Livšic's Theorem revisited

Mark Pollicott
(joint work with Richard Sharp)

University of Warwick

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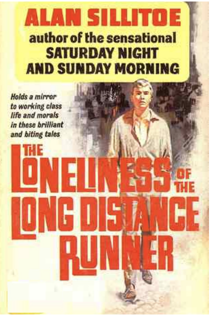

This is a well known book (and a better known film) written by an author from my home town in the UK

Discrete Livšic type theorems

First setting: shift spaces of finite type

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- We denote the (one-sided) *shift space* by

$$\Sigma_A = \left\{ \underline{x} = (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} \{1, 2, \ldots, k\} : A(x_n, x_{n+1}) = 1 \text{ if } n \in \mathbb{Z}_+ \right\}$$

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which is a compact set with respect to the metric

$$d(\underline{x}, \underline{x}') = \sum_{n=0}^{\infty} \frac{e(x_n, x_n')}{2^n} \text{ where } e(i,j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

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where $\underline{x} = (x_n)_{n=0}^{\infty}$ and $\underline{x}' = (x_n')_{n=0}^{\infty}$.
Thus if $\underline{x}$ and $\underline{x}'$ agree in (exactly) first N places then $d(\underline{x}, \underline{x}') \asymp \frac{1}{2^N}$.

The dynamics: The shift map

Definition

We define the *subshift of finite type* $\sigma : \Sigma_A \to \Sigma_A$ by $(\sigma \underline{x})_n = x_{n+1}$,

i.e., if $\underline{x} = (x_0, x_1, x_2, \ldots, x_n, \ldots)$ then $\sigma \underline{x} = (x_1, x_2, x_3, \ldots, x_{n+1}, \ldots)$

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- Of course, a periodic point $\sigma^n \underline{x} = \underline{x}$ takes the simple form

$$\underline{x} = (x_0, x_1, x_2, \dots, x_{n-1}, x_0, x_1, x_2, \dots, x_{n-1}, x_0, x_1, \dots) \in \Sigma_A,$$

i.e., the sequence repeats.

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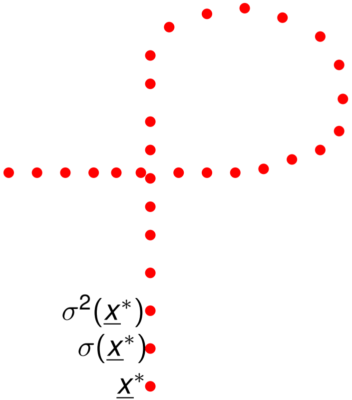
I will give a sketch of the proof in a couple of slides (= time to get a quick cup of coffee).

Proof of Livšic's theorem for subshifts



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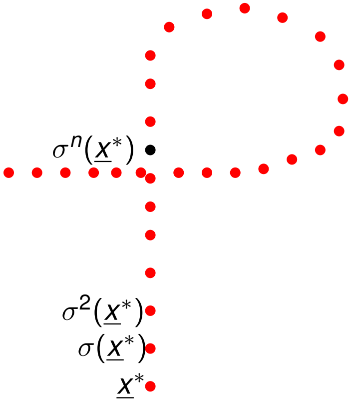

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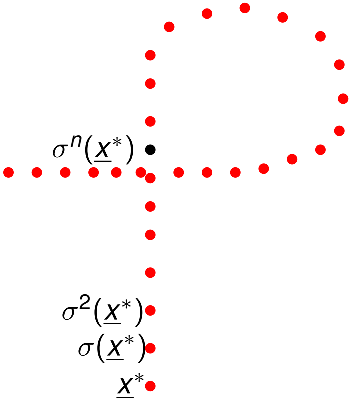

