FIRST YEAR REPORT

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ABSTRACT. We begin by briefly reviewing foundational material in higher algebraic K-theory: the notion of an exact category, Quillen's Q-construction, and the K-theory space of an exact category are all defined. We use this material to define the higher K-groups of a ring R. We then prove a version of Quillen's localization theorem for the case when R is a Dedekind domain.
This is followed by some material defining the Witt group $W(R)$ of a ring R, and then by a proof of an analogous localization theorem for the Witt groups of a Dedekind domain.
We then turn to the Hermitian K-theory version of the previous material, discussing, among other things, a localization theorem for Hermitian K-theory. We show how this gives an analogue of the algebraic K-theory and Witt theory localization sequences for Dedekind domains. We conclude by outlining a proof of the Hermitian localization theorem and giving some possible directions to enlarge its scope.

CONTENTS

1. Introduction

The earliest work done on algebraic K-theory centred on the Grothendieck group K_0 of a category. The "classical" theory used explicit algebraic presentations for its definitions; for example, given a ring R, $K_0(R)$ was defined to be the group completion of the abelian monoid consisting of isomorphism classes of projective R-modules, where the monoid operation is direct sum. This means, for instance, that $K_0(F)$ where F is any field is isomorphic to the integers, since all projective modules over a field are free. Explicit definitions were also given for K_1 and K_2: see ([10], Chapter III) for details.

During the search for a characterization of $K_n(R)$ for general n, several incompatible definitions were suggested. The "correct" definition was eventually given by Quillen in two steps; first, he used the *+-construction*, first introduced by Kervaire in [5], to define

$$K_n(R) = \pi_n(BGL(R)^+),$$

where π_n is a homotopy group, $GL(R)$ is the infinite general linear group given by $\bigcup_{n\in\mathbb{N}} GL_n(R)$, and B denotes taking the classifying space of the group. This definition only agrees with the previous definitions of K_n for $n > 0$. Quillen's next step was to repair this. He did so by defining

$$K_n(R) = \pi_n(BGL(R)^+ \times K_0(R)).$$

This is the correct definition, since $BGL(R)^+$ is path-connected and $K_n(R)$ is discrete, so it is equal to our previous definition in higher degrees and is also equal to the earlier definition of K_0. Later, in his seminal paper [7], Quillen defined the *Q-construction*, which we discuss in detail in the sequel (Definition 2.1.9). The general idea is the same as the +-construction: one constructs a topological space whose homotopy groups are K-groups. This time, however, the construction takes an exact category $\mathcal{E}$ (an additive category with a class of exact sequences) and builds a new category $Q\mathcal{E}$. One then constructs a CW complex $BQ\mathcal{E}$ from this category, and then the homotopy groups of the loop space $\Omega BQ\mathcal{E}$ are the K-groups of the exact category $\mathcal{E}$. Quillen proved that the +-construction gives the same K-groups as the Q-construction when the exact category $\mathcal{E}$ is the category of projective modules over R. However, the Q-construction has the advantages of being functorial immediately from the definition and applying to a wider range of situations. This technique can be used for other forms of K-theory; for example, the K-theory of a topological space X can be defined via a similar process, except the exact category one uses is the category of vector bundles over X.

Also in [7], Quillen proved a collection of fundamental theorems; for example, the results known as the "Devissage" (Theorem 2.2.2) and "Localization" (Theorem 2.2.1) theorems. The Devissage Theorem gives conditions under which two abelian categories have the same K-theory, and the Localization Theorem gives a long exact sequence of K-groups. These results are very important, since directly calculating K-groups can be very difficult. These theorems can also be formulated, using techniques inspired by Quillen's work, in the setting of *Hermitian K-theory*, a type of algebraic K-theory which is an invariant of rings with involution. It turns out to be the case that Hermitian K-theory has deep connections with another collection of invariants of rings, called the *Witt groups* (see [8] for a full account of this.) Indeed, one may say, as a rule of thumb, that if there is a theorem which has an analogue in both algebraic K-theory and Witt theory, then there should be a version for Hermitian K-theory. In the present paper, we illustrate this principle by discussing three different localization theorems for Dedekind domains; first for algebraic K-theory, then for Witt theory, and finally for Hermitian K-theory.

2. The Localization Theorem for Dedekind Domains

In this section, we discuss the 'classical' version of our main result. We begin by briefly reviewing some foundational material, first formulated by Quillen in [7].

2.1. **The K-Theory Space of an Exact Category.** First, we will need to define the notion of an exact category; as will be clear from the definition, this is a generalization of the notion of an exact sequence in an abelian category.

Definition 2.1.1. An *exact category* is a pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is an additive category and $\mathcal{M}$ is a family of sequences in $\mathcal{E}$ of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$$

satisfying the following: there exists an embedding of $\mathcal{E}$ as a full subcategory of an abelian category $\mathcal{A}$ such that

(i) $\mathcal{M}$ is the class of all sequences of the form above which are exact in $\mathcal{A}$,
(ii) $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, in the sense that if there is an exact sequence of the form above in $\mathcal{A}$ with $B, D \in \mathcal{E}$, then C is isomorphic to an object in $\mathcal{E}$.

The members of $\mathcal{M}$ are called the *short exact sequences* of $\mathcal{E}$. When the class $\mathcal{M}$ is clear, we will use the abbreviated notation $\mathcal{E}$. We call maps which occur as one of the i above *admissible monomorphisms*, and we call maps which occur as one of the j above *admissible epimorphisms*.
As one may expect, we will also need the notion of a structure preserving map between exact categories:

Definition 2.1.2. An *exact functor* $F : \mathcal{C} \to \mathcal{D}$ between exact categories is an additive functor which maps short exact sequences in $\mathcal{C}$ to short exact sequences in $\mathcal{D}$. If $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, and the exact sequences in $\mathcal{C}$ are precisely those sequences which are exact in $\mathcal{D}$, we call $\mathcal{C}$ an *exact subcategory* of $\mathcal{D}$.

Definition 2.1.3. Let $\mathcal{E}$ be a small exact category. Then $K_0(\mathcal{E})$ is defined to be the abelian group with generators $[C]$, one for each object C of $\mathcal{E}$, and relations $[C] = [B] + [D]$ for every short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{E}$.

Example 2.1.4. Every abelian category $\mathcal{A}$ is an exact category in an obvious way: simply take $\mathcal{M}$ to be the class of exact sequences in $\mathcal{A}$.

Example 2.1.5. The category $\mathbf{P}(R)$ of finitely generated projective R-modules is exact, since it embeds in the abelian category R-**mod**. As every exact sequence of projective modules splits, it follows from Definition 2.1.3 that we have $K_0\mathbf{P}(R) = K_0(R)$.

Remark. The example above is an important motivation for the notion of an exact category. The main point is that there are two different but related notions both denoted by K_0, and which both bear the name "Grothendieck group." One is the left adjoint of the forgetful functor from abelian groups to abelian monoids, and is used in the most classical definition of $K_0(R)$; one considers the isomorphism classes of projective R-modules, together with direct sum. This structure is an abelian monoid, and applying K_0 to it, as the notation suggests, yields the abelian group $K_0(R)$.
The second notion of Grothendieck group works like K_0 in Definition 2.1.3, except that it was initially defined only in an abelian category. For someone who wanted to put the K-theory of a ring in a categorical framework, this would pose a problem, since the category $\mathbf{P}(R)$ is not abelian. However, $\mathbf{P}(R)$ is certainly equipped with a collection of exact sequences, even though not all of its morphisms have kernels and cokernels, and the Grothendieck group of an abelian category is defined only in terms of exact sequences. An important innovation of Quillen was to exploit this, and introduce the more general idea of an exact category, which solves the aforementioned problem, and also, as we will see, opens the door to an elegant definition of the higher K-groups of a ring.

Our goal is to define the higher K-groups of a small exact category, such that, when the exact category is the category $\mathbf{P}(R)$, we obtain a definition of the higher K-groups of the ring R. To do this, we must first define the notion of the classifying space of a category.

Definition 2.1.6. Let $\mathcal{C}$ be a small category. Its *nerve*, denoted $N_*\mathcal{C}$, is the simplicial set whose p-simplices are the diagrams in $\mathcal{C}$ of the form

$$X_0 \to X_1 \to \cdots \to X_p.$$

The i-th face map of this simplex is obtained by deleting the object X_i, and the i-th degeneracy is obtained by replacing X_i with $\mathrm{id} : X_i \to X_i$.
The *classifying space* of $\mathcal{C}$, denoted $B\mathcal{C}$ or $|\mathcal{C}|$, is the geometric realization of $N\mathcal{C}$. This is a CW complex with p-cells in bijection with the nondegenerate p-simplices of the nerve. The precise definition is as follows.

