Cantor sets, Hausdorff dimension and their applications

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London Analysis Seminar - Bonfire night

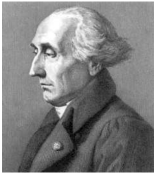
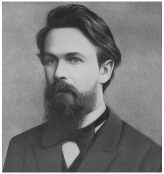


Overview of the talk

The plan is to look at special subsets in the real line and to try to estimate their size (Hausdorff Dimension).

Part of the motivation will be the applications to:

1. The Zaremba Conjecture on finite continued fractions

2. The difference between the Lagrange and Markov Spectra in the context of diophantine approximation

Sets and their dimension

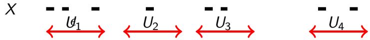
Let $X \subset \mathbb{R}$ be a (bounded zero Lebesgue measure) subset of the real line.

The Hausdorff dimension $\dim_H(X) \in [0, 1]$ quantifies the "size" of these sets.

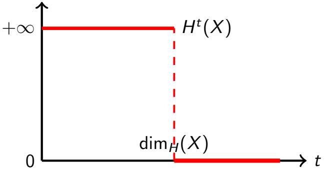
- A single point, or any countable set X has zero Hausdorff dimension.
- An interval $X = [a, b]$ has Hausdorff dimension one.
- If $X \subset Y$ then $\dim_H(X) \leq \dim_H(Y)$.

A formal definition of the Hausdoff Dimension

Given a set X:



- Fix a scale $\epsilon > 0$.
- Consider (finite) covers of X by open intervals $\mathcal{U} = \{U_n\}$ where each interval has diameter at most ϵ (i.e., $\sup_n \mathrm{diam}(U_n) < \epsilon$).
- For each exponent $t > 0$, denote $H^t_\epsilon(X) = \inf_{\mathcal{U}} \{\sum_n \mathrm{diam}(U_n)^t\}$ (where the infimum is over all of such covers).
- Now we let the scale tend to zero and denote $H^t(X) = \lim_{\epsilon \to 0} H^t_\epsilon(E_N) \in [0, +\infty]$.



- Finally, we let $\dim_H(X) = \inf\{t > 0 : H^t(X) = 0\}$.

Easy example: Middle third Cantor set

The middle third Cantor set can be written in terms of "deleted digits"

$$X = \left\{ \sum_{n=1}^{\infty} \frac{i_n}{3^n} \, : \, i_1, i_2, i_3, \cdots \in \{0, 2\} \right\}$$

i.e., in the base 3 expansion we delete the digit 1

One can (almost) see that the definition that X has Hausdorff dimension

$$\frac{\log 2}{\log 3} = 0.6309297535714573\ldots$$

- When $\epsilon = \frac{1}{3^k}$ then we can use a cover $\mathcal{U}$ of 2^k intervals of size approximately $\frac{1}{3^k}$ (i.e., after deleting "middle thirds" to level k).
- Then $H^t_\epsilon(X) \approx 2^k/3^{tk} \to 0$ as $\epsilon \to 0$ (i.e., $k \to +\infty$) if $t > \frac{\log 2}{\log 3}$, and
- and $H^t_\epsilon(X) \approx 2^k/3^{tk} \to +\infty$ as $\epsilon \to 0$ (i.e., $k \to +\infty$) if $t < \frac{\log 2}{\log 3}$.

Thus $\dim_H(X) = \inf\{t > 0 : \, H^t(X) = 0\} = \frac{\log 2}{\log 3}$.

Our main setting: Continued fractions

To specify the Cantor sets X we want to study we will use continued fractions. Recall the following classical result:

Lemma

Any irrational number $x \in (0, 1)$ can be written in the form of a continued fraction:

$$x = [a_1, a_2, a_3, \cdots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with the coefficients $a_1, a_2, a_3, \cdots \in \mathbb{N}$.

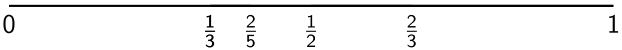
This essentially comes from the Euclidean algorithm.