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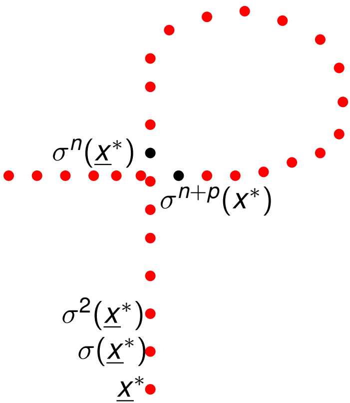

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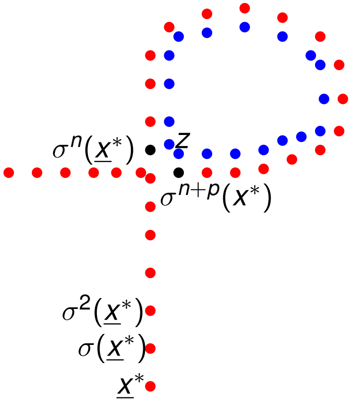

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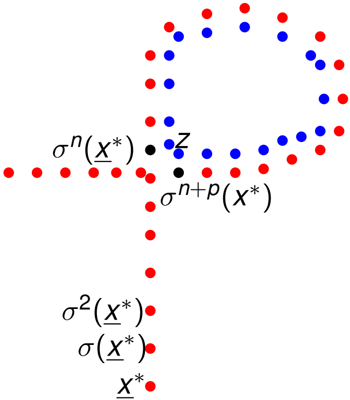

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Finally

$$\underbrace{u(\sigma^{n+p} \underline{x}^*) - u(\sigma^n \underline{x}^*)}_{\sum_{k=0}^{n+p-1} f(\sigma^k \underline{x}^*) - \sum_{k=0}^{n-1} f(\sigma^k x^*)} = \sum_{k=n}^{n+p-1} f(\sigma^k \underline{x}^*) \approx \sum_{k=0}^{p-1} f(\sigma^k \underline{z}) = 0.$$

Holder Comments

References for the proof

The original proof was published by A. N. Livšic in 1971.

HOMOLOGY PROPERTIES OF Y-SYSTEMS

A. N. Livshits

UDC 513.83

Suppose that a Y-system $T^t(T^k)$ acts on a manifold M^n. We present a criterion of zero homology for Hölder functions with respect to this dynamical system, as well as some consequences of this criterion and a generalization for functions taking their values in a Lie group.

Let M^m be a smooth closed Riemannian manifold of class C^2 with a metric ρ, let $T^t\,(T^k)$ be a smooth flow (cascade) [1], and f a real function on M^m that satisfies a Hölder condition; f is said to be homologous to zero in the class of Hölder functions if there exists a Hölder function g, such that

$$f(x) = \frac{d}{dt} g(T^t x)_{t=0} \qquad (f(x) = g(Tx) - g(x)).$$

The main result of this note is the following criterion of zero homology of a function.

THEOREM 1. If T^t is a Y-flow (T^k a Y-cascade) [1] with everywhere-dense trajectories, then for f to be homologous to zero in the class of Hölder functions, it is necessary and sufficient that the following condition hold for any periodic trajectory $\{T^t x\}_{t=0}^{t=\tau}$ ($\{T^n x\}_{n=1}^{n=n_0}$):

$$\int_0^\tau f(T^t x)\,dt = 0 \qquad \left(\sum_1^{n=0} f(T^n x) = 0\right), \tag{1}$$

and if the Hölder modulus of continuity of the function f is $\omega(\rho) = C\rho^\delta$, then the modulus of continuity of the function g will not exceed $C_\delta C \rho^\delta$, where C_δ is a constant that depends on δ and on the dynamical system.

Proof. The necessity is evident. Let us prove the sufficiency. For simplicity we shall consider the case of a cascade. The manifold M^m is assumed to be endowed with a Lyapunov metric which is matched with the Y-condition for our cascade [2]. Let us prove the principal lemma.