Definition 2.1.7. Let S be a simplicial set. Its *geometric realization*, denoted $|S|$, is a topological space constructed in the following manner.
For each $p \geq 0$, make the product $S_p \times \Delta^p$ into a topological space by viewing it as the disjoint union of $|S_p|$ copies of Δ^p. We index these copies by the elements s of S_p, so that the number of copies is equal to the cardinality of S_p. Now, consider the disjoint union of each $S_p \times \Delta^p$ as a topological space, and denote this larger space by $\bar{S}$. Define the equivalence relation $\sim$ on $\bar{S}$ by the rule that $(s, x) \in S_m \times \Delta^m$ and $(t, y) \in S_n \times \Delta^n$ are equivalent if and only if there exists a map $\alpha : [m] \to [n]$ in the simplex category Δ such that $\alpha_*(t) = s$ and $\alpha^*(x) = y$. That is to say:

$$(\alpha_*(t), x) \sim (t, \alpha^*(s))$$

The space $\bar{X}/\sim$ is the geometric realization $|X|$.

Example 2.1.8. Let $\mathbf{n}$ be the poset $\{1 < \cdots < n\}$, viewed as a category. Then $N_p\mathbf{n}$ consists of the diagram $1 < \cdots < p$ if $1 \leq p \leq n$, and is empty otherwise. Thus, the classifying space of $\mathbf{n}$ is homeomorphic to the standard $n-1$ simplex.

We are now ready to define the higher K-groups of a small exact category $\mathcal{E}$: the idea is that they will be the homotopy groups of the classifying space of a category $Q\mathcal{E}$, which is known as the *Q-construction*, and was first defined by Quillen in [7].

Definition 2.1.9 (Quillen's Q-Construction)**.** Let $\mathcal{E}$ be a small exact category. Then the category $Q\mathcal{E}$ has the same objects as $\mathcal{E}$. A morphism from C to D in $Q\mathcal{E}$ is an equivalence class of diagrams

$$C \overset{j}{\twoheadleftarrow} D_2 \overset{i}{\rightarrowtail} D$$

where j is an admissible epimorphism and i is an admissible monomorphism.
Two such diagrams are equivalent if there is an isomorphism between them which is the identity on C and D. The composition of the morphism above with a morphism $D \twoheadleftarrow E_2 \rightarrowtail E$ is $C \twoheadleftarrow E_1 \rightarrowtail E$, where E_1 is the pullback of the diagram

$$\begin{tikzcd} & E_2 \arrow[d, two heads] \\ D_2 \arrow[r, tail] & D \end{tikzcd}$$

It is a fact ([10], IV.6.2) that the geometric realization $BQ\mathcal{E}$ is a connected CW complex with $\pi_1(BQ\mathcal{E}) \cong K_0(\mathcal{A})$. In light of this, we make the following definition.

Definition 2.1.10. Let $\mathcal{E}$ be a small exact category. Then $K(\mathcal{E})$, which we call the *K-theory space* of $\mathcal{E}$, denotes the loop space $\Omega BQ\mathcal{E}$, and we set

$$K_n(\mathcal{E}) = \pi_n K(\mathcal{E}) = \pi_{n+1}(BQ\mathcal{E}) \quad \text{for } n \geq 0.$$

In particular, if we let $\mathcal{E}$ be $\mathbf{P}(R)$, this gives a definition of the K-groups of the ring R which is in agreement with other definitions.

2.2. **Localization, Devissage, and Applications.** In this subsection, we discuss the 'classical' version of our main result. We begin by stating a fundamental theorem of algebraic K-theory, first proved by Quillen in [7]:

Theorem 2.2.1 (Localization)**.** *Let $\mathcal{B}$ be a Serre subcategory of a small abelian category $\mathcal{A}$. Then the sequence of K-theory spaces*

$$K(\mathcal{B}) \to K(\mathcal{A}) \xrightarrow{\text{loc}} K(\mathcal{A}/\mathcal{B})$$

is a homotopy fibration. Therefore, there is a long exact sequence of homotopy groups

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n(\mathcal{B}) \to K_n(\mathcal{A}) \xrightarrow{\text{loc}} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{n-1}(\mathcal{B}) \to \cdots \tag{2.1}$$

which ends with $K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$.

Proof. See [7]. □

Remark. In fact, localization also holds for certain types of exact category, but we will only need the abelian case in the present paper.

On a number of occasions, we will require another fundamental theorem due to Quillen:

Theorem 2.2.2 (Devissage)**.** *Let $i : \mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories such that $\mathcal{A}$ is an exact abelian subcategory of $\mathcal{B}$ and $\mathcal{A}$ is closed in $\mathcal{B}$ under subobjects and quotients. Suppose that every object B of $\mathcal{B}$ has a finite filtration*

$$0 = B_r \subset \cdots \subset B_1 \subset B_0 = B$$

by objects in $\mathcal{B}$ such that every subquotient B_i/B_{i-1} lies in $\mathcal{A}$. Then

$$K(\mathcal{A}) \simeq K(\mathcal{B}) \ \ \text{and} \ \ K_n(A) \cong K_n(B) \text{ for all } n.$$

Proof. See [7]. □

Now, let R be a Noetherian ring, and let S be a central, multiplicatively closed subset of R. Denote by $\mathbf{M}(R)$ the category of finitely generated left R-modules, and denote by $\mathbf{M}_S(R)$ the full subcategory of finitely generated S-torsion R-modules. Also, let $G(R)$ denote the K-theory space of $\mathbf{M}(R)$ and define the G-groups of R by $G_n(R) = K_n\mathbf{M}(R)$. It is a fact ([10], II.6.4.1) that $\mathbf{M}_S(R)$ is a Serre subcategory of $\mathbf{M}(R)$, with quotient category $\mathbf{M}(S^{-1}R)$. The localization theorem 2.2.1 gives a homotopy fibration

$$K\mathbf{M}_S(R) \to G(R) \to G(S^{-1}(R)). \tag{2.2}$$

It is observed in ([10], V.4.4) that $\mathbf{M}_S(R)$ is the colimit over all $s \in S$ of the categories $\mathbf{M}(R/sR)$, and that

$$K_*\mathbf{M}_S(R) \cong \varinjlim G_*(R/sR).$$

Now, let $\mathbf{Ch}^b_S\mathbf{M}(R)$ be the category of bounded chain complexes of S-torsion finitely generated left R-modules. Also, define $K\mathbf{Ch}^b_S\mathbf{M}(R) := G(R \text{ on } S)$. Comparing (2.2) to the homotopy fibration

$$G(R \text{ on } S) \to G(R) \to G(S^{-1}R)$$

from ([10], V.2.6.1), we see that the canonical map $\mathbf{M}_S(R) \to \mathbf{Ch}^b_S\mathbf{M}(R)$ induces a homotopy equivalence

$$K\mathbf{M}_S(R) \xrightarrow{\simeq} G(R \text{ on } S)$$

The prototypical case is when $S = \{s^n \mid n \in \mathbb{N}\}$, for some element $s \in R$ and $n \in \mathbb{N}$. In this situation, the maps $G(R/sR) \to G(R/s^nR)$ are homotopy equivalences by Devissage, so that we have $G(R/sR) \simeq K\mathbf{M}_S(R)$. At this point, we will require the notion of a transfer map.

Definition 2.2.3. For any ring R, let $\mathbf{H}(R)$ denote the category consisting of all R-modules which possess a finite resolution consisting of finitely generated projective R-modules.

Definition 2.2.4. Let $f : R \to S$ be a ring homomorphism such that S has a finite resolution consisting of finitely generated projective R-modules. Let M be an S-module; we can make M into an R-module, with action given by $r \cdot m = f(r) \cdot m$.
In fact, this defines a functor $\mathbf{P}(S) \to \mathbf{H}(R)$. We call the induced map

$$f_* : K(S) \to K\mathbf{H}(R) \cong K(R)$$

the *transfer map.* Obviously, the transfer map induces a map $f_* : K_n(S) \to K_n(R)$ for each n.

By inspection, the map $G(R/sR) \to G(R)$ identifying $G(R/sR)$ with the homotopy fibre of $G(R) \to G(R[1/s])$ is the transfer i_* associated to the map $i : R \to sR$. Thus, in this situation, the long exact localization sequence 2.1 becomes:

$$\cdots \to G_{n+1}(R[s^{-1}]) \xrightarrow{\partial} G_n(R/sR) \xrightarrow{i_*} G_n(R) \to G_n(R[s^{-1}]) \xrightarrow{\partial} \cdots \tag{2.3}$$

This is a sequence of $K_*(R)$ modules, because $\mathbf{P}(R)$ acts on the sequence of abelian categories

$$\mathbf{M}_S(R) \to \mathbf{M}(R) \to \mathbf{M}(S^{-1}R).$$

2.3. **Dedekind Domains.** Now, suppose D is a Dedekind domain, let F denote its field of fractions, and let $\mathfrak{p}$ denote a nonzero prime ideal of D. Then D, F, and $D/\mathfrak{p}$ are all regular, so that we have $K_*(D) \cong G_*(D), K_*(F) \cong G_*(F)$, and $K_*(D/\mathfrak{p}) \cong G(D/\mathfrak{p})$ ([10], V.3.3). This proves the following:

Theorem 2.3.1. *Using the notation immediately above, and setting $S = R \setminus \{0\}$, the long exact localization sequence 2.3 of the previous subsection becomes*

$$\cdots K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_n(R/\mathfrak{p}) \xrightarrow{\oplus (i_{\mathfrak{p}})_*} K_n(R) \to K_n(F) \xrightarrow{\partial} \cdots,$$

where the direct sums run over all nonzero prime ideals of R, and the maps $(i_{\mathfrak{p}})_*$ are transfer maps.

