Zeta functions for closed geodesics

Mark Pollicott

Warwick University

September 2022

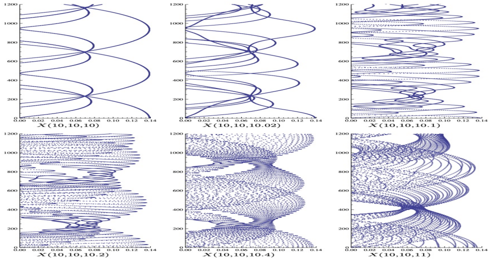

FIGURE 7. Evolution of resonance patterns away from the symmetric case.

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- The zeta function $\zeta(s)$ should carry information about some countable set of real numbers of interest. Rather like a "generating function" .

Since this is all rather vague, it is probably better to be more specific about the two types of zeta functions we might want to consider.

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Aim

Zeta functions are complex functions and we want to describe their poles and zeros.

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The Riemann zeta function is the complex function (actually introduced by Euler for s real)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

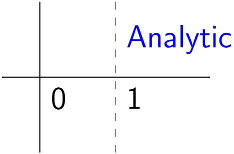
which converges for $Re(s) > 1$. However, it is convenient for us to write this in the equivalent form as an Euler product

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

where the product is over all primes $p = 2, 3, 5, 7, 11, \cdots$.

Properties of the Riemann zeta function $\zeta(s)$

The Riemann zeta function is analytic and non-zero for $Re(s) > 1$.



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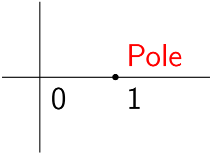

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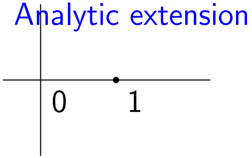

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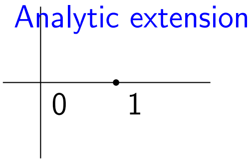

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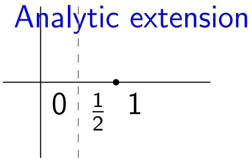

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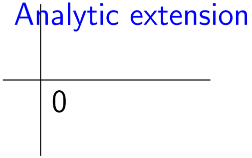

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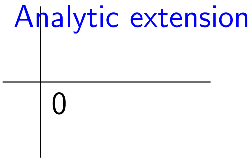

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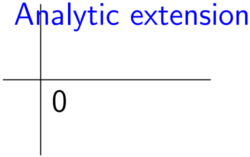

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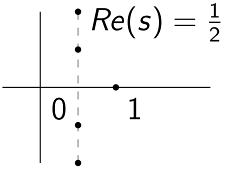

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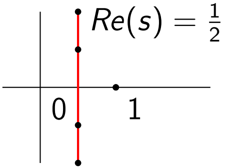

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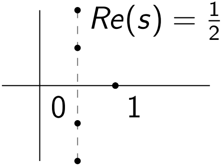

The conjecture appears in Riemann's only work on number theory, in his report to the Berlin Academy of Sciences on his election in 1859:

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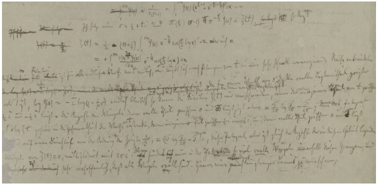

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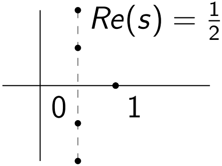

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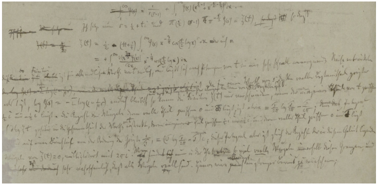

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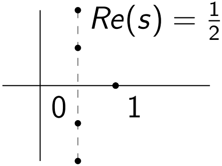

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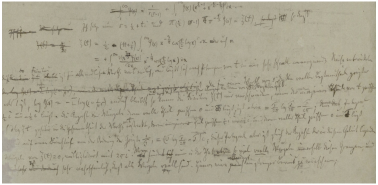

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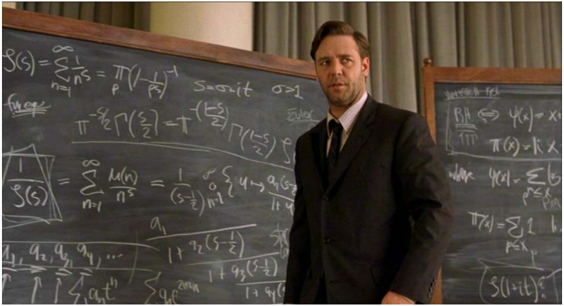

Left board: On the top line different different forms of $\zeta(s)$ are given; and underneath the extension to $\mathbb{C}$ comes from the functional equation.