Continued fractions with bounded digits

Fix $N \geq 2$. We can restrict our attention to those irrational $x \in (0,1)$ whose coefficients are bounded by N, i.e.,

$$E_N := \{x = [a_1, a_2, a_3, \cdots] : a_i \in \{1, 2, \cdots, N\} \text{ for } i \geq 1\}.$$

Example. Consider the case $N = 2$:



Since $a_1 \in \{1,2\}$ we have $x \in [\frac{1}{3}, 1]$.
Since $a_1 \in \{1,2\}$ *and* $a_2 \in \{1,2\}$ we have $x \in [\frac{1}{3}, \frac{2}{5}] \cup [\frac{1}{2}, \frac{2}{3}]$.

More generally, $a_i \in \{1,2\}$ for $i = 1, \cdots, n$ gives a family of 2^n intervals and intersecting these families (over n) gives the set E_2.

Dimension of the sets E_N

Fix $N \geq 2$. It is not very difficult to show that:

1. E_N is a Cantor set; and
2. E_N has Lebesgue measure zero (i.e., $\mathrm{leb}(E_N) = 0$).

Question

How "large" a set of zero measure is it? More precisely, what is the value of the Hausdorff Dimension $\dim_H(E_N)$*?*

Unfortunately, there is no closed form expression for $\dim_H(E_N)$.

Therefore, we need to find a numerical approximation - *and we will return to the motivation for this soon.*

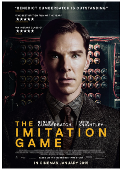
We begin with the example $N = 2$ and E_2 (i.e., the Cantor set consisting of points whose continued fraction expansion contains only the digits 1s and 2s).

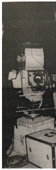
Example 1: A Good estimate for E_2

The first estimate when $N = 2$ appears in the thesis of Jack Good (under the supervision of Hardy and Besicovitch) in 1941:

$$0.5306 < \dim(E_2) < 0.5320$$





During the war Good was a code breaker at Bletchley Park and featured as a character in the 2014 movie about the life of Alan Turing, as the guy in glasses who solves the recruitment puzzle at the same as Kiera Knightley.

In another cinemagraphic connection, Good also worked with Stanley Kubrick as a technical advisor for the movie *2001: A space odyssey*

A better then Good estimate for $\dim(E_2)$

With improved computational resources, but also because of the use of better techniques, better estimates appeared.

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Where the estimates are presented to the number of places they are known to be accurate.

Question

How can we further improve on these estimates?

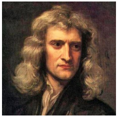
A better estimate for $\dim_H(E_2)$

In 2018, Oliver Jenkinson (QM-UL) and I used a zeta function approach to compute this to 100 decimal places.

$$\begin{aligned}\dim_H(E_2) = 0.&5312805062\,7720514162\,4468647368\,4717854930\,5910901839\\ &8779888397\,8039275295\,3564383134\,5918109570\,1811852398\\ &8042805724\,3075187633\,4223893394\,8082230901\,7869596532\\ &8712235464\,2997948966\,3784033728\,7630454110\,1508045191\\ &3969768071\,3 \pm 10^{-201}\end{aligned}$$

In 2020, Polina Vytnova and I computed this to over 200 decimal places.

This isn't a matter of having a "bigger computer", the better estimates come from a different approach which, in this case, happens to work quite well.



"I am ashamed to tell you to how many figures I carried these computations, having no other business"
- Isaac Newton (on computing 15 digits for π in 1666)

Example 2: Estimates on $\dim(E_5)$

Let $N = 5$ then E_5 is the Cantor set of numbers whose continued fraction expansions whose digits all lie in $\{1, 2, 3, 4, 5\}$.

In 2018 Oliver Jenkinson (QM-UL) and I showed that

$$\dim_H(E_5) = 0.836829445 \pm 5 \cdot 10^{-9}$$

and in 2020 Polina Vytnova and I improved this to

$$\dim_H(E_5) = 0.83682944368120882244159438727 \pm 10^{-29}.$$

The first estimate is sufficient for the application (on the next slide).

Question

Who cares about the Hausdorff dimension of these sets?