Bibliographical Note. The preceding section is based primarily on [7] and [10].

3. Localization for Witt groups

We now turn our attention to an analogous theorem for Witt groups of Dedekind domains. Throughout this section, except where otherwise stated, D will denote a Dedekind domain and F will denote its quotient field. We will begin by reviewing some basic definitions and results on Witt groups.

3.1. **Witt groups.** Throughout this subsection, R will be a commutative unital ring and X will be a left R-module.

Definition 3.1.1. A *bilinear form* on a left R-module X is a function

$$\beta : X \times X \to R$$

such that $\beta(x, y)$ is R-linear as a function of x for each fixed y, and R-linear as a function of y for each fixed x. Furthermore, a bilinear form β is called an *inner product* on X if it satisfies the following

non-degeneracy conditions.
Firstly, for each R-linear map

$$\varphi : X \to R$$

there should exist an element $x_0 \in X$ such that the homomorphism

$$x \mapsto \beta(x_0, y)$$

is equal to φ. Similarly, there should exist a unique $y_0 \in X$ such that the homomorphism $x \mapsto \beta(x, y_0)$ is equal to φ.

Example 3.1.2. Letting $R = \mathbb{C}$, any complex inner product space is a bilinear form module in the obvious way.

Our usual notation for an inner product will be $\beta(x, y) = x \cdot y$. If β is a bilinear form or inner product on X, we call the pair (X, β) a *bilinear form module* or an *inner product module* over R; also, when there is no danger of confusion, we will use the abbreviated notation X.

Definition 3.1.3. Two bilinear form modules (X, β) and (X', β') are *isomorphic* if there exists an R-linear bijection $f : X \to X'$ satisfying the identity $\beta'(f(x), f(y)) = \beta(x, y)$ for all x and y in X.

Finitely generated projective R-modules are of particular interest for our purposes:

Definition 3.1.4. An inner product module X will be called an *inner product space* if X is finitely generated and projective over R.

Also of particular interest are symmetric inner product spaces, which are defined as follows:

Definition 3.1.5. An inner product space X is *symmetric* if its inner product has the property that $x \cdot y = y \cdot x$ for all $x, y \in X$.

Finally, before defining the Witt group $W(R)$ of R, we will require the notion of a split inner product space, defined as follows:

Definition 3.1.6. A symmetric inner product space S over a ring R is *split* if there exists a submodule $N \subset S$ such that $N = N^{\perp}$ and N is a direct summand of S. ($N^{\perp} = \{s \in S \mid s \cdot N = 0\}$.)

We can now define the notion of the Witt group of R; for proof that $W(R)$ is a group, see ([4], Chapter I.)

Definition 3.1.7. We say that two symmetric inner product spaces X and X' over R are in the same *Witt class*, and denote this by $X \sim X'$, if there exist split inner product spaces S and S' such that $X \oplus S$ is isomorphic to $X' \oplus S'$. The Witt classes together with the orthogonal sum operation form a group $W(R)$, which we call the *Witt group* of R.

In order to prove the main theorem of this section, we will require some material on valuations.

3.2. **Valuations and the residue class form homomorphisms.**

Definition 3.2.1. A *discrete valuation* on a field F is a group homomorphism v from $F^{\times}$ to $\mathbb{Z}$ satisying the equation

$$v(\alpha + \beta) \geq \mathrm{Min}(v(\alpha), v(\beta))$$

for $\alpha, \beta, \alpha + \beta \neq 0$. Furthermore, we set $v(0) = \infty$.

Example 3.2.2. Let F be the field of rational functions with complex coefficients, in one indeterminate X. We can write $\frac{f(X)}{g(X)} \in F$ as $X^n \frac{s(X)}{t(X)}$, where $s(0)$ and $t(0)$ are non-zero and n is an integer. Then, the map $d : F^\times \to \mathbb{Z}$ which sends $\frac{f(X)}{g(X)}$ to n is a discrete valuation.

Notation. In this subsection, a discrete valuation on a general field F will be denoted v. The associated valuation ring, which consists of all $\alpha \in F$ such that $v(\alpha) \geq 0$, will be denoted $\mathcal{D}$. The unique maximal ideal of $\mathcal{D}$ (which consists of all elements $\alpha \in F$ such that $v(\alpha) > 0$) will be denoted B, and the residue class field $\mathcal{D}/B$ will be denoted $\bar{F}$. Finally, the image of $u \in \mathcal{D}^\times$ will be denoted $\bar{u} \in \bar{F}^\times$, and, for $u \in F^\times$, $\langle u \rangle$ will denote the symmetric inner product space with one element e_1 in its basis such that $e_1 \cdot e_1 = u$.

We aim to construct a group homomorphism $\partial_v : W(F) \to W(\bar{F})$, which will be well defined up to multiplication by units of the form $\langle \bar{u} \rangle$ in $W(\bar{F})$. In order to define ∂_v, we will make use of the following lemma, which will allow us to give a presentation of $W(F)$.

Lemma 3.2.3. *The abelian group $W(F)$ is generated by the elements $\langle u \rangle$ with $u \in F$, subject only to the following relations.*

(i) $\langle u \rangle = \langle u\alpha^2 \rangle$ *for* $\alpha \neq 0$.
(ii) $\langle u \rangle + \langle -u \rangle = 0$
(iii) $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u + v) \rangle$ *for* $u + v \neq 0$.

Proof. See ([4], Chapter IV, Lemma 1.1) □

Next, we choose a *prime element* $\pi \in \mathcal{D}$; that is, an element with $v(\pi) = 1$, so that $\pi\mathcal{D} = B$. Then, every element of $F^\times$ can be written uniquely as a product of the form $\pi^i u$ for $u \in \mathcal{D}^\times (i > 0.)$

Lemma 3.2.4. *Fixing π, and fixing $k \in \{0, 1\}$, there is a unique additive homomorphism*

$$\psi^k : W(F) \to W(\bar{F})$$

which maps each generator $\langle \pi^i u \rangle$ to $\langle \bar{u} \rangle$ when $i \equiv k$ mod 2 and to 0 when $i \not\equiv k$ mod 2.

Proof. By Lemma 3.2.3, we only need to check that each one of the relations which define $W(F)$ map to a valid relation in $W(\bar{F})$. This is immediately clear for all of the relations except the third. To show our assetions holds for relation (iii), we let ε_i denote 1 when $i \equiv k$ mod 2 and 0 when $i \not\equiv k$ mod 2. We must show that if $\pi^h u_1 + \pi^i u_2 = \pi^j u_3$, then $\varepsilon\langle \bar{u}_1 \rangle + \varepsilon\langle \bar{u}_2 \rangle = \varepsilon_j \langle \bar{u}_3 \rangle + \varepsilon_{h+i+j} \langle \bar{u}_1 \bar{u}_2 \bar{u}_3 \rangle$ in $W(\bar{F})$. After dividing by a suitable power of π (and re-indexing if necessary) we may assume that two of the three integers h, i, j are 0 and that the third is greater than 0. The argument splits into three cases here.

(1) When $h = i = j = 0, \bar{u}_1 + \bar{u}_2 = \bar{u}_3$, and the required equation follows.
(2) When $h > i = j = 0, \bar{u}_2 = \bar{u}_3$, hence $\langle \bar{u}_1 \rangle = \langle \bar{u}_1 \bar{u}_2 \bar{u}_3 \rangle$, and the required equation follows. The case $i > 0$ is totally analogous.
(3) When $0 = h = i < j, \bar{u}_1 + \bar{u}_2 = 0$, hence $\langle \bar{u}_1 \rangle + \langle \bar{u}_2 \rangle = 0, \langle \bar{u}_3 \rangle + \langle \bar{u}_1 \bar{u}_2 \bar{u}_3 \rangle = 0$, and the required equation follows.

□

Definition 3.2.5. The group homomorphisms ψ^0 and ψ^1 are called the two *residue class form homomorphisms* associated with the valuation v. In the present paper we will only require the use of ψ^1, and therefore use the alternative notation ∂_v.

Remark. The homomorphism ∂_v depends on the choice of prime element π; however, none of the following proofs will depend on it, so that our arguments are valid for the ∂_v obtained from any choice of π.

Now, let $\mathcal{D} \subset F$ be the valuation ring associated with v, and let $W(\mathcal{D}) \to W(F)$ be the natural induced group homomorphism.

Lemma 3.2.6. *The composition*

$$W(\mathcal{D}) \to W(F) \xrightarrow{\partial_v} W(\bar{F})$$

is zero.