LEMMA. There exists a positive K such that for any ε there exists an N_ε, and if $n > N_\varepsilon$, then it follows from $\rho(T^n x, x) < \varepsilon$ that there exists an x_0: $T^n x_0 = x_0$ and a $\rho(T^l x_0, T^l x) < K_\varepsilon$ for $1 \leq l < n$.

Proof. Let $\mathfrak{S}^k$ and $\mathfrak{S}^l$ be invariant contracting and expanding foliations, respectively. We shall use the following assertion, a consequence of the continuous dependence of the fibers (foliations) on the initial points: there exists a positive a such that if Π_0 and Π_1 are smooth areas that lie in the fibers $\mathfrak{S}^k$ and such that any point $\omega_0 \in \Pi_0$ can be connected by a path of length $< a$ that lies in a fiber $\mathfrak{S}^l$ with a point $\omega_1 \in \Pi_1$, then the mapping $U: \Pi_0 \to \Pi_0$, $u(\omega_0) = \omega_1$ will be continuous, and in the case of a small continuous deformation of these areas this mapping will vary continuously ([1], p. 26). Hence, it follows from the compactness of M^m that all the U constructed in this way have moduli of continuity that do not exceed a common $\delta(\varepsilon)$ ($\delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$), where an induced metric is taken in the fibers. Moreover, there exist positive γ and C such that for any two points A and B of the manifold M^m with $\rho(A, B) < \gamma$ it is possible to find a point S that lies in one contracting fiber containing A and in one expanding fiber containing B, with the distance from S to A and from S to B in the fibers being smaller than $C_\rho(A, B)$.

Leningrad State University. Translated from Matematicheskie Zametki, Vol. 10, No. 5, pp. 555-564, November, 1971. Original article submitted March 23, 1970.

References for the proof

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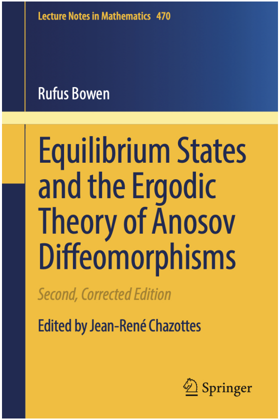

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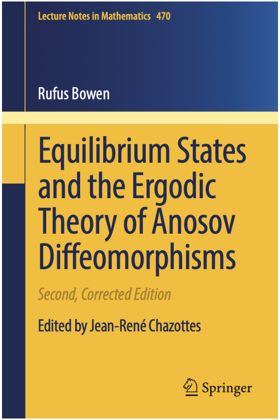

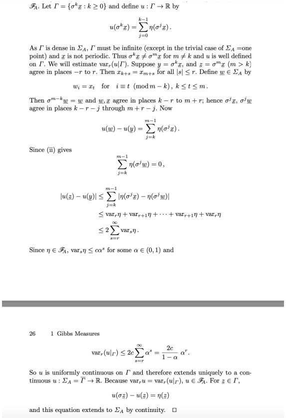

$\mathscr{F}_A$. Let $\Gamma = \{\sigma^k \underline{x} : k \geq 0\}$ and define $u : \Gamma \to \mathbb{R}$ by

$$u(\sigma^k \underline{x}) = \sum_{j=0}^{k-1} \eta(\sigma^j \underline{x}) .$$

As Γ is dense in Σ_A, Γ must be infinite (except in the trivial case of Σ_A =one point) and $\underline{x}$ is not periodic. Thus $\sigma^k \underline{x} \neq \sigma^m \underline{x}$ for $m \neq k$ and u is well defined on Γ. We will estimate $\mathrm{var}_r(u|\Gamma)$. Suppose $\underline{y} = \sigma^k \underline{x}$, and $\underline{z} = \sigma^m \underline{x}$ $(m > k)$ agree in places $-r$ to r. Then $x_{k+s} = x_{m+s}$ for all $|s| \leq r$. Define $\underline{w} \in \Sigma_A$ by

$$w_i = x_t \quad \text{for} \quad i \equiv t \pmod{m-k}, \ k \leq t \leq m .$$

Then $\sigma^{m-k}\underline{w} = \underline{w}$ and $\underline{w}, \underline{x}$ agree in places $k-r$ to $m+r$; hence $\sigma^j \underline{x}$, $\sigma^j \underline{w}$ agree in places $k-r-j$ through $m+r-j$. Now

$$u(\underline{w}) - u(\underline{y}) = \sum_{j=k}^{m-1} \eta(\sigma^j \underline{x}) .$$