Application I: Zaremba Conjecture

Any rational $\frac{p}{q} \in \mathbb{Q}$ (p,q coprime) can be written as a *finite* continued fraction

$$\frac{p}{q} = [a_1, \cdots, a_n] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} \qquad \text{where } a_i \in \mathbb{N}.$$

(This uses the Euclidean algorithm cf. Hardy and Wright)

The Zaremba conjecture asks if we can still get all the denominators if we bound the digits. More precisely:

Conjecture (Zaremba, 1972)

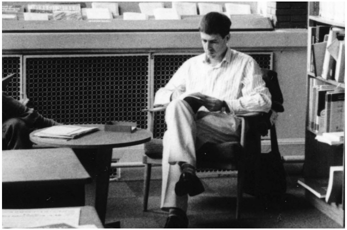
For any natural number $q \in \mathbb{N}$ there exists p (coprime to q) and $a_1, \cdots, a_n \in \{1, 2, 3, 4, 5\}$ such that

$$\frac{p}{q} = [a_1, \cdots, a_n].$$

Unfortunately, this conjecture is still open.

The Bourgain-Kontorovich-Huang Theorem

However, the conjecture is true *for most denominators*, i.e., a density one result.



Theorem (Bourgain-Kontorovich, Huang)

$$\lim_{Q \to +\infty} \frac{1}{Q} Card \left\{ \begin{array}{r} 1 \leq q \leq Q \,|\exists p \in \mathbb{N},\ with\ \dfrac{p}{q} = [a_1, \cdots, a_n] \\ with\ a_1, \cdots, a_n \in \{1,2,3,4,5\} \end{array} \right\} = 1$$

However, the proof is conditional on the fact $\dim_H(E_5) = 0.8368\cdots > \frac{5}{6} = 0.833\cdots$.

Computation and accuracy

For the applications we need to have complete confidence in the accuracy of our estimates.

This depends on:

- having a theoretical method which gives precise bounds; and
- rigorously bounding errors in the actual numerical computation.

The latter is well understood.
The former is the more interesting.

"Fast is fine, but accuracy is everything." - Wyatt Earp

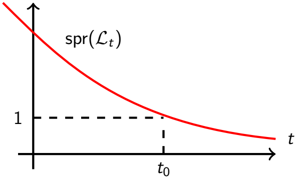
Estimates on $\dim_H(E_N)$ - Step 1: Enter the operator

The estimates on $\dim_H(E_N)$ come the study of bounded linear operators.

Consider the Banach space of continuous functions $C([0,1])$ and the family of linear operators $\mathcal{L}_t : C([0,1]) \to C([0,1])$:

$$\mathcal{L}_t w(x) = \sum_{i=1}^{N} w\left(\frac{1}{x+i}\right)\frac{1}{(x+i)^{2t}} \qquad (t > 0)$$

Let $\text{spr}(\mathcal{L}_t)$ denote the spectral radius of $\mathcal{L}_t$.



Lemma (after Bowen, Ruelle)

The map $t \to spr(\mathcal{L}_t)$ is strictly monotone decreasing and the solution $spr(\mathcal{L}_t) = 1$ corresponds to $t = \dim_H(E_N)$

The operator and the dimension

The heuristic for the connection between $\dim_H(E_N)$ and the spectral radius of the the operator $\mathcal{L}_t : C([0,1]) \to C([0,1])$ comes from considering the n-power to get

$$\mathcal{L}_t^n w(x) = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} w\left(\frac{1}{i_1 + \frac{1}{i_2 + \cdots \frac{1}{i_n + x}}}\right)\left(\frac{1}{i_1 + \frac{1}{i_2 + \cdots \frac{1}{i_n + x}}}\right)^{2t} \quad w \in C([0,1])$$

Letting $w = 1$ (constant function) and $x = 0$ we can compare

$\mathcal{L}_t^n 1(0) \approx H_\epsilon^t(X)$ where we consider the (optimal) cover by N^n intervals

$$I = \left[\frac{1}{i_1 + \frac{1}{i_2 + \cdots \frac{1}{i_n + 1}}}, \frac{1}{i_1 + \frac{1}{i_2 + \cdots \frac{1}{i_n}}}\right] \text{ with } \mathrm{diam}(I) \asymp \left(\frac{1}{i_1 + \frac{1}{i_2 + \cdots \frac{1}{i_n + x}}}\right)^2.$$

Therefore, we are left with having to estimate $\mathrm{spr}(\mathcal{L}_t)$ *for different* t *to find* t *with* $\mathrm{spr}(\mathcal{L}_t) = 1$.