Proof. Since $\mathcal{D}$ is a local ring, all projective $\mathcal{D}$-modules are free; this means that any inner product space over $\mathcal{D}$ can be expressed as an orthogonal sum consisting of inner product spaces of rank 1 with inner product matrix (u), and of inner product spaces of rank 2 with inner product matrices of the form

$$\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$$

with $\alpha \in B$, and with β and 1 in F.
In the first case, the corresponding element $\langle u \rangle$ in $W(F)$ clearly satisfies $\partial_v \langle u \rangle = 0$, since $u = \pi^0 u$ and $0 \not\equiv 1 \bmod 2$.
In the second case, if $\alpha \neq 0$, the corresponding element in $W(F)$ an be written as a sum $\langle \alpha \rangle + \langle \alpha(\alpha\beta - 1) \rangle$, where $\alpha\beta - 1 \equiv -1 \bmod B$ since $\alpha \in B$. Clearly, ∂_v annihilates any such sum.
Finally, if $\alpha = 0$, the given summand is split: thus, each possible orthogonal summand maps to zero in $W(\bar{F})$ and the result follows. ☐

3.3. **Witt groups of Dedekind domains.** Towards our goal, we can use the material in the previous subsection, since, for a Dedekind domain D, every nonzero prime ideal $\mathfrak{p} \subset D$ gives rise to a $\mathfrak{p}$-*adic valuation* on the quotient field F, which we will now define in two steps.

Definition 3.3.1. Let R be a principal ideal domain with field of fractions F, and let π be an irreducible element of R. Since every principal ideal domain is a unique factorization domain, every nonzero $a \in R$ can be written uniquely as

$$a = \pi^{e_a} p_1^{e_1} \dots p_n^{e_n},$$

where the e_i are non-negative integers and the p_i are irreducible elements of R, under the condition that there is no unit u of R such that $\pi = up_i$ or $p_i = up_j$ for any i, j with $i \neq j$. Then, the π-*adic* valuation v_π on F is given by $v_\pi(0) = \infty$ and $v_\pi(a/b) = e_a - e_b$ for nonzero a and b in R.

Definition 3.3.2. Let D be a Dedekind domain with field of fractions F, and let $\mathfrak{p}$ be a non-zero prime ideal of D. Then the localization $D_\mathfrak{p}$ is a principal ideal domain with field of fractions F. The construction of the previous definition applied to the prime ideal $\mathfrak{p}D_\mathfrak{p}$ of $D_\mathfrak{p}$ yields the $\mathfrak{p}$-*adic valuation* on F.

The $\mathfrak{p}$-adic valuation has residue class field $D/\mathfrak{p}$. We denote the associated homomorphism $W(F) \to W(D/\mathfrak{p})$ by $\partial_\mathfrak{p}$.
Now, let X be an inner product space over F. Given a finite subset $\{x_1, \dots, x_k\} \subset X$ containing a basis for X over F, we form the D-submodule

$$L = Dx_1 + \cdots + Dx_k \subset X.$$

Definition 3.3.3. We call any D-submodule of the form above a *lattice* or a D-*lattice* in X.

Given a lattice $L \subset X$, the dual lattice $L^\sharp \subset X$ is defined thus:

$$L^\sharp := \{x \in X \mid x \cdot l \in D \ \forall\, l \in L\},$$

where $\cdot$ denotes the inner product of X.

Remark. For any lattice L, the dual lattice $L^\sharp$ is a D-module, canonically isomorphic to $\mathrm{Hom}_D(D, L)$; indeed, every D-linear map from L to D extends uniquely to an F-linear map from X to F, which must have the form $x \mapsto x \cdot x_0$ for some unique $x_0 \in L^\sharp$.

At this point, we will require a general lemma from homological algebra.

Lemma 3.3.4. *Let D be a Dedekind domain. Then every finitely generated torsion free D-module is projective.*

Proof. See ([2], VII.5). $\square$

We use this lemma to conclude that L and $L^\sharp$ are projective. We will now state and prove a key theorem, from which the main result of this section will follow quickly.

Theorem 3.3.5. *An inner product space X over F contains a lattice L such that $L = L^\sharp$ if and only if the Witt class of X is in the kernel of the homomorphism $\partial_{\mathfrak{p}} : W(F) \to W(D/\mathfrak{p})$ for every nonzero prime ideal $\mathfrak{p}$ of D.*

Proof. We begin by noting that L is self-dual if and only if the inner product on X, when restricted to L, makes L into an inner product space over D.
One direction of the proof is straightforward; if X contains a self-dual lattice, it is in the image of the map $W(D) \to W(F)$: it therefore follows immediately from Lemma 3.2.6 that X is in the kernel of $\partial_{\mathfrak{p}}$ as required.
For the converse, we will work on a case-by-case basis. First, we assume that D has a unique nonzero prime ideal $\mathfrak{p}$, so that D is the valuation ring associated with the $\mathfrak{p}$-adic valuation on F. Now, if X is an inner product space over F, we can write it in the following way:

$$X \cong \langle \pi u_1 \rangle \oplus \cdots \oplus \langle \pi u_m \rangle \oplus \langle u_{m+1} \rangle \oplus \cdots \oplus \langle u_n \rangle,$$

where π is defined as above and the u_i are units of F. This is valid by Lemma 3.2.3 and the discussion immediately after; taken together, these say that $W(F)$ is generated by elements of the form $\langle \pi^i u \rangle$ where i is an integer and u is a unit of F. Also, we have the relation $\langle u \rangle = \langle u\alpha^2 \rangle$ for $\alpha \neq 0$. Combining all of this information allows us to consider the powers of π in the generators modulo 2, which lets us write X in the form above.
If the Witt class of X is in $\ker \partial_{\mathfrak{p}}$, then, by definition, the inner product space over $D/\mathfrak{p}$ defined by

$$\langle \bar{u}_1 \rangle \oplus \cdots \oplus \langle \bar{u}_m \rangle$$

is split; the generators with a factor of π go to 0 since $0 \not\equiv 1 \bmod 2$. Thus, this inner product space has an inner product matrix of the form

$$\begin{bmatrix} 0 & I \\ I & B \end{bmatrix}$$

with respect to a suitable basis.
Lifting to the ring D, it follows that the inner product space $\langle u_1 \rangle \oplus \cdots \oplus \langle u_m \rangle$ has an inner product matrix of the form

$$\begin{bmatrix} A & I \\ I & B \end{bmatrix}$$

such that every entry of the matrix A is in $\mathfrak{p}$. Hence, this is also true of the inner product space $\langle u_1 \rangle \oplus \cdots \oplus \langle u_m \rangle$ over F.
Tensoring with the inner product space $\langle \pi \rangle$, we obtain that the inner product space $\langle \pi u_1 \rangle \oplus \cdots \oplus \langle \pi u_m \rangle$ has inner product matrix

$$\begin{bmatrix} \pi A & \pi I \\ \pi I & \pi B \end{bmatrix}$$

Multiplying each of the first $m/2$ basis vectors by π^{-1}, we obtain

$$\begin{bmatrix} \pi^{-1}A & I \\ I & \pi B \end{bmatrix}$$

This is a matrix with entries in D, whose determinant is a unit of D.
Now, consider the lattice L in $\langle \pi u_1 \rangle \oplus \cdots \oplus \langle \pi u_m \rangle$ spanned by our modified basis. The inner product defined by the matrix immediately above, when restricted to L, gives an inner product in L; this is implied by the combination of the facts that it has entries in D and that its determinant is a unit of D. Furthermore, we have an obvious self-dual lattice in $\langle u_{m+1} \rangle \oplus \cdots \oplus \langle u_n \rangle$; simply take the D-submodule spanned by $u_{m+1}, \ldots, u_n$. Forming the direct sum of the two self-dual lattices we have defined gives the required self-dual lattice in X.
Now, we turn our attention to the case where D has more than one nonzero prime ideal. For each $\mathfrak{p}$, let $D_{\mathfrak{p}} \subset F$ be the associated valuation ring. Choosing a basis $e_1, \ldots, e_n$ for X, we note that each inner product $e_i \cdot e_j$ belongs to $D_{\mathfrak{p}}$ for all but a finite number of nonzero prime ideals. Similarly, the determinant of the inner product matrix $(e_i \cdot e_j)$ belongs to $D_{\mathfrak{p}}$ for all but a finite number of nonzero prime ideals.
Thus, there exists a finite set S of nonzero prime ideals such that the $D_{\mathfrak{p}}$-lattice

$$D_{\mathfrak{p}} e_1 + \cdots + D_{\mathfrak{p}} e_n$$

is self-dual for all $\mathfrak{p} \notin S$. Now, suppose that $\partial_{\mathfrak{p}}(X) = 0$ for all $\mathfrak{p}$. Then, by arguments above, for each $\mathfrak{p} \in S$, we can choose a self-dual $D_{\mathfrak{p}}$-lattice.
At this point, we will require a lemma.

Lemma 3.3.6. *Let X be an F-vector space with basis $e_1, \ldots, e_n$. Given a $D_{\mathfrak{p}}$-lattice $L_{\mathfrak{p}}$ in X for each prime $\mathfrak{p}$, subject to the restriction that*

$$L_{\mathfrak{p}} = D_{\mathfrak{p}} e_1 + \cdots + D_{\mathfrak{p}} e_n$$

for all but a finite number of $\mathfrak{p}$, there is a unique D-lattice

$$L = \bigcap_{\mathfrak{p}} L_{\mathfrak{p}}$$

such that the $D_{\mathfrak{p}}$-lattice spanned by L is equal to $L_{\mathfrak{p}}$ for every $\mathfrak{p}$.