Since (ii) gives

$$\sum_{j=k}^{m-1} \eta(\sigma^j \underline{w}) = 0 ,$$

$$\begin{aligned} |u(\underline{z}) - u(\underline{y})| &\leq \sum_{j=k}^{m-1} |\eta(\sigma^j \underline{x}) - \eta(\sigma^j \underline{w})| \\ &\leq \mathrm{var}_r \eta + \mathrm{var}_{r+1} \eta + \cdots + \mathrm{var}_{r+1} \eta + \mathrm{var}_r \eta \\ &\leq 2 \sum_{s=r}^{\infty} \mathrm{var}_s \eta . \end{aligned}$$

Since $\eta \in \mathscr{F}_A$, $\mathrm{var}_s \eta \leq c\alpha^s$ for some $\alpha \in (0,1)$ and

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$$\mathrm{var}_r(u|_\Gamma) \leq 2c \sum_{s=r}^{\infty} \alpha^s = \frac{2c}{1-\alpha}\, \alpha^r .$$

So u is uniformly continuous on Γ and therefore extends uniquely to a continuous $u : \Sigma_A = \overline{\Gamma} \to \mathbb{R}$. Because $\mathrm{var}_r u = \mathrm{var}_r(u|_\Gamma)$, $u \in \mathscr{F}_A$. For $\underline{z} \in \Gamma$,

$$u(\sigma \underline{z}) - u(\underline{z}) = \eta(\underline{z})$$

and this equation extends to Σ_A by continuity. □

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Skew products

Let G be an infinite countable group (e.g., $G = \mathbb{Z}^k$)

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Example (Trivial Example)

Let $\Sigma = \{1, 2, 3\}^{\mathbb{Z}}$ and $\sigma : \Sigma \to \Sigma$ is a full shift (on 3 symbols). Let $G = \mathbb{Z}$ then this is non-compact. Define $\psi : \Sigma \to \mathbb{Z}$ by

$$\psi(\underline{x}) = \begin{cases} 1 & \text{if } x_0 = 1 \\ 0 & \text{if } x_0 = 2 \\ -1 & \text{if } x_0 = 3 \end{cases} \qquad \text{where } \underline{x} = (x_n)_{n=0}^{\infty}.$$

The skew product $\widehat{\sigma}$ is easily seen to be transitive.

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The proof is exactly the same as for the case of the usual Livšic theorem (as observed by Ruhr and Sarig for example):

(The proof of the Livshits Theorem for subshifts of finite type given in [Bow75, Theorem 1.28] works verbatim in the countable alphabet case.) ☐

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Conclusion. $\widehat{\sigma}$-periodic points $(\underline{x}, g)$ correspond to σ-periodic points $\underline{x}$ satisfying the additional condition $\psi^n(\underline{x}) = e$ (identity).

Back to the trivial example

Example (Trivial Example)

Let $\Sigma = \{1,2,3\}^{\mathbb{Z}}$ and $\sigma : \Sigma \to \Sigma$ is a full shift (on 3 symbols). Let $G = \mathbb{Z}$ then this is non-compact. Define $\psi : \Sigma \to \mathbb{Z}$ by

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Back on Σ_A: For a periodic point $\underline{z} = (z_0, z_1, \cdots, z_{p-1}, z_0, z_1, \cdots)$ with $\sigma^p \underline{z} = \underline{z}$ we have that the restriction $\sum_{i=0}^{p-1} F(\sigma^i \underline{z}) = 0$ applies precisely when

$$\#\{0 \leq i \leq p-1 : z_i = 1\} = \#\{0 \leq i \leq p-1 : z_i = 3\},$$

i.e., the digits 1 and 3 appear the same number of times.