Right board: The Riemann Hypothesis in an alternative asymptotic form: $\#\{primes \leq T\} = \int_2^T \frac{du}{\log u} + \mathrm{O}(\mathrm{T}^{1/2+\epsilon})$ as $T \to +\infty$.

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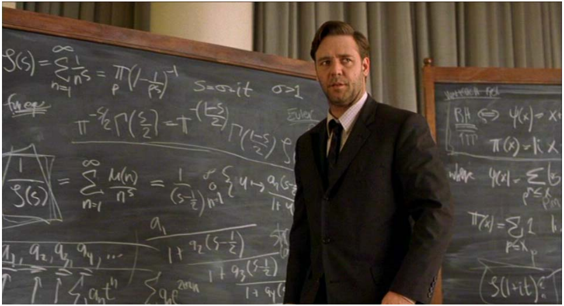

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Now we can forget primes and repeat everything for closed geodesics ...

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We can now forget the number theory and instead consider a little geometry.

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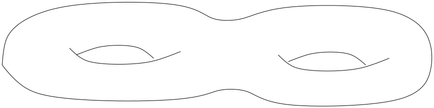

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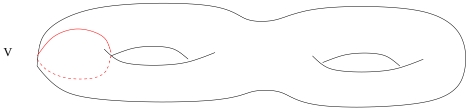

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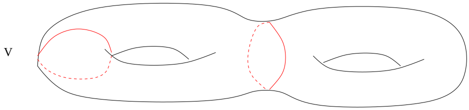

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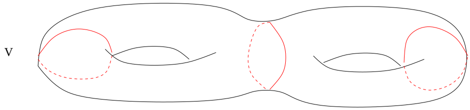

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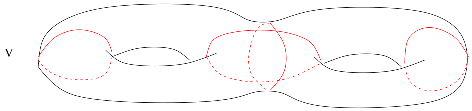

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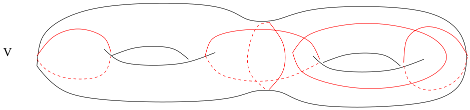

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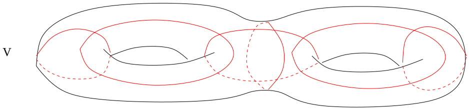

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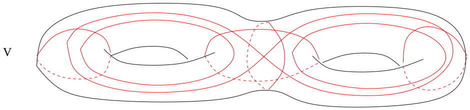


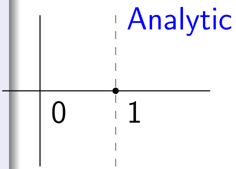
The Selberg Zeta function

Definition

Given a closed geodesic γ we again denote its length by $\ell(\gamma)$. The (Selberg) Zeta function

$$Z(s) = \prod_{\gamma} \left(1 - e^{-sl(\gamma)}\right)^{-1}, \quad s \in \mathbb{C}$$

converges for for $Re(s) > 1$.



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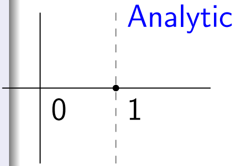

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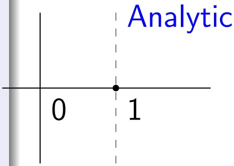

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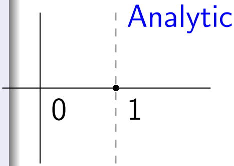

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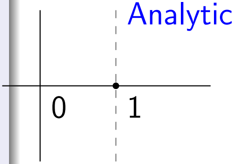

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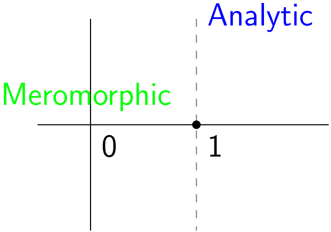