Proof. See ([6], §81 : 14.) □

We can combine this lemma with the previous discussion to conclude that there exists a D-lattice L with the property that the induced $D_{\mathfrak{p}}$-lattice $D_{\mathfrak{p}}L$ is self-dual for every non-zero prime ideal $\mathfrak{p}$. Our aim is to show that L is self-dual.
To see this, first let x and y be elements of L. Then $x \cdot y \in D_{\mathfrak{p}}$ for every $\mathfrak{p}$, so $x \cdot y \in D$. This proves that $L \subset L^\sharp$.
Conversely, if $x \in L^\sharp$, then $x \cdot D_{\mathfrak{p}} L \subset D_{\mathfrak{p}}$ for every prime $\mathfrak{p}$, hence

$$x \in \bigcap_{\mathfrak{p}} (D_{\mathfrak{p}} L)^\sharp = \bigcap_{\mathfrak{p}} D_{\mathfrak{p}} L = L.$$

Therefore, the lattice L is self-dual, as required. □

In light of this theorem, we can proceed immediately to our main result:

Corollary 3.3.7. *For any Dedekind domain D with field of fractions F, the sequence*

$$0 \to W(D) \to W(F) \xrightarrow{\oplus \partial_{\mathfrak{p}}} \oplus_{\mathfrak{p}} W(D/\mathfrak{p})$$

is exact, where the direct sum is over all nonzero prime ideals of D.

Proof. Exactness at $W(F)$ follows immediately from Theorem 3.3.5, so we only need to show that the map $W(D) \to W(F)$ is injective. This is the same as showing that, if an inner product space L over D corresponds to a split inner product space over F, then L is itself split.
To this end we think of L as a self-dual lattice in the inner product space $X = F \otimes_D L$. Let $N \subset X$ be a subspace of half the dimension, with $N \cdot N = 0$, so that $N = N^\perp$. Then, we have that $N \cap L$ is a self-orthogonal subspace of L; indeed, let x be an element of L orthogonal to all of $N \cap L$. Then it is orthogonal to all of N, and hence belongs to $N^\perp \cap L = N \cap L$, as required.
It only remains to show that $N \cap L$ is a direct summand of L, but this is the case since the quotient

$$L/(N \cap L) \subset X/N$$

is finitely generated and torsion free, hence projective by Lemma 3.3.4. Thus, L is split, as required. □

Example 3.3.8. In some situations, the maps $\partial_\mathfrak{p}$ above are surjective, so that we have exactness at $\oplus_\mathfrak{p} W(D/\mathfrak{p})$. In particular, this is true when $D = \mathbb{Z}$, so that we have an exact sequence

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \to \oplus W(\mathbb{Z}/p\mathbb{Z}) \to 0,$$

where the direct sum runs over all primes p.

Bibliographical Note. The preceding section is based primarily on [4].

4. Localization for Hermitian K-theory

In this concluding section, we turn our attention to the analogue of the localization theorem for Hermitian K-theory.

4.1. **Preliminaries.**

Before we can state the main theorem of the section, we will require some preliminary material. First, let R be a ring, and write R-free for the category of finitely generated left R-modules. Further, we write $\mathbf{F}(R)$ for the category whose objects are the natural numbers, with morphisms from n to m being given by the abelian group of $m \times n$ matrices (r_{ij}) with entries in R, and with composition of morphisms being given by matrix multiplication. Clearly, $\mathbf{F}(R)$ and R-free are equivalent categories.
Similarly, we will write R-proj for the category of finitely generated projective left R-modules.

Definition 4.1.1. Let $\mathcal{C}$ be a category. Then the *idempotent completion* of $\mathcal{C}$, denoted $\tilde{\mathcal{C}}$, is the category whose objects are pairs (C, p) with C an object of $\mathcal{C}$ and with p a morphism from C to C such that $p^2 = p$. A map in $\tilde{\mathcal{C}}$ from (C, p) to (D, q) is a map $f : C \to D$ such that $f \circ p = f = q \circ f$, and composition is given by composition of maps in $\mathcal{C}$. Further, the category $\mathcal{C}$ is called *idempotent complete* if the functor $\mathcal{C} \to \tilde{\mathcal{C}}$ given by $C \mapsto (C, \mathrm{id}_C)$ is an equivalence of categories.
Agreeing with notation used earlier in the paper, we write $\mathbf{P}(R)$ for the idempotent completion of $\mathbf{F}(R)$: this is valid because $\mathbf{P}(R)$ is equivalent to R-proj. To see this, let (n, p) be an object of $\mathbf{P}(R)$. Then the module pR^n is projective, since we have the direct sum decomposition $R^n \cong pR^n \oplus (1-p)R^n$. Defining a functor which sends (n, p) to pR^n gives the desired equivalence.

Remark. We use functorial versions of R-free and R-proj (denoted $\mathbf{F}(R)$ and $\mathbf{P}(R)$) respectively) because doing so will make it easier to check the existence of dualities and duality-preserving functors, which are fundamental to Hermitian K-theory and will be defined shortly. In fact, we will work with explicit functorial versions of most of the categories defined in what follows.

Now, let $\Sigma \subset R$ be a multiplicative subset of central non-zero divisors, and let $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ be the category whose objects are the left R-modules for which there is an exact sequence of left R-modules

$$0 \to P_1 \xrightarrow{i} P_0 \to M \to 0$$

such that P_0 and P_1 are in R-proj and $\Sigma^{-1}i : \Sigma^{-1}P_1 \to \Sigma^{-1}P_0$ is an isomorphism. (Equivalently, $\Sigma^{-1}M = 0$.)
We make $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ into an exact category by declaring a sequence to be exact if it is exact as a sequence of R-modules.
Now, we wish to work with a functorial version of $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$, which we denote by $\mathcal{T}_\Sigma$ and define as follows:

- The objects of $\mathcal{T}_\Sigma$ are monomorphisms $i : P_1 \to P_0$ fitting into an exact sequence of the form above, such that $\Sigma^{-1}i$ is an isomorphism.
- The abelian group of morphisms from $P_1 \xrightarrow{i} P_0$ to $Q_1 \xrightarrow{j} Q_0$ is the abelian group of pairs (f_1, f_0) of morphisms $f_i : P_i \to Q_i$ $(i = 0, 1)$ with $f_0 i = j f_1$ modulo pairs of the form (hi, jh) for some map $h : P_0 \to Q_1$.

Objects for which i is the identity map of some projective module P are zero objects in $\mathcal{T}_\Sigma$; we identify all of these, and refer to the resulting zero object as the base point. It is not difficult to see that the functor from $\mathcal{T}_\Sigma$ to $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ given by sending i to its cokernel is an equivalence of categories. We aim to use this equivalence to give $\mathcal{T}_\Sigma$ the structure of an exact category. First, we will define a fundamental notion of hermitian K-theory; that of a category with duality.

Definition 4.1.2. A *category with duality* is a triple $(\mathcal{C}, \sharp, \eta)$, where $\mathcal{C}$ is a category, $\sharp$ is a functor from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$, and $\eta : \mathrm{id}_{\mathcal{C}} \to \sharp\sharp$ is a natural isomorphism, such that for all objects C of $\mathcal{C}$, we have $1_{C^\sharp} = \eta_C^\sharp \circ \eta_{C^\sharp}$. Given two categories with duality $(\mathcal{C}, \sharp, \eta)$ and $(\mathcal{D}, \sharp, \tau)$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called *duality-preserving* if $\sharp \circ F = F^{\mathrm{op}} \circ \sharp$ and $F(\eta_C) = \tau_{F(C)}$ for every object C of $\mathcal{C}$.

As one may expect, this notion can be extended, so that the category $\mathcal{C}$ has extra structure, and this structure is compatible with the duality:

Definition 4.1.3. An *exact (resp. preadditive) category with duality* is a category with duality $(\mathcal{E}, \sharp, \eta)$ such that $\mathcal{E}$ is an exact (resp. preadditive) category with duality, and such that the functor $\sharp : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$ is exact (resp. additive).

Definition 4.1.4. Given a category with duality $(\mathcal{C}, \sharp, \eta)$, its associated *hermitian category* $\mathcal{C}_h$ is defined as follows.
The objects are pairs (M, ϕ), with M an object of $\mathcal{C}$ and $\phi : M \to M^\sharp$ an isomorphism such that $\phi = \phi^\sharp \eta$. A morphism $\alpha : (M, \phi) \to (N, \psi)$ is a morphism from M to N in $\mathcal{C}$ such that $\alpha^\sharp \psi \alpha = \phi$.

Let ϵ be either 1 or -1, and let $(\mathcal{C}, \sharp, \eta)$ be a preadditive category with duality. Then we denote by ${}_\epsilon\mathcal{C}_h$ the hermitian category associated to the category with duality $(\mathcal{C}, \sharp, \epsilon\eta$.)

