Lectures on Affine Algebraic geometry: Affine fibration, Epimorphism/Embedding and Cancellation Problems

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Abstract

This series of talks is primarily focused on three important problems in Affine Algebraic geometry and the interplay among them, namely, **Affine fibration problem, Epimorphism/Embedding problem and Zariski Cancellation problem**. A finitely generated flat R-algebra A is said to be an $\mathbb{A}^n$**-fibration over** R if for every prime ideal $p \in \mathrm{Spec}(R)$, $A \otimes_R \kappa(p) = \kappa(p)^{[n]}$, where $\kappa(p) := R_p/pR_p$. **The Affine fibration problem asks if every $\mathbb{A}^n$-fibration over a d dimensional regular local ring R is necessarily a polynomial ring over R.**

We will start with the pioneering work of Avinash Sathaye in 1983 on positive solution to "$\mathbb{A}^2$-fibration problem" over Discrete valuation rings containing $\mathbb{Q}$ and subsequently see some sufficient condition for "triviality" of "$\mathbb{A}^2$-fibrations" over a Discrete valuation rings (not necessarily containing $\mathbb{Q}$), which is shown in a recent joint work with Neena Gupta. We will then focus on the the landmark result by Teuro Asanuma in 1987 which says that Affine fibrations over regular local rings are "stably polynomial algebras". We will demonstrate counterexample to Affine Fibration problem over fields of positive characteristic constructed by Teuro Asanuma. These counterexamples were further studied by Neena Gupta in 2014 and proven to be counterexample in positive characteristic to the **Zariski Cancellation problem which asks if $\mathbb{A}^1$-cylinder over an affine variety is an affine space, then whether the variety is an affine space itself.**

Further a theory has been developed by her for threefolds in 2014 and later on generalized in higher dimensions in my joint works with her in 2023, which connects the construction of counterexample to the Zariski Cancellation problem to the study of **Epimorphism/Emedding problem for affine spaces which asks whether every embedding of $\rho : \mathbb{A}_k^{n-1} \hookrightarrow \mathbb{A}_k^n$ rectifiable, i.e., whether there exist an automorphism of ϕ of $\mathbb{A}_k^n$ such that $\phi \circ \rho = \eta$, where η is the canonical embedding of $\mathbb{A}_k^{n-1}$ in $\mathbb{A}_k^n$.**

We will see how the study of this embedding problem for certain "linear hyperplanes" influenced the study of Zariski Cancellation Problem and further it yielded some family of affine varieties which are examples of smooth $\mathbb{A}^1$-contractible familities of varieties not-isomorphic to the affine spaces in dimensions $\geqslant 3$, and thus connects to the **Characterization problem of affine spaces, which seeks for suitable properties of an affine variety that ensure that it is isomorphic to the affine space**. The subsequent talks will be focused on these works, and explore some important open directions for further research on these problems.