is its own inverse. We now must show that the diagram commutes. To do so, we will consider a group homomorphism $f : G \to H$ and show that $\eta_H \circ f^{\text{op}} = f \circ \eta_G$:

$$\eta_H \circ f^{\text{op}}(a) = \eta_H \circ f(a) = (f(a))^{-1} = f(a^{-1}) = f \circ \eta_G.$$

Therefore, the identity functor on **Grp** is naturally isomorphic to the opposite functor $\textbf{Grp} \to \textbf{Grp}$.

Definition A.1.3. Let $\mathcal{C}$ be a small category (that is to say, one whose objects and morphisms form sets rather than proper classes) and let $\mathcal{D}$ be a general category. We can form a category, which we will denote by $\mathcal{D}^{\mathcal{C}}$, whose objects are the covariant functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are the natural transformations between them. The composition of morphisms in $\mathcal{D}^{\mathcal{C}}$ is the composition of natural transformations, which is defined in the expected way, via composition of the individual components.

We will now define a crucial notion; the notion of an *equivalence of categories.*

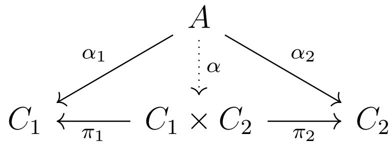
Definition A.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We will call F an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms $\text{id}_{\mathcal{C}} \cong G \circ F$ and $\text{id}_{\mathcal{D}} \cong F \circ G$.

Remark. Although the notion of an equivalence of categories somewhat resembles classical notions of isomorphism, the distinction is important: $G \circ F$ and $F \circ G$ are only $\text{id}_{\mathcal{C}}$ and $\text{id}_{\mathcal{D}}$ up to natural isomorphism. In this respect, equivalences of categories are more akin, to give one example, to homotopy equivalences of topological spaces. One can define the stricter notion of *isomorphism of categories*, in which natural isomorphisms of functors are replaced by equalities, but this condition is very strong and is hence seldom satisfied in practice. For categories, the most useful notion of "sameness" is equivalence, not isomorphism.

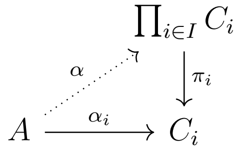
We now proceed to a discussion of *products* and, dually, *coproducts*: these generalize such constructions as the direct sum of abelian groups and the cartesian product of topological spaces.

A.2 Products and coproducts

Definition A.2.1. Let $\{C_i\}$ be a set of objects in a category $\mathcal{C}$, indexed by some set I. Their *product*, when it exists, is an object of $\mathcal{C}$, denoted $\prod_{i\in I} C_i$, together with maps $\pi_j : \prod_{i\in I} C_i \to C_j$ $(j \in I)$ such that, for every object A in $\mathcal{C}$ and every family of morphisms $\alpha_i : A \to C_i$ there exists a unique morphism $\alpha : A \to \prod_{i\in I} C_i$ such that $\pi_j\alpha = \alpha_j$ for every $j \in I$. We call the family of maps (π_j) the *projections*. In a familiar use of notation, if I is a two-element set, we write $C_1 \times C_2$ for the product. This situation is indicated by the following commutative diagram:



In the general case, we have a family of commutative diagrams, indexed by I, all of which have the form:



Remark. Notice that we referred to the product in singular terms above. Technically, a family of objects can have more than one product. However, as is usual for universal constructions, any two products of a family of objects are isomorphic; furthermore, the isomorphism is unique.

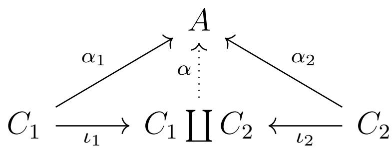
Example A.2.2. The prototypical example comes from **Set**: in this category, the product is the familiar cartesian product of sets. If X_i is a family of sets, we define $\pi_j : \prod_{i\in I} X_i \to X_j$ by $\pi_j((x_i)_{i\in I}) = x_j$. Given any set Y with a family of functions $\alpha_i : Y \to X_i$, the unique morphism $\alpha : Y \to \prod_{i\in I} X_i$ making everything commute is given by $\alpha(y) = (\alpha_i(y))_{i\in I}$.

Example A.2.3. In **Top**, the product of a family of spaces is the space whose underlying set is the Cartesian product. The topology we choose on this space is known as the product topology; it is the coarsest topology (that is to say, the topology with the fewest open sets) which ensures all the projections are continuous maps.

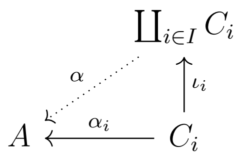
Example A.2.4. In **Grp**, the product is the direct product of groups.