Example 4.1.5. Let R be a ring with involution; that is to say, a ring with a map $* : R \to R^{\mathrm{op}}$ satisying $(a+b)^* = a^* + b^*, (ba)^* = a^* b^*$, and $(a^*)^* = a$.
In other words, $(R, *, \mathrm{id})$ is a preadditive category with duality with one object. This duality $*$ can be extended to a duality $\sharp$ on $\mathbf{F}(R)$ and $\mathbf{P}(R)$ as follows:

- On $\mathbf{F}(R)$, $n = n^\sharp$, and $(r_{ij})^\sharp = (r^*_{ji})$. (For example, when $R = \mathbb{C}$, with $*$ given by complex conjugation, this is the same as taking the conjugate transpose of a matrix.)
- On $\mathbf{P}(R)$, we use the duality on $\mathbf{F}(R)$; $(n, p)^\sharp = (n^\sharp, p^\sharp)$.

With these definitions, one can verify by direct inspection that $(\mathbf{F}(R), \sharp, \mathrm{id})$ and $(\mathbf{P}(R), \sharp, \mathrm{id})$ are additive (in fact, exact) categories with duality.

Remark. Since the two categories are equivalent, one may expect the duality $\sharp$ on $\mathbf{P}(R)$ to have an analogue in the category R-proj; this is indeed the case, and it turns out to have a familar description, which we will now outline. Let M be a left R-module, and let N be an R-bimodule. We define the following left R-module:

$$\mathrm{Hom}_{\text{skew R}}(M, N) := \{f \in \mathrm{Hom}_{\mathbb{Z}}(M, N) \mid f(rm) = r^* f(m)\}$$

which is an R-module with the action $r \cdot f(m) = rf(m)$.
Then, the duality on R-proj is given by $M^\sharp := \mathrm{Hom}_{\text{skew R}}(M, R)$. The natural isomorphism $\mathrm{id} \to \sharp\sharp$ is given at each M by the map $M \to M^{\sharp\sharp}$ which sends m to ev_m^*, where ev_m is the evaluation at m. If R is commutative with trivial involution, then $\sharp$ is simply $\mathrm{Hom}_R(-, R)$.

We now wish to define a duality on the category $\mathcal{T}_\Sigma$. As above, let $(R, \sharp)$ be a ring with involution and let $\Sigma \subset R$ be a multiplicative subset of non-zero divisors; in addition, let Σ be closed under the involution. Then the localization $\Sigma^{-1}R$ exists and $\sharp$ defines an involution on $\Sigma^{-1}R$ by $(s^{-1}r)^\sharp = (s^\sharp)^{-1}a^\sharp$. We denote by $\mathrm{Ext}^1_{\text{skew}R}(-, R)$ the first right derived functor of $\mathrm{Hom}_{\text{skew R}}(-, R)$: this functor induces an exact duality on $\mathcal{H}^1_{\Sigma,\text{proj}}$, and there exists a natural isomorphism $\eta : \mathrm{id} \to (\mathrm{Ext}^1)^2$. By our assumptions on Σ, the localization map $R \to \Sigma^{-1}R$ is injective and respects the involution structure. This implies that, for i an object of $\mathcal{T}_\Sigma$, we have that $i^\sharp : P_0^\sharp \to P_1^\sharp$ is injective, and that the cokernel of $i^\sharp$ is Σ-torsion, so that $i^\sharp$ is an object of $\mathcal{T}_\Sigma$. Thus, the assignment $i \mapsto i^\sharp$ defines a duality on $\mathcal{T}_\Sigma$, which makes $(\mathcal{T}_\Sigma, \sharp, \mathrm{id})$ into an additive category with duality. Moreover, since $\sharp$ is simply an explicit version of $\mathrm{Ext}^1_{\text{skew}R}(-.R)$, which is exact, $\sharp$ is also exact, so that $(\mathcal{T}_\Sigma, \sharp, \mathrm{id})$ is an exact category with duality.
For a given exact category $\mathcal{E}$, we must also define a simplicial exact category with duality (a simplicial object in the category of exact categories with duality) $\mathcal{R}_*\mathcal{E}$. The starting point for this is in [9]; therein, Waldhausen constructs a simplicial exact category $S_*\mathcal{E}$ such that the classifying space of $iS_*\mathcal{E}$ (the category with the same objects as $S_*\mathcal{E}$ and morphisms the isomorphisms of $S_*\mathcal{E}$) is homotopy equivalent to $Q\mathcal{E}$. If $\mathcal{E}$ happens to be an exact category with duality $\sharp$, then pointwise application of $\sharp$ makes $S_n\mathcal{E}$ an exact category with duality for each n. A slight difficulty arises here: one can show that the assignment $n \mapsto S_n\mathcal{E}$ is not a simplicial exact category with duality, since the face and degeneracy maps do not commute with the dualities. However, this problem can be overcome by considering the *edge-wise subdivison* $n \mapsto S_{n+1}\mathcal{E}$, which is a simplicial exact category with duality. The aforementioned category $\mathcal{R}_*\mathcal{E}$ will be a version of the edge-wise subdivision.
To make this precise, let $n \geq 0$ be an integer, and let $\mathbf{n}$ be the totally ordered set

$$\{n' < (n-1)' < \cdots < 0' < 0 < \cdots < (n-1) < n\}.$$

We make $\mathbf{n}$ into a category with duality by setting $l^\sharp = l'$ and $(l')^\sharp = l$ for $0 \leq l \leq n$. We will write $'$ for the duality $\sharp$.
Now, let $\theta : [n] \to [m]$ be a map in the simplicial category Δ. Then sending $(\theta : [n] \to [m])$ to $(\underline{\theta} : \mathbf{n} \to \mathbf{m})$, where $\underline{\theta}(l) = \theta(l)$ and $\underline{\theta}(l') = \theta(l)'$ makes the assignment $[n] \mapsto \mathbf{n}$ into a cosimplicial category with duality.
Now, denote by $I(n)$ the category of arrows in $\mathbf{n}$; that is to say, the objects of $I(n)$ are pairs $(p, q) \in \mathbf{n} \times \mathbf{n}$ such that $p \leq q$, and the morphisms of $I(n)$ are commutative squares in $\mathbf{n}$. The duality on $\mathbf{n}$ induces a duality on $I(n)$. Also, the cosimplicial structure defined by $[n] \mapsto \mathbf{n}$ makes $I(n)$ into a cosimplicial category with duality. With all of this information in place, we can now define the category $\mathcal{R}_*\mathcal{E}$.

Definition 4.1.6. Let $(\mathcal{E}, \sharp, \eta)$ be an exact category with duality. Fix a zero object with $0 = 0^\sharp$ and call it the base point. Then the simplicial exact category $(\mathcal{R}_*\mathcal{E}, \sharp, \eta)$ is defined as follows.
The objects of $R_n\mathcal{E}$ are functors $A : I(n) \to \mathcal{E}$ where all the sequences $A_{pq} \to A_{pr} \to A_{qr}$ are admissible exact sequences in $\mathcal{E}$ whenever $p \leq q \leq r$ in $\mathbf{n}$, and where $A_{pp} = 0$, the base point object of $\mathcal{E}$. The morphisms of $R_n\mathcal{E}$ are natural transformations. The dual of an object is given by $(A^\sharp)_{pq} = (A_{q'p'})^\sharp$, and the dual of a morphism is given by taking the point-wise dual and re-indexing. We set $(\eta_A)_{pq} = \eta_{A_{pq}}$. Finally, the exact structure on $\mathcal{R}_n\mathcal{E}$ is defined pointwise by the additive split exact structure on $\mathcal{E}$; that is, the exact structure where the admissible exact sequences are exactly the split exact sequences.

Remark. We may write $\mathcal{T}_\Sigma^\oplus$ for the category $\mathcal{T}_\Sigma$ equipped with the split exact structure; evidently, this category is equivalent as an exact category to $\mathcal{R}_0\mathcal{T}_\Sigma$.

Now, the simplicial structure on $(\mathcal{R}_*\mathcal{E}, \sharp, \eta)$ is induced by the cosimplicial structure on $I(*)$. We write $\mathcal{R}_*^h\mathcal{E}$ for the hermitian category of $\mathcal{R}_*\mathcal{E}$. Forgetting the duality, one easily sees that $\mathcal{R}_n\mathcal{E}$ is equivalent to $S_{n+1}\mathcal{E}$.

