# MA3G6 Assignment 3

due Thursday, 19 November 2026, 12:00 PM, submit on moodle

All rings are assumed to be commutative with 1.

**Problem 1.**

(a) Let $S \subset \mathbb{Z}$ be the multiplicatively closed subset $S = \{5^n | \ n \in \mathbb{N}\}$. Find $n \in \mathbb{Z}$ such that $S^{-1}(\mathbb{Z}/10) \cong \mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}$-modules.

(b) Decide if the sequence of $\mathbb{Z}$-modules

$$0 \xrightarrow[f_0]{} \mathbb{Z}^2 \xrightarrow[f_1]{\begin{pmatrix} -3 & -2 \\ 6 & 4 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow[f_2]{\begin{pmatrix} -8 & -4 \\ 4 & 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow[f_3]{} 0$$

is exact. At the places where the sequence is not exact, compute, if possible, the default of exactness $H_i = \frac{\ker(f_i)}{\mathrm{Im}(f_{i-1})}$ as a direct sum of cyclic abelian groups.

**Problem 2.** Let $R_1, R_2$ be two rings, and $S \subset R_1 \times R_2$ a multiplicative subset. Let $S_1 = p_1(S)$, $S_2 = p_2(S)$ be the images of $S$ under the projection maps $p_i(s_1, s_2) = s_i$.

(a) Show that $S_i \subset R_i$ is multiplicative subset, $i = 1, 2$ and that

$$\phi : S^{-1}(R_1 \times R_2) \longrightarrow S_1^{-1}R_1 \times S_2^{-1}R_2 : \frac{(x_1, x_2)}{(s_1, s_2)} \mapsto \left(\frac{x_1}{s_1}, \frac{x_2}{s_2}\right)$$

is a well-defined ring homomorphism.

(b) Show that the ring homomorphism $\phi$ is an isomorphism.

**Problem 3.** For $x \in R$ show that the following are equivalent.

(a) $x$ is nilpotent in $R$ .

(b) $\frac{x}{1}$ is nilpotent in $R_P$ for all prime ideals $P \lhd R$.

(c) $\frac{x}{1}$ is nilpotent in $R_m$ for all maximal ideals $m \lhd R$.