A restricted Livšic theorem on Σ_A

Let $\psi : \Sigma_A \to G$ be continuous function into a countable group.

Theorem (Restricted Livšic's Theorem)

Assume $\widehat{\sigma}$ is transitive. Let $F : \Sigma_A \to \mathbb{R}$ be a Hölder continuous function. Assume that **restricting** *to those periodic points $\sigma^n \underline{x} = \underline{x}$ for which $\psi^n(\underline{x}) = e$ we have*

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So with less periodic orbit data we have an extra term $\alpha \circ \psi$.

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Somehow we need to "push down" $\widehat{u}$ from $\widehat{\Sigma}_A$ to Σ_A ...

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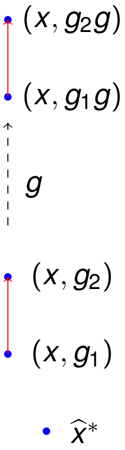

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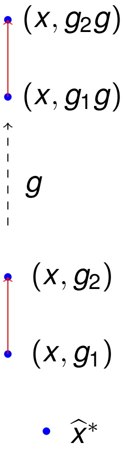

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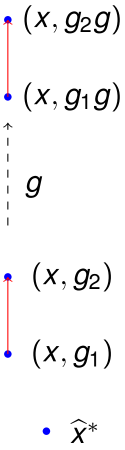

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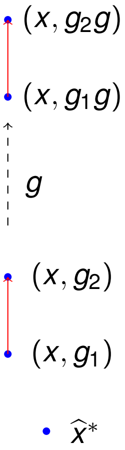

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\end{aligned}$$

Proof of the Livšic theorem III

Third step (Pushing down $\widehat{u}$ introduces $\alpha : G \to \mathbb{R}$).
By Livšic theorem for skew products $\widehat{u} \in \Sigma_A \times G$ satisfies

$$\underbrace{\widehat{F}(x,g)}_{=F(x)} = \underbrace{(\widehat{u}\widehat{\sigma})(x,g) - \widehat{u}(x,g)}_{\text{Independent of } g} \;\forall (x,g) \in \widehat{\Sigma}_A = \Sigma_A \times G. \qquad (*)$$

By independence, for any $g_1, g \in G$, replace g in (*) by g_1 and $g_1 g$:

$$(\widehat{u}\widehat{\sigma})(x,g_1) - \widehat{u}(x,g_1) = (\widehat{u}\widehat{\sigma})(x,g_1 g) - \widehat{u}(x,g_1 g)$$
$$\implies \underbrace{(\widehat{u}\widehat{\sigma})(x,g_1 g) - (\widehat{u}\widehat{\sigma})(x,g_1)}_{=(\Delta_g \widehat{u})\circ\widehat{\sigma}(x,g_1)} = \underbrace{\widehat{u}(x,g_1 g) - \widehat{u}(x,g_1)}_{=:(\Delta_g \widehat{u})(x,g_1)} =: \alpha(g)$$

1. By transitivity, $\Delta_g u : \widehat{\Sigma}_A \to \mathbb{R}$ takes a constant value $\alpha(g)$.
2. Define $u : \Sigma_A \to \mathbb{R}$ by $u(x) = \widehat{u}(x,e)$. Then

$$\begin{aligned}
\boxed{F(x)} &= \widehat{F}(x,e) = (\widehat{u}\widehat{\sigma})(x,e) - \widehat{u}(x,e) && \text{by (*) with } g = e\\
&= \widehat{u}(\sigma x, \psi(x)) - \widehat{u}(x,e) && \text{by definition of } \widehat{\sigma}\\
&= (\widehat{u}(\sigma x, e) + \alpha(\psi(x))) - \widehat{u}(x,e) = \boxed{u(\sigma x) - u(x) + \alpha(\psi(x))}
\end{aligned}$$

Proof of the Livšic theorem IV

We have almost finished. We have established

$$F = u \circ \sigma - u + \alpha \circ \psi$$

where $\alpha(g) = \widehat{u}(x, g) - \widehat{u}(x, e)$.