Definition 4.1.7. Given an exact category with duality $(\mathcal{E}, \sharp, \eta)$, we define a topological space ${}_\epsilon\mathcal{W}(\mathcal{E})$ to be the geometric realization of a certain bisimplicial set:

$$ {}_\epsilon\mathcal{W}(\mathcal{E}) := |(p,q) \mapsto N_p^i\,{}_\epsilon\mathcal{R}_q^h\mathcal{E}| $$

As earlier, N_* denotes the nerve of a category. We call this realization the *$\mathcal{W}$-theory space.* We may write ${}_\epsilon W(R)$ for the W-theory space associated with $(P(R), \sharp, \mathrm{id})$, where R is a ring with involution. Finally, the *U-theory space* of an exact category with duality $\mathcal{E}$ is defined thus:

$$ {}_\epsilon U(\mathcal{E}) := \Omega_\epsilon\mathcal{W}(\mathcal{E}) $$

4.2. **The Hermitian K-theory Space of an Exact Category.** Analogously to standard K-theory, we wish to construct a space $K^h(\mathcal{E})$, where $\mathcal{E}$ is an exact category with duality, such that the homotopy groups of $K^h(\mathcal{E})$ are the hermitian K-groups of $\mathcal{E}$. In particular, we will define the hermitian K-groups of a ring with involution R to be the homotopy groups of $K^h(\mathbf{P}(R))$. To do this, we will have to define a Hermitian analogue of Quillen's Q-construction.

Definition 4.2.1. Let $(\mathcal{E}, \sharp, \eta)$ be an exact category with duality, and consider the hermitian category $\mathcal{E}_h$. We define a category $Q^h(\mathcal{E}, \sharp, \eta)$ as follows. The objects are the objects of $\mathcal{E}_h$. A map $(M, \phi) \to (N, \psi)$ is an equivalence class of diagrams

$$ M \xleftarrow{\;j\;} U \xrightarrow{\;i\;} N $$

as in the ordinary Q-construction, but, as well as being either admissible epimorphisms or admissible monomorphisms respectively, j and i must also be morphisms in $\mathcal{E}_h$, and i must induce an isomorphism from $\ker(j)$ to $\ker(i^\sharp\psi)$.

Remark. Per Remark 1.11 in [3], the category $Q^h\mathcal{E}$ has classifying space homotopy equivalent to the space $\mathcal{W}(\mathcal{E})$ of the previous subsection.

With this definition in place, we can define the Hermitian K-theory space of an exact category with duality.

Definition 4.2.2. Let $(\mathcal{E}, \sharp, \eta)$ be an exact category with duality. The obvious forgetful functor from $Q^h\mathcal{E} \to Q\mathcal{E}$ which sends (M, ϕ) to M induces a map

$$ BQ^h\mathcal{E} \to BQ\mathcal{E} $$

on classifying spaces whose homotopy fibre (considering a zero object of $\mathcal{E}$ to be a base point of $Q\mathcal{E}$) is defined to be the *Hermitian K-theory space* $K^h(\mathcal{E}, \sharp, \eta)$ of $\mathcal{E}$. The Hermitian K-groups of $\mathcal{E}$ are defined to be the homotopy groups of this space.

Definition 4.2.3. For a ring with involution R and ϵ equal to 1 or -1, the Hermitian K-theory space is defined thus:

$$ {}_\epsilon K^h(R) = K^h(\mathbf{P}(R), \sharp, \mathrm{id}), $$

where $(\mathbf{P}(R), \sharp, \mathrm{id})$ is the exact category with duality defined in Example 4.1.5.

Remark. Hermitian K-theory is a generalization of algebraic K-theory in the following way: let R be a ring, and consider the ring with involution $R \times R^{\mathrm{op}}$, where the involution is given by swapping the factors. It is a fact ([3], Remark 3.3) that $K^h(R \times R^{\mathrm{op}})$ is homotopy equivalent to $K(R)$.

4.3. **Localization Theorems for Hermitian K-Theory.** We now have all the elements in place required to state localization theorems for Hermitian K-theory. We begin with the most general case.

Theorem 4.3.1 (Hermitian Localization)**.** *Let $(R, \sharp)$ be a ring with involution, in which 2 is a unit, and, as above, let $\Sigma \subset R$ be a multiplicative subset of non-zero divisors closed under the involution $\sharp$. Then, there is a homotopy fibration*

$$_{\epsilon}U(\mathcal{T}_{\Sigma}) \to {}_{\epsilon}K^h(R) \to {}_{\epsilon}(\Sigma^{-1}R),$$

where the map $_{\epsilon}K^h(R) \to_{\epsilon} (\Sigma^{-1}R)$ is obtained by applying the Hermitian K-theory functor to the localization map $(R, \sharp) \to (\Sigma^{-1}R, \sharp)$ which is a map of rings with invoution by assumption.

Corollary 4.3.2. *Under the hypotheses of Theorem 4.3.1, we have a long exact sequence*

$$\cdots \to {}_{\epsilon}K^h_{n+1}(\Sigma^{-1}R) \to {}_{\epsilon}U_n(\mathcal{T}_{\Sigma}) \to {}_{\epsilon}K^h_n(R) \to {}_{\epsilon}K^h_n(\Sigma^{-1}R) \to \cdots$$

Proof. This is the long exact sequence associated to the homotopy fibration of Theorem 4.3.1. □

Remark. The long exact sequence above ends at $_{\epsilon}K^h_0(\Sigma^{-1}R)$, since the map $_{\epsilon}K^h_0(R) \to {}_{\epsilon}K^h_0(\Sigma^{-1}R)$ is not surjective in general. This turns out to be implied by the fact that the map of Witt groups $W(R) \to W(\Sigma^{-1}R)$ is not surjective in general.
If R is a Dedekind domain and $\Sigma = R\backslash\{0\}$, we proved (Corollary 3.3.7) that the map on Witt groups is injective; however it is not an isomorphism in general.

There also exists a version of Devissage (Theorem 2.2.2) for Hermitian K-theory, which we will now outline; it will be necessary in considering localization for Dedekind domains.
As before, let $(R, \sharp)$ be a ring with involution, let $f \in R$ be a central non-zero divisor with $f^{\sharp} = f$, and let $\Sigma = \{f^n \mid n \in \mathbb{N}\}$. Then there exists a functor from R to $\mathcal{T}_{\Sigma}$ (considering R to be a preadditive category with one object) which sends R to $f : R \to R$ and sends a map $r : R \to R$ to the map of arrows $(r, r) : f \to f$. This is a map of arrows since f is central, so the required diagram commutes. Also, since we assume $f^{\sharp} = f$, the functor preserves dualities. Since the map $f : R \to R$ is sent to the zero object in $\mathcal{T}_{\Sigma}$, we obtain a functor of categories with duality from the quotient R/fR to $\mathcal{T}_{\Sigma}$.
At this point, we will require some definitions.

Definition 4.3.3. Let $\mathcal{A}$ be a preadditive category with duality. An $\mathcal{A}$-*module* is an additive functor from $\mathcal{A}^{\mathrm{op}}$ to the category of abelian groups. Denote the category of $\mathcal{A}$-modules by $\mathcal{A}$-mod. Recall that the Yoneda embedding $\mathcal{A} \to \mathcal{A}$-mod sending A to $\mathrm{Hom}_{\mathcal{A}}(-, A)$ is fully faithful, and write A for the representable functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$.
Now, let $\mathcal{A}$-free be the full subcategory of $\mathcal{A}$-mod consisting of those modules which are finite direct sums of representable modules. Analogously to our practice in Section 4.1, we define a functorial version $\mathbf{F}(\mathcal{A})$ of $\mathcal{A}$-free as follows. The objects of $\mathbf{F}(\mathcal{A})$ are sequences $(A_1, \ldots, A_n)$ of objects of $\mathcal{A}$ and maps are matrices of maps $A_i \to B_j, i = 1, \ldots, n, = 1, \ldots, m$, where $(B_1, \ldots, B_m)$ is another object of $\mathbf{F}(\mathcal{A})$. Composition is given by matrix multiplication. We declare the empty sequence to be the zero object, call it the basepoint, and identify it with the objects $(0, \ldots, 0)$. Thus, $\mathbf{F}(\mathcal{A})$ is a preadditive category with duality. Going further, $\mathbf{F}(\mathcal{A})$ has a direct sum operation $\oplus$, which sends $(A_1, \ldots, A_n) \times (B_1, \ldots, B_m)$ to $(A_1, \ldots, A_n, B_1, \ldots, B_m)$.
Now, let $\mathcal{A}$-proj be the full subcategory of $\mathcal{A}$-mod consisting of those modules which are direct factors of finitely generated free $\mathcal{A}$-modules, and let $\mathbf{P}(\mathcal{A})$ be the idempotent completion of $\mathbf{F}(A)$. Clearly, these two categories are equivalent.

Now, since $\mathcal{T}_{\Sigma}$ is idempotent complete, the aforementioned functor from R/fR to $\mathcal{T}_{\Sigma}$ extends to a duality preserving functor from $\mathbf{P}(R/fR)$ to $\mathcal{T}_{\Sigma}$. More precisely, there exist two duality-preserving functors $\mathbf{P}(R/fR) \to \mathbf{P}(\mathcal{T}_{\Sigma}) \leftarrow \mathcal{T}_{\Sigma}$, where the right arrow is an equivalence because $\mathcal{T}_{\Sigma}$ is idempotent complete.