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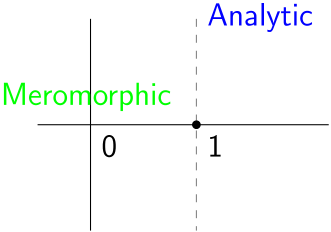

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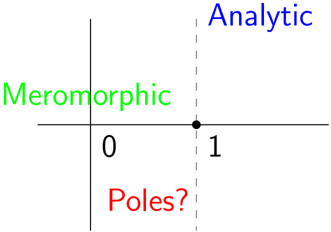

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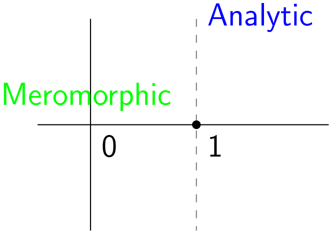

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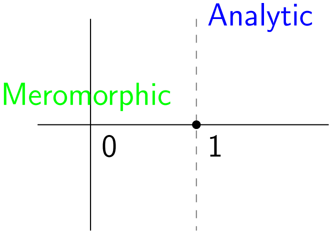

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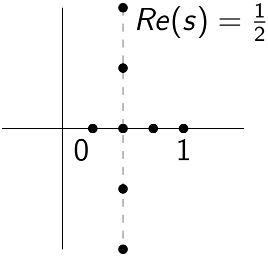

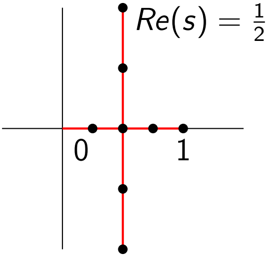

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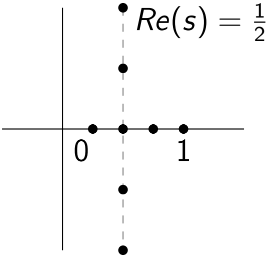

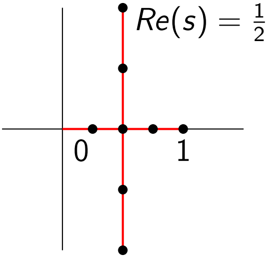

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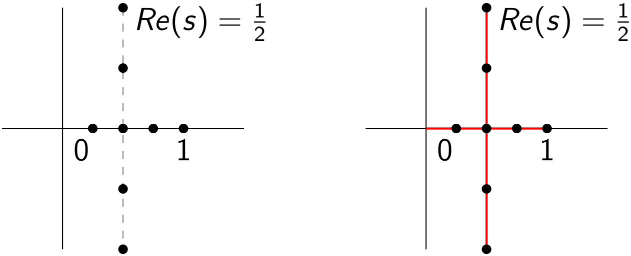

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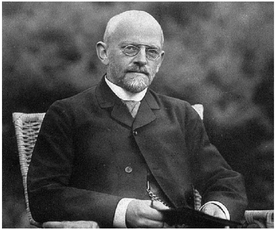
What causes the poles to lie on these two lines?

Hilbert-Polya and the "Riemann hypothesis"



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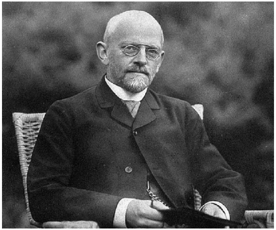

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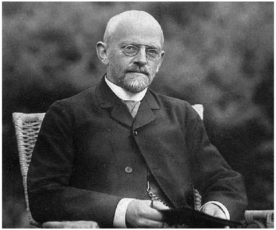 

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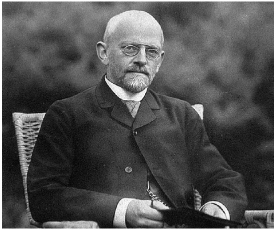

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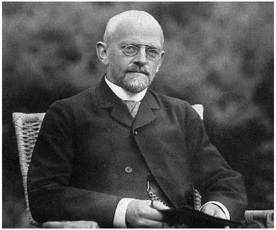