Proof of the Livšic theorem IV

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Given $g_1, g_2 \in G$ then for any $x \in \Sigma_A$

$$\begin{aligned} \alpha(g_1 g_2) &= \widehat{u}(x, g_1 g_2) - \widehat{u}(x, e) \\ &= (\widehat{u}(x, g_1 g_2)\widehat{u}(x, g_2) + (\widehat{u}(x, g_2) - \widehat{u}(x, e)) \\ &= \alpha(g_1) + \alpha(g_2). \end{aligned}$$

This completes the proof.

“Don’t worry—our EV will win in the long run.”

Continuous Livšic type theorems

Anosov flows

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There exists a continuous splitting of the unit tangent bundle $TM = E^0 \oplus E^u \oplus E^s$. and there exist $C, \lambda > 0$ with

1. E^0 a one dimensional bundle tangent to the flow direction.
2. For $v \in E^s$ we have
$$\|D\phi_t|_{E^s}\|, \|D\phi_{-t}|_{E^u}\| \leq Ce^{-\lambda t} \text{ for } t \geq 0.$$
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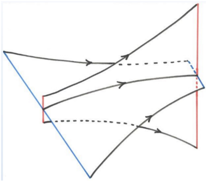

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$$\int_0^{\lambda(\tau)} F(\phi_t x_\tau)dt = 0 \text{ for } x_\tau \in \tau \qquad (\text{"}F \text{ vanishes around } \tau\text{"}).$$

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The proof is similar to that for the discrete version (for subshifts of finite type, Anosov diffeomorphisms, etc.)

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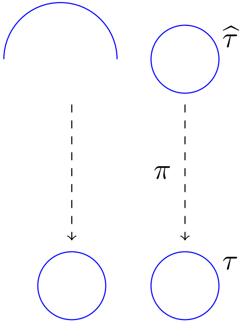
A closed orbit τ for $\phi_t : M \to M$ has least period $\lambda(\tau) > 0$ (i.e., for $x \in \tau$ we have $\phi_{\lambda(\tau)}(x) = x$ and we take the smallest such value).



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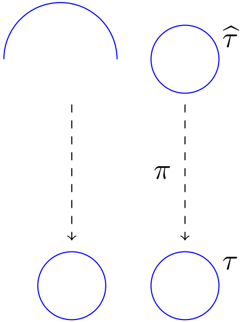

The restriction. We consider only closed ϕ-orbits τ which lift to closed $\widehat{\phi}$-orbits $\widehat{\tau}$ of the same least period (i.e., $\lambda(\tau) = \lambda(\widehat{\tau})$).

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Smoothness

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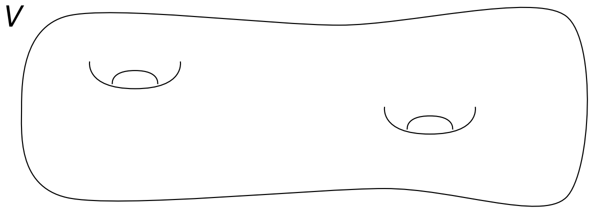
Let V be a (compact) surface with a Riemannian metric ρ of negative curvature and a three dimensional unit tangent bundle SV.