As is done commonly in category theory, we can *dualize* the above discussion, which means that we reverse the direction of each morphism. This leads to the following definition:

Definition A.2.5. Let $\{C_i\}$ be a set of objects in a category $\mathcal{C}$, indexed by some set I. Their *coproduct*, when it exists, is an object of $\mathcal{C}$, denoted $\coprod_{i\in I} C_i$, together with maps $\iota_j : C_j \to \coprod_{i\in I} C_i$ $(j \in I)$ such that, for every object A in $\mathcal{C}$ and every family of morphisms $\alpha_i : C_i \to A$ there exists a unique morphism $\alpha : \coprod_{i\in I} C_i \to A$ such that $\alpha\iota_j = \alpha_j$ for all $j \in I$.. We call the family of maps (ι_j) the *coprojections*. If I is a two-element set, we write $C_1 \coprod C_2$ for the product. This situation is indicated by the following commutative diagram:



In the general case, we have a family of commutative diagrams, indexed by I, all of which have the form:



Example A.2.6.
- In **Set**, the coproduct is the disjoint union operation. The coprojections are the standard inclusion functions.
- In **Grp**, the coproduct is the free product of groups. In **Ab**, the category consisting of abelian groups with group homomorphisms, the coproduct of the direct sum, which coincides exactly with the direct product when the family of abelian groups under consideration is finite. This will be discussed further in Section A.4.
- In $\mathbf{Top}_*$, the category of based topological spaces, the coproduct is the wedge sum.

We will now discuss a central concept: the concept of *adjointness* of two functors. As the examples will go some way towards making clear, this notion is ubiquitous.

A.3 Adjoint functors

Definition A.3.1. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of functors. We call this pair *adjoint* if, for every pair of objects C and D in $\mathcal{C}$ and $\mathcal{D}$ respectively, we have the following bijection:

$$\tau = \tau_{CD} : \mathrm{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, R(D))$$

Furthermore, this bijection is required to be natural in C and D in the sense that, for each $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}(L(C'), D) & \xrightarrow{Lf_*} & \mathrm{Hom}_{\mathcal{D}}(L(C), D) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{D}}(L(C), D') \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\
\mathrm{Hom}_{\mathcal{C}}(C', R(D)) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathcal{C}}(C, R(D)) & \xrightarrow{Rg_*} & \mathrm{Hom}_{\mathcal{C}}(C, R(D'))
\end{array}$$

In other words, τ is a natural isomorphism between the bifunctors $\mathrm{Hom}_{\mathcal{D}}(L(-), -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, R(-))$ which map from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to **Set**. As well as referring to (L, R) as an adjoint pair, we will also call L the *left adjoint* of R, and R the *right adjoint* of L. Here, f_* and g_* are symbols for the maps induced on the hom-sets by $f : C \to C'$ and $g : D \to D'$.

Example A.3.2. Let K be a field, and consider $L : \mathbf{Set} \to \mathbf{Vect}_K$, the functor which sends a set X to the vector space with basis the elements of X. This functor is left adjoint to the forgetful functor $U : \mathbf{Vect}_K \to \mathbf{Set}$ which sends every K-vector space to its underlying set. This is due to the fact that $\mathrm{Hom}_{\mathbf{Vect}_K}(L(X), V)$ has the same number of elements as $\mathrm{Hom}_{\mathbf{Set}}(X, U(V))$.

Remark. The preceding example is a particular case of a very general phenomenon: whenever we have a free object generated by a set, be it an abelian group, an R-module, an algebra over a field, or any other type of algebraic structure, the functor determined by forming the free object from its generating set is left adjoint to the forgetful functor which sends the free object to its underlying set. This is one contributing factor to the ubiquity of adjoint pairs.

Example A.3.3. Let R be a ring, and let B be a left R-module. For every abelian group A, $\mathrm{Hom}_{\mathbf{Ab}}(B, A)$ is a right R-module with action given by $(fr)(b) = f(rb)$. This defines a functor $\mathrm{Hom}_{\mathbf{Ab}}(B, -) : \mathbf{Ab} \to \textbf{mod-}\boldsymbol{R}$, which, together with the functor $- \otimes_R B : \textbf{mod-}\boldsymbol{R} \to \mathbf{Ab}$, forms an adjoint pair.

Example A.3.4. Consider the inclusion functor $\mathbf{Ab} \to \mathbf{Grp}$, along with the abelianization functor $\mathbf{Grp} \to \mathbf{Ab}$, which sends a group G to its abelianization $G/[G, G]$ These two functors form an adjoint pair.

A.4 Abelian categories

Definition A.4.1. A category $\mathcal{C}$ is called *additive* if the following conditions hold:

1. Each hom-set has an abelian group structure, which we will write additively. Furthermore, the composition of morphisms distributes over the addition; that is to say, for any morphisms $f : A \to B, f' : A \to B, g : B \to C$, and $g' : B \to C$, we have the following identities:

$$(g + g') \circ f = g \circ f + g' \circ f$$
$$g \circ (f + f') = g \circ f + g \circ f'$$

2. $\mathcal{C}$ has a zero object; in other words, an object which is both initial and terminal.