Step 2: Estimates on $spr(\mathcal{L}_t)$

We can use a sort of "min-max" estimate:

Lemma

Let $t_0 < t_1$

1. *If there exists (positive) polynomial $f : [0,1] \to \mathbb{R}^+$ such that*

$$\inf_x \frac{\mathcal{L}_{t_0} f(x)}{f(x)} > 1 \implies \text{ then } spr(\mathcal{L}_{t_0}) > 1.$$

2. *If there exists (positive) polynomial $g : [0,1] \to \mathbb{R}^+$ such that*

$$\sup_x \frac{\mathcal{L}_{t_1} g(x)}{g(x)} < 1 \implies \text{ then } spr(\mathcal{L}_{t_1}) < 1.$$

The two lemmas give us a way to estimate the dimension.

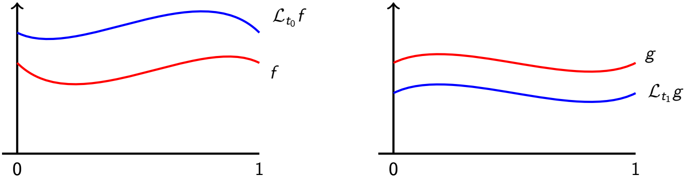
Corollary

If we can find f, g as above then $t_0 < \dim_H(E_m) < t_1$.

Thus estimating $\dim_H(E_m)$ is reduced to finding polynomials f, g as above.

Summary - so far

Given $N \geq 2$ and $t_0 < t_1$, to show that $\dim_H(E_N) \in [t_0, t_1]$ if suffices to ...guess (or construct) positive polynomials $f, g : [0,1] \to \mathbb{R}^+$ such that



$$\mathcal{L}_{t_0} f \geq f \implies t_0 \leq \dim_H(E_N) \qquad \mathcal{L}_{t_1} g \geq g \implies \dim_H(E_N) \leq t_1$$

It only remains to try to find such functions f and g, which is the final step.

Step 3: Cooking up test functions

We could just try and guess the functions f and g (and hope we get lucky) but a more systematic approach is to use a little interpolation theory.

- Fix a natural number m (e.g., $m = 6$).
- We can denote
  1. $p_k(x) \in C([0,1])$ be Lagrange polynomials ($1 \leq k \leq m$), and
  2. let $x_k \in [0,1]$ be Chebyshev points ($1 \leq k \leq m$)

  so that $p_i(x_j) = \delta_{ij}$. for $1 \leq i, j \leq m$
- Given t consider the $m \times m$ matrix $A(i,j) = (\mathcal{L}_t p_i)(x_j)$ for $1 \leq i, j \leq m$.
- Let $w = (w_1, \cdots, w_m)$ be a (left) eigenvector for the largest eigenvalue.
- Finally, choose $f(x) = \sum_{k=1}^m w_k p_k(x)$ (or $g(x) = \sum_{k=1}^m w_k p_k(x)$).

Application II: Diophantine approximation

Recall a classical result in number theory.

Theorem (Dirichlet, 1840)

There infinitely many rational numbers $\frac{p}{q}$ ($p, q \in \mathbb{Z}$, $q \neq 0$) satisfying

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}$$



Question

Can one improve on Dirichet's theorem for individual x?

Better approximations: Lagrange spectrum

For different irrational x we can choose the largest values $c(x) > 1$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{c(x)q^2}$$

still has infinitely many solutions with $\frac{p}{q} \in \mathbb{Q}$ i.e.,

$$c(x) = \liminf\{|q|.|qx - p| \,:\, p, \in \mathbb{Z}, q \neq 0\}.$$

For example, $c\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

Definition

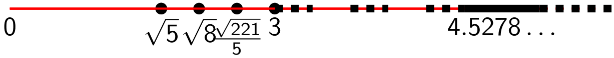
The *Lagrange spectrum* $\mathcal{L} \subset \mathbb{R}^+$ is defined by

$$\mathcal{M} = \{c(x) \,:\, x \in \mathbb{R} - \mathbb{Q}\}.$$

Properties of the Lagrange spectrum $\mathcal{L}$

Question

What does $\mathcal{L} \subset \mathbb{R}^+$ look like?