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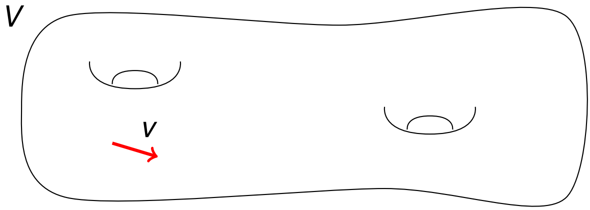

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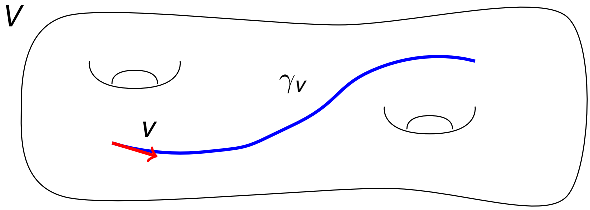

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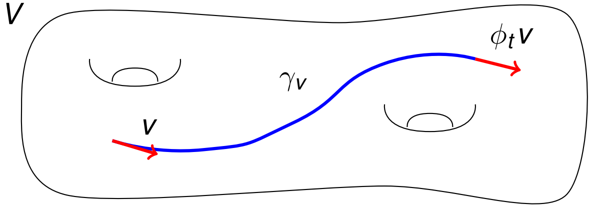

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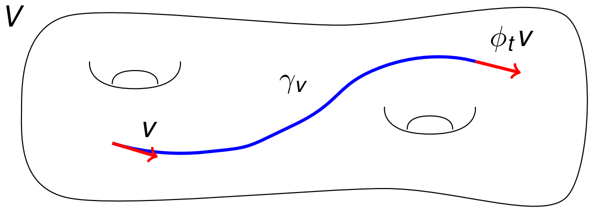

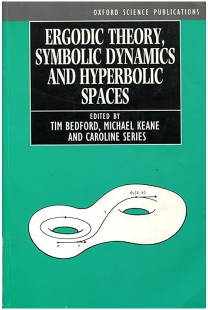

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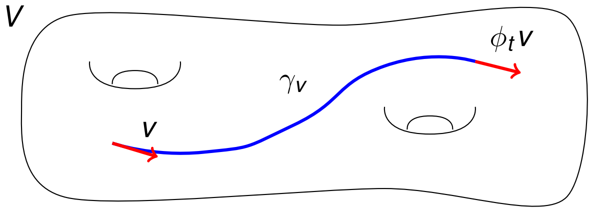

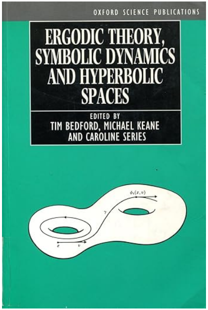

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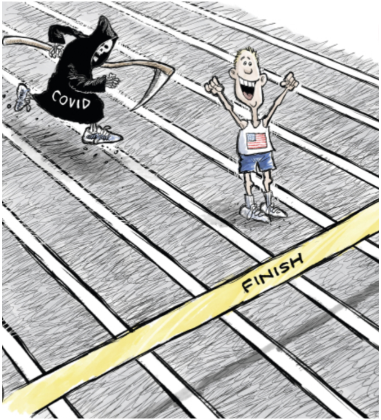

Thank you for your time.

Applications

This can be combined with rigidity results which depend on information on weights on closed orbits to conclude the same results with information on fewer closed orbits (determined by a cover). For example,

- Otal's theorem for negatively curved surfaces (marked length spectrum of closed geodesics determines the surface up to isometry)
- Guillarmou and Lefeuvre's theorem for negatively curved manifolds (locally the marked length spectrum determines the manifold up to isometry)
- Butt, Erchenko, Humbert, Lefeuvre and Wilkinson's theorem for negatively curved manifolds (locally the marked Poincaré determinant spectrum determines the manifold up to homothety)

Back

Some questions

1. The original Livšic theorem was extended to matrix valued functions $F : \Sigma_A \to GL(d, \mathbb{R})$ by Kalinin (2011). Does the restricted Livsic theorem generalize to this setting?
2. Is there a version of