Theorem 4.3.4 (Hermitian Devissage)**.** *Let $(R, \sharp)$ be a commutative ring with involution in which 2 is a unit, and let f be a non-zero divisor with $f^\sharp = f$. Assume that R and R/fR are regular rings, and, as above, let $\Sigma = \{f^n \mid n \in \mathbb{N}\}$.*
Then the inclusion of exact categories with duality $\boldsymbol{P}(R/fR) \to \mathcal{T}_\Sigma$ defined above induces homotopy equivalences

$$ {}_\epsilon\mathcal{W}(R/fR) \xrightarrow{\simeq}{}_\epsilon\, \mathcal{W}(\mathcal{T}_\Sigma) $$

and

$$ {}_\epsilon U(R/fR) \xrightarrow{\simeq}{}_\epsilon\, U(\mathcal{T}_\Sigma). $$

We now turn our attention to the case of Dedekind domains. Let R be a Dedekind domain with trivial involution, and let $\Sigma = R\backslash\{0\}$. For $\mathfrak{p}$ a nonzero prime ideal of R, let $\Sigma_\mathfrak{p} = R_\mathfrak{p}\backslash\{0\}$, where $R_\mathfrak{p}$ is the localization at $\mathfrak{p}$.
The localization maps $R \to R_\mathfrak{p}$ induce, by functoriality, maps of categories with duality from $\mathcal{T}_\Sigma$ to $\mathcal{T}_{\Sigma_\mathfrak{p}}$ which assemble to a duality-preserving functor

$$ \mathcal{T}_\Sigma \to \bigoplus_{(0)\neq\mathfrak{p}\subset R} \mathcal{T}_{\Sigma_\mathfrak{p}} $$

since the support of a finitely generated torsion module (recall that $\mathcal{T}_\Sigma$ is equivalent to the category of Σ-torsion modules) is a finite set of non-zero prime ideals. In fact, this functor is an equivalence of categories.
Now, choosing a local parameter $\pi_\mathfrak{p}$ for the discrete valuation ring $R_\mathfrak{p}$, we have that $R/\mathfrak{p} = R_\mathfrak{p}/\pi_\mathfrak{p}$ and that $\Sigma_\mathfrak{p}^{-1}R_\mathfrak{p} = R[\pi_\mathfrak{p}^{-1}]$. Applying Theorem 4.3.4 to the situation where $R = R_\mathfrak{p}$ and $f = \pi_\mathfrak{p}$, we obtain a homotopy equivalence

$$ {}_\epsilon\mathcal{W}(R/\mathfrak{p}) \to_\epsilon \mathcal{W}(\mathcal{T}_{\Sigma_\mathfrak{p}}). $$

Thus, we have the following:

Corollary 4.3.5. *Let R be a Dedekind domain with trivial involution, and let $\Sigma = R\backslash\{0\}$. Suppose 2 is a unit of R. Then the inclusions $\mathbf{P}(R/\mathfrak{p}) \to \mathcal{T}_\Sigma$ of exact categories with duality induce isomoprhisms*

$$ \bigoplus_{(0)\neq\mathfrak{p}\subset R} {}_\epsilon U_n(R/\mathfrak{p}) \to {}_\epsilon U_n(\mathcal{T}_\Sigma) $$

Now, combining Theorem 4.3.1 and Corollary 4.3.5 we obtain the following Hermitian K-theory version of Theorem 2.3.1 and Corollary 3.3.7:

Theorem 4.3.6. *Let R be a Dedekind domain with trivial involution, and let $\Sigma = R\backslash\{0\}$. Suppose 2 is a unit of R. Then we have a long exact sequence*

$$ \cdots \to {}_\epsilon K^h_{n+1}(\Sigma^{-1}R) \to \oplus\, {}_\epsilon U_n(R/\mathfrak{p}) \to {}_\epsilon K^h_n(R) \to {}_\epsilon K^h_n(\Sigma^{-1}R) \to \cdots $$

where the direct sums are over every non-zero prime ideal of R.

4.4. **Proof sketch and further directions.** In this concluding subsection, we outline a proof of localization for Hermitian K-theory, as well as possible directions for further research. The first step is to construct a simplicial additive category with duality G_*, defined as follows:

Definition 4.4.1. Recall the cosimplicial category with duality $\mathbf{n}$ of Section 4.1, and consider the category of additive functors $P : \mathbf{n} \to \mathbf{P}(R)$ such that $P(i \leq j) : P_i \to P_j$ is an inclusion with Σ-torsion cokernel. We define G_n to be this category. Moreover, the duality $\sharp$ on $\mathbf{P}(R)$ induces a duality $\sharp$ on G_n by $P^\sharp(i \leq j) = P(j' \leq i')^\sharp$, so that $(G_n, \sharp, \mathrm{id})$ is an additive category with duality for each n. The cosimplicial structure $[n] \mapsto \mathbf{n}$ imbues $(G_*, \sharp, \mathrm{id})$ with the structure of a simplicial additive category with duality.

There exists an additive duality preserving functor $\iota_n : \mathbf{P}(R) \to G_n$, which sends a projective module Q to the constant functor $P(i \leq j) = \mathrm{id}_Q$. Now, recall the construction $\mathcal{R}_n\mathcal{T}_\Sigma$; that is, the category of functors A from $I(n)$ to $\mathcal{T}_\Sigma$ such that, for $p \leq q \leq r$, all of the sequences $A_{pq} \to A_{pr} \to A_{qr}$ are short exact sequences in $\mathcal{T}_\Sigma$, and such that A_{pp} is zero for all p.
We also have an additive duality preserving functor $\rho_n : G_n \to \mathcal{R}_n\mathcal{T}_\Sigma$ which sends a functor P to the functor $\rho_n(P)$, which is defined such that $\rho_n(P)_{i,j}$ is equal to $P(i \leq j)$, and such that maps from $\rho_n(P)_{i,j} \to \rho_n(P)_{k,l}$ are given by the inclusions $P(i) \subset P(j)$, $P(k) \subset P(l)$.
Now, considering $\mathbf{P}(R)$ to be a constant simplicial additive category with duality, the two families of functors ι_n and ρ_n assemble to give functors of simplicial additive categories with duality and a sequence

$$\mathbf{P}(R) \xrightarrow{\iota_*} G_* \xrightarrow{\rho_*} \mathcal{R}_*\mathcal{T}_\Sigma. \tag{4.1}$$

This sequence forms the basis of the proof of Localization in [3]; first one shows that, in each degree, the sequence 4.1 induces a homotopy fibration of Hermitian K-theory spaces. Then, by a theorem of Bousfield and Friedlander (see [1], Theorem B.4 for details) the geometric realization of 4.1

$$K^h(R) \xrightarrow{\iota_*} |K^h(G_*)| \xrightarrow{\rho_*} |K^h(\mathcal{R}_*\mathcal{T}_\Sigma)|$$

is still a a homotopy fibration. Finally, one identifies (up to π_0) $|K^h(G_*)|$ and $|K^h(\mathcal{R}_*\mathcal{T}_\Sigma)|$ with $K^h(\Sigma^{-1}R)$ and $\mathcal{W}(\mathcal{T}_\Sigma)$, respectively. This proves Theorem 4.3.1.
A feature of the proof outlined above is that it relies on 2 being invertible in the ring R. This assumption is common enough in Hermitian K-theory, but unfortunate, since it limits the scope of the theory. One possible direction for further research would be to attempt to avoid this assumption.
A possible way of pursuing this would be to consider the sequence

$$\mathbf{P}(R) \xrightarrow{\iota_*} G_* \xrightarrow{\rho_*} \mathcal{R}_*\mathcal{T}_\Sigma$$

again, but, this time, apply the Hermitian Q-construction (Definition 4.2.1) to it, to obtain

$$Q^h\mathbf{P}(R) \xrightarrow{Q^h\iota_*} Q^hG_* \xrightarrow{Q^h\rho_*} Q^h\mathcal{R}_*\mathcal{T}_\Sigma,$$

and then to show that this sequence is a homotopy fibration for each n. This will still lead to a proof of the localization theorem, but it will not require 2 to be a unit, since the Bousfield-Friedlander Theorem and the identifications of $|K^h(G_*)|$ with $K^h(\Sigma^{-1}R)$ and of $|K^h(\mathcal{R}_*\mathcal{T}_\Sigma)|$ with $\mathcal{W}(\mathcal{T}_\Sigma)$ do not depend on the invertibility of 2.

Bibliographical Note. The preceding section is based primarily on [3] and [8].

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Appendix A. Description of Academic Activites

In this section, we outline activities undertaken during the 2016-2017 academic year. Firstly, we list all attended modules, courses, seminars, and study groups; examined modules are in italics.

- *TCC - Homological Algebra*
- *MA4A5 - Algebraic Geometry*
- *MA4J7- Cohomology and Poincaré Duality*
- Weekly seminars with supervisor (Marco Schlichting) and his other students
- Winter School on Bordism, L-Theory, and Real Algebraic K-Theory, University of Regensburg, December 5-9, 2016.

Secondly, we give a list of books and papers read.

- Localization in Hermitian K-Theory of Rings - Jens Hornbostel and Marco Schlichting
- Symmetric Bilinear Forms - Dale Husemoller and John Milnor
- Higher Algebraic K-theory I - Daniel Quillen
- Hermitian K-Theory of Exact Categories - Marco Schlichting
- The K-book: An Introduction to Algebraic K-theory - Charles A. Weibel