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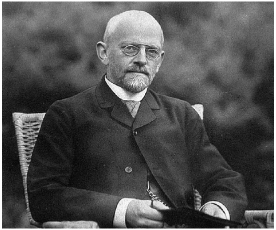 

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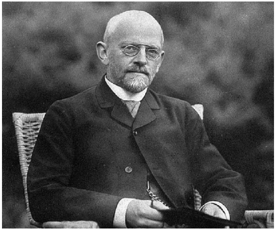 

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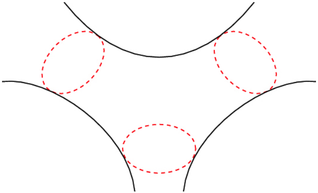

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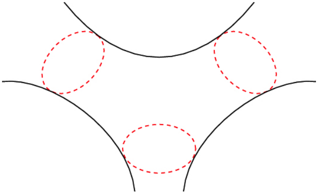

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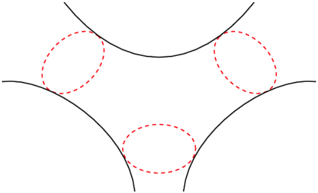

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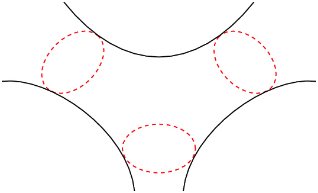

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- X (again) has a countable infinity of closed geodesics $\{\gamma\}$ with lengths $\ell(\gamma)$ tending to infinity.

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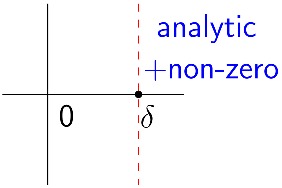

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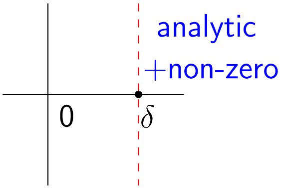

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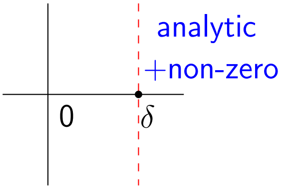

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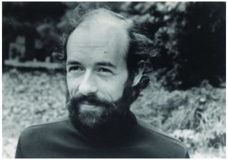
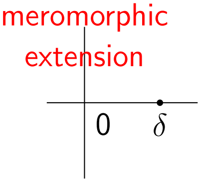
Compared with the compact case we have "fewer" closed geodesics.

Extension of $Z(s)$ to $\mathbb{C}$





The analogue of Selberg's result for compact surfaces.

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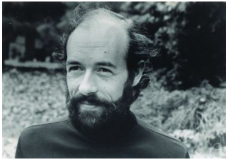

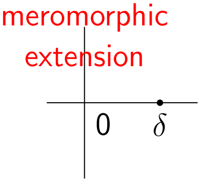

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However, the method of proof is very different to that of Selberg. It is "dynamical" and in the absence of the (self-adjoint) laplacian in the proof there is now **no** control on the location of the poles.

Locations of poles for $Z(s)$ in infinite area case

Question

For infinite area, where are the poles for $Z(s)$ in $0 < Re(s) < \delta$?

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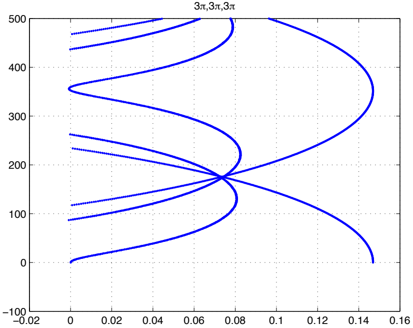
The zeta function seems far from satisfying a "Riemann Hypothesis". Plots of the zeros were made by David Borthwick circa 2013. In this example $\ell_1 = \ell_2 = \ell_3 = 3\pi$, say.



To avoid confusion. Despite its appearance this is **not** a continuous curve. To see the "pattern" we need to squash the vertical axis so the poles are just very close to each other.