- The smallest value for $\mathcal{L}$ is $\sqrt{5}$ and below 3 there countably many values:

$$\mathcal{L} \cap [\sqrt{5}, 3] = \{\sqrt{5}, \sqrt{8}, \sqrt{221}/5 \cdots\}$$

- In 1947 Hall showed that $[4.5278\ldots, +\infty) \subset \mathcal{L}$.
- In between the set $\mathcal{L} \cap (3, 4.5278\ldots)$ is complicated.

Markov spectrum $\mathcal{M}$

To define another related subset of $\mathbb{R}^+$ consider those binary quadratic forms

$$f(x,y) = ax^2 + bxy + cy^2 \ (a,b,c \in \mathbb{R})$$

with (discriminant) $b^2 - 4ac = 1$.

Definition (Markov, 1879)

$$\lambda(f) := \inf\{|f(x,y)| : (x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}\}.$$

The *Markov spectrum* is defined by $\mathcal{M} = \{1/\lambda(f) : f \text{ as above }\}$.

Surprisingly (or not) the sets $\mathcal{L}$ and $\mathcal{M}$ are very similar. More precisely,

- Tornheim (1955) showed $\mathcal{L} \subset \mathcal{M}$;
- Freiman (1968) showed $\mathcal{L} \neq \mathcal{M}$.

Question

How large is the difference $\mathcal{M} \setminus \mathcal{L}$?

A lower bound on $\dim_H(\mathcal{M} \setminus \mathcal{L})$

The difference $\mathcal{M} \setminus \mathcal{L}$ has zero Lebesgue, i.e., $\text{leb}(\mathcal{M} \setminus \mathcal{L}) = 0$. Therefore one can ask what is $\dim_H(\mathcal{M} \setminus \mathcal{L})$?

Theorem (Matheus-Moreira)

$\mathcal{M} \setminus \mathcal{L}$ contains a (diffeomorphic) copy of E_2. In particular, $\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim(E_2)$

Corollary

$\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq 0.513\cdots$ *(to 200 decimal places ...)*

An upper bound on $\dim_H(\mathcal{M} \setminus \mathcal{L})$

Question

Can we get an upper bound on $\dim_H(\mathcal{M} \setminus \mathcal{L})$*?*

Matheus and Moreira developed a method which is based on the Hausdorff dimension of sets given by continued fraction expansions with restrictions. Using this they estimated.

Theorem (Matheus-Moreira)

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.9869\ldots$$

They also conjectured

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.888$$

which is true:

Theorem (P.-Vytnova)

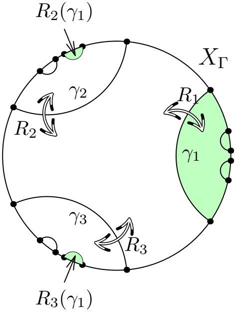
$$\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.882325$$

The two dimensional hyperbolic space can represented as the Poincaré disc

$$\mathbb{D}^2 = \{z \in \mathbb{C} \colon |z| < 1\}$$

with the Poincaré metric $ds^2 = 4(1 - |z|^2)^{-2}$.

McMullen considered the group $\Gamma = \langle R_1, R_2, R_3 \rangle$ of isometries generated by reflections $R_1, R_2, R_3 : \mathbb{D}^2 \to \mathbb{D}^2$ in three symmetrically placed geodesics.



The limit set X_Γ is the Cantor set accumulation points of $\Gamma 0$ in $\partial \mathbb{D}$.

Lowest Eigenvalue of the Laplacian

The quotient space $\mathbb{D}^2/\Gamma$ is a surface of constant curvature $\kappa = -1$.

The smallest eigenvalue λ of the Laplacian $-\Delta : L^2(\mathbb{D}^2/\Gamma) \to L^2(\mathbb{D}^2/\Gamma)$ is related to $\dim_H(X_\Gamma)$ by

$$\lambda_\Gamma = \min \left\{ \dim_H(X_\Gamma)\left(1 - \dim_H(X_\Gamma)\right), \frac{1}{4} \right\}.$$

Using the method we have described one can easy compute:

Theorem

The dimension of the limits set of Γ satisfies

$$\dim_H(X_\Gamma) = 0.295546475 \pm 5 \cdot 10^{-9}$$

and the smallest value of the Laplacian satisfies

$$\lambda_\Gamma = 0.2081987565 \pm 2.5 \cdot 10^{-9}$$

And the result can easily be made much more accurate.

Thank you for your time