3. Given any two objects, their product and coproduct both exist.

We call categories where the first condition holds *preadditive*.

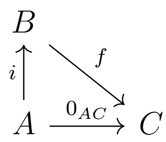
Example A.4.2. **Ab** is an additive category, as is ***R*-mod** for any ring R.

Example A.4.3. Any unital ring can be viewed as a preadditive category with one object: this is because of the abelian group structure and distributive law on the single hom-set, which consists of all endomorphisms of the category's single object.

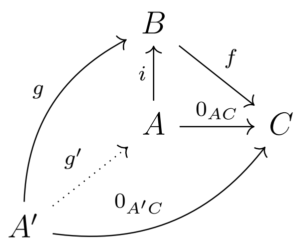
Definition A.4.4. Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if each $f : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$ is a homomorphism of abelian groups.

Remark. In the two examples of additive categories previously mentioned, finite direct products are the same as finite direct sums. This is also true for the category of all vector spaces over a field K, denoted $\mathbf{Vect}_K$. It is a fact, proven in Section 2.2 of [13], that this property is possessed by all additive categories.

Definition A.4.5. Let $\mathcal{A}$ be an additive category, and let $f : B \to C$ be a morphism in $\mathcal{A}$. A *kernel* of f, if it exists, is an object A together with a map $i : A \to B$ such that fi is equal to the 0 morphism from A to C. That is to say, the following diagram commutes for a unique g':



In addition, every morphism $g : A' \to B$ in $\mathcal{A}$ such that $fg = 0$ factors through A as $g = ig'$ for a unique $g' : A' \to A$. That is to say, the following diagram commutes:



Reversing all the arrows in the above diagrams, we obtain the dual notion of the *cokernel* of a map.

Remark. As is the case with the products and coproducts of Definitions A.2.1 and A.2.5, any two kernels or cokernels of a map f are uniquely isomorphic. Furthermore, every kernel is a monomorphism, and each cokernel is an epimorphism. In categories such as **Ab**, $\boldsymbol{R}$**-mod**, and $\mathbf{Vect}_K$, kernel and cokernel have their usual meanings, except that we often take the simplifying measure of identifying the kernel or cokernel with the object, which, in actuality, only comprises a part of it: for example, we often say that the kernel of the abelian group homomorphism $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}_2$ is merely $2\mathbb{Z}$, rather than $2\mathbb{Z}$ together with its inclusion in $\mathbb{Z}$.

Using the notions we have discussed in this section, we are now ready to make the definition on which the section hinges; that of an *abelian category.*

Definition A.4.6. Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ is an *abelian category* if the following conditions are true:

1. Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

2. Every monomorphism in $\mathcal{A}$ is the kernel of some morphism.

3. Every epimorphism in $\mathcal{A}$ is the cokernel of some morphism.

The prototypical examples of abelian categories are those which we have often discussed in examples: $\boldsymbol{R}$**-mod**, $\mathbf{Vect}_K$, and **Ab**. Chain complexes are often seen to come from these categories precisely because they are abelian. If $\mathcal{A}$ is any abelian cateory, we can form chain complexes in A, and chain maps between these chain complexes. These form an additive category, which we denote by $\mathbf{Ch}(\mathcal{A})$; homology is then a functor from $\mathbf{Ch}(\mathcal{A})$ to $\mathcal{A}$.

The following lemma is a famous result of homological algebra, which has myriad applications, although we will only use it once.

Lemma A.4.7 (The 5-Lemma)**.** *Let the diagram*

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E' \end{array}$$

be commutative in an abelian category. If the two rows are exact, m and p and isomorphisms, l is an epimorphism, and q is a monomorphism, n is an isomorphism.

Proof. See Chapter 2 of [16]. □

A.5 The Yoneda Embedding

We now proceed to briefly discuss the famous *Yoneda lemma*; if $\mathcal{C}$ is a locally small category, this provides a way to "represent" the structure of $\mathcal{C}$ in terms of the familiar category **Set**.

Lemma A.5.1 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, and let F be an arbitrary functor from $\mathcal{C}$ to **Set**. For each object C of $\mathcal{C}$, the natural transformations from the hom-functor $Hom(C,-)$ to F are in one-to-one correspondence with the elements of $F(C)$.*

Proof. See [1], Chapter 8. □

Of particular importance is the case where the functor F is another hom-functor, say $\mathrm{Hom}(D,-)$. In this situation, the natural transformations between the two hom-functors are in one-to-one correspondence with the set $\mathrm{Hom}(D,C)$.

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