More pictures : For values $\ell_1 = \ell_2 < \ell_3$

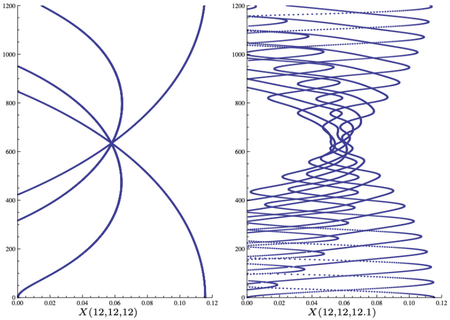


FIGURE 6. Resonances of 3-funnel surfaces.

Borthwick also observed that if the lengths ℓ_1, ℓ_2, ℓ_3 are not quite equal then the "pattern" of poles is more complicated.

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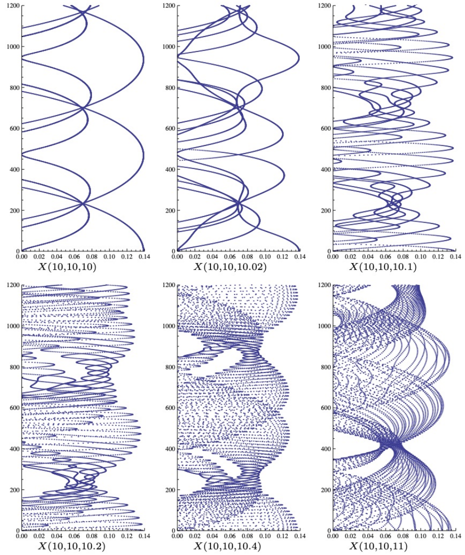

FIGURE 7. Evolution of resonance patterns away from the symmetric case.

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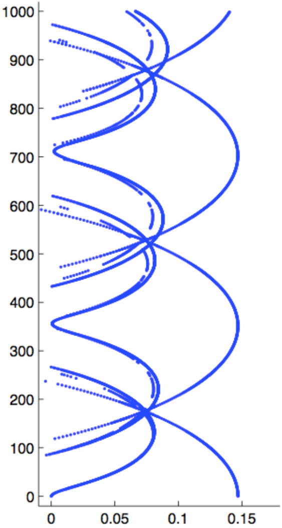
Moreover increasing one length ℓ_3 we can see the "pattern" of poles getting more exotic.

Empirical observations



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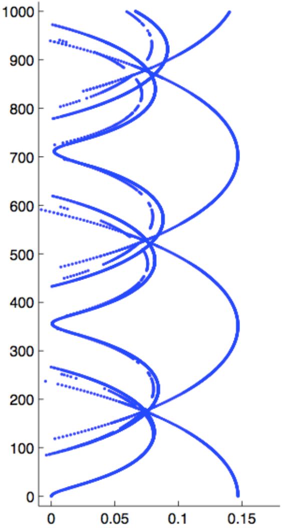

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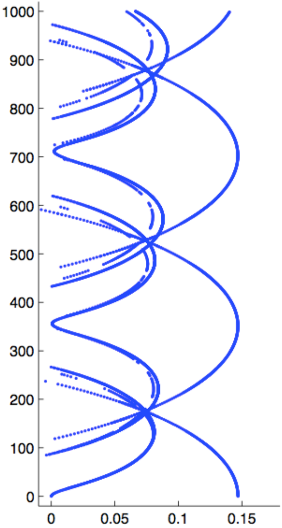

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1. The apparent vertical "aperiodicity" of the "pattern" of poles is approximately πe^b.

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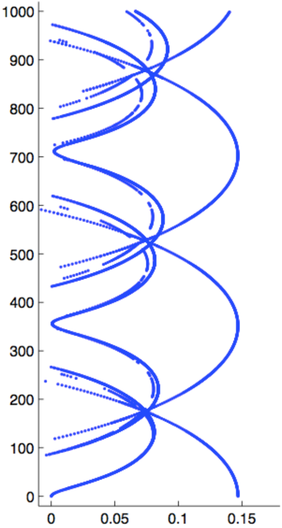

Let us return to the (simplest) case where $\ell_1 = \ell_2 = \ell_3$.

There are some observations which it would be nice to have (at least) an heuristic explanation.

1. The apparent vertical "aperiodicity" of the "pattern" of poles is approximately πe^b.
2. The "pattern" of poles appears to lie close to a finite number of curves.

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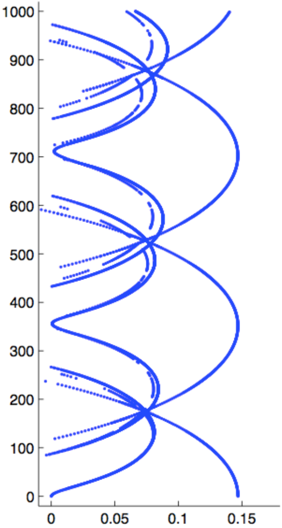

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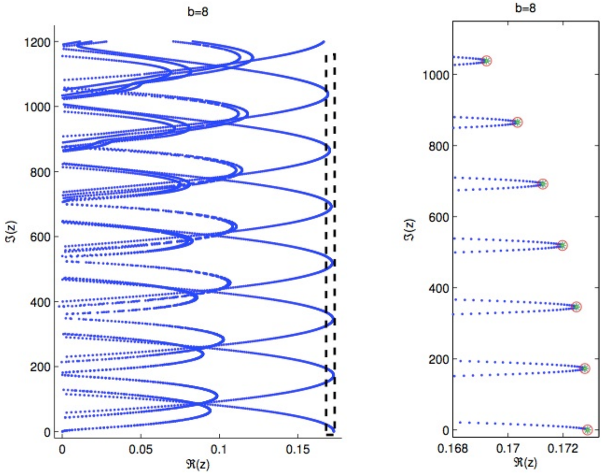
On closer inspection the plot isn't really periodic in the imaginary direction. We can blow up the picture near $Re(s) = \delta$.



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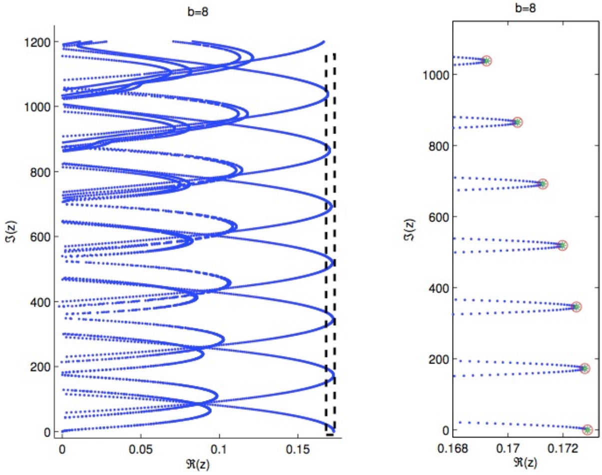

Actually, one can prove easy there isn't real periodicity since otherwise there would be infinitely many poles above $s = \delta$ which is relatively easy to show isn't the case.

4. Explaining the “pattern” of the poles

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- the apparent vertical period $\sim \frac{\pi}{2} e^{\ell}$ tends to infinity.

Therefore, we just rescale the pictures (by ℓ in the horizontal direction and $e^{-\ell}$ in the vertical direction)...

Rescaling the rectangle

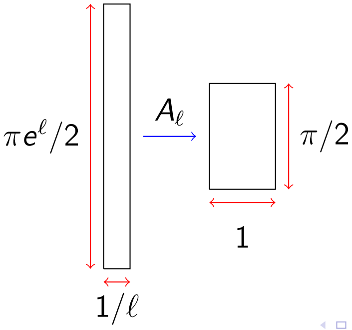
Let us affinely transform the portion of the vertical strip

$$[0, 1/\ell] \times i[0, \frac{\pi}{2}e^{\ell}]$$

to a standard rectangle

$$[0, 1] \times i[0, \frac{\pi}{2}]$$

defined by $A_\ell : \sigma + it \mapsto u + iv = (\sigma\ell) + i(te^{-\ell})$.



Finally, the curves for the “pattern” of poles

These limiting curves in $[0,1] \times i[0,\pi/2]$ are explicitly given by the formulae

$$\mathcal{C}_1 = \left\{ \log |e^{2it} + 1| + it \mid t \in \mathbb{R} \right\};$$

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Theorem (P.-Vytnova)

As $\ell \to +\infty$ the rescaled poles $Z(s)$ converge on $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

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$$\left\{ \ell\sigma + ie^{-\ell} t : \sigma + it = a \text{ pole for } Z(s) \right\} \subset B(\cup_{i=1}^{4} \mathcal{C}_i, \epsilon)$$

Comparing the curves with the zeta poles

Using Mathematica we can plot the four curves: $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and their union $\cup_{i=1}^{4}\mathcal{C}_i$.

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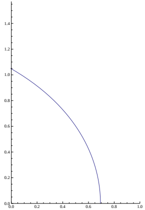

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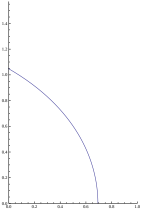

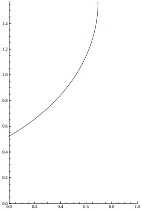

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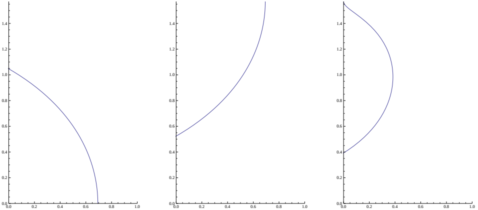

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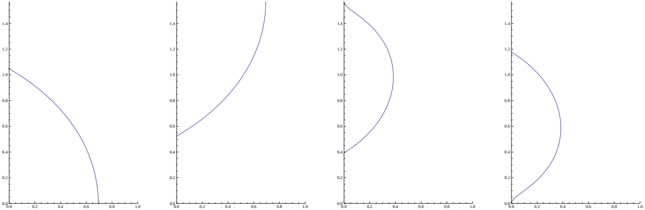

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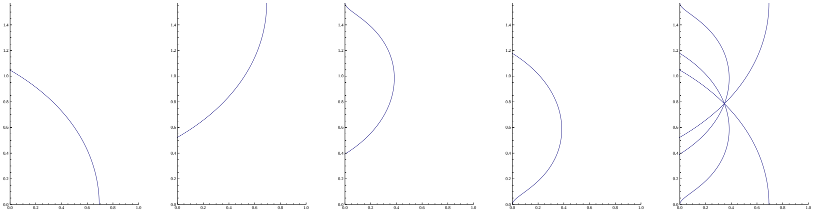

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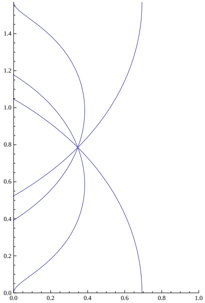

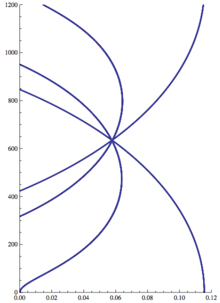

Finally, we can compare the four curves with Borthwick's original experimental plot of poles for $Z(s)$ where $\ell = 3\pi$.

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In Borthwick's original plots of poles for $Z(s)$ the lengths ℓ_1, ℓ_2, ℓ_3 determining the surface X were relatively large ($\sim 3\pi \sim +\infty$) which is why we saw the "patterns".

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Actually, Borthwick choose ℓ_1, ℓ_2, ℓ_3 relatively large simply because it made approximations to the zeta function $Z(s)$ for the computations better.

Summary: Comparing zeta functions for geodesics

We get very different behaviour of the poles for the zeta function depending on whether surface is compact or infinite volume.

Setting	Complex function	Strip
Closed geodesics γ on a compact surface	Selberg ζ-function $Z(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$	$0 < Re(s) < 1$ Poles satisfy "Riemann H."
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Perhaps one day ...

The end

Thank you for your attention

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We can consider the more general case of a compact surface V of *negative* curvature. There is an analogue of the Selberg extension result(s).

Theorem (Giulietti-Liverani-P.)

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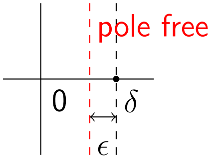
There exists $\epsilon > 0$ such that $Z(s)$ has an analytic pole-free extension to $Re(s) > \delta - \epsilon$, except for the simple pole at $s = \delta$.



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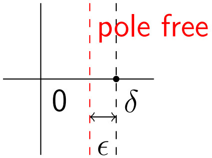

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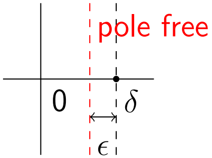

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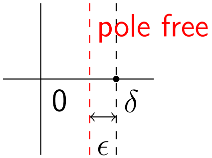

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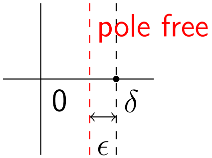

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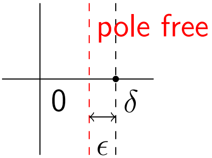

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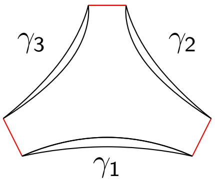
Infinite area revisited: Sketch proof



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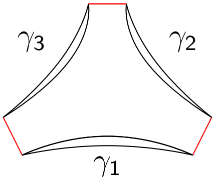

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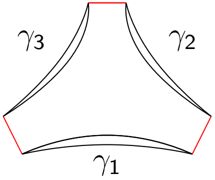
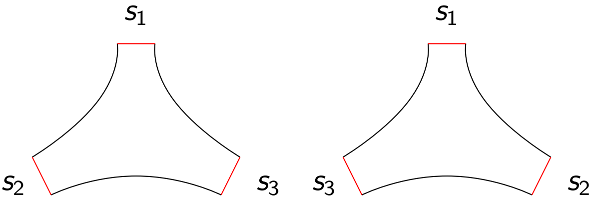


- If we cut off the three funnels by the three closed geodesics $\gamma_1, \gamma_2, \gamma_3$ then we are left with a surface V with three boundary components (which Thurston called a "pair of pants").
- X is the union of two hexagons which are glued along the (short red) edges s_1, s_2, s_3.



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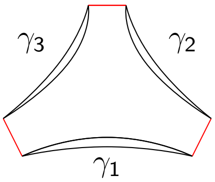

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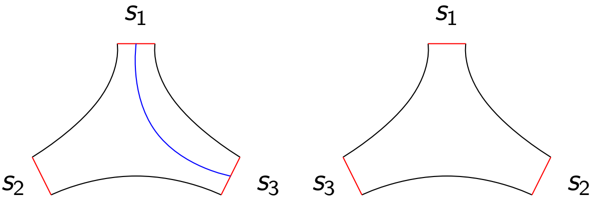
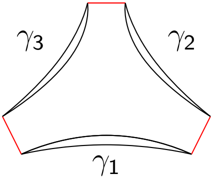


- Closed geodesics can only cross the hexagon and leave or enter by the short red edges.

(ii) Length of geodesics crossing hexagons

(Periodic) geodesics on V which don't escape across the boundary $\gamma_1 \cup \gamma_2 \cup \gamma_3$ are in bijection with (periodic) sequences of "even" edges.



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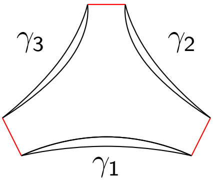

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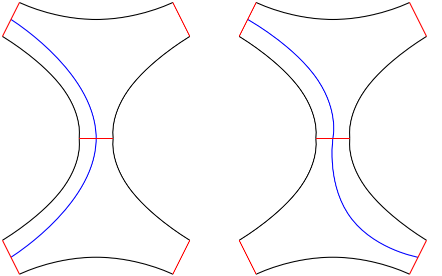
The contributions of each pair of successive crossings of a hexagon are either $\sim \ell$ or $\sim \ell + e^{-\ell}$ as $\ell \to +\infty$.



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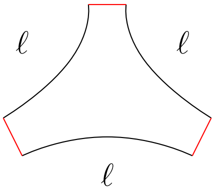

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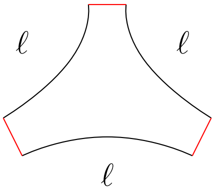

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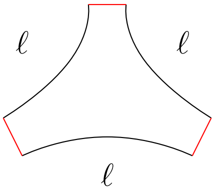

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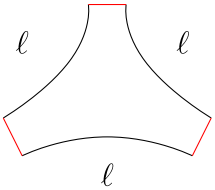

Conclusion. Poles s correspond to $1/w = \mu \in \{\text{eigenvalues} B(z)\}$. Curves correspond to $|1/w| = |\mu|$.

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For infinite surfaces and finite $\ell > 0$ there are much stronger results, more in the spirit of the Riemann Hypothesis, which illustrate this.

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Thank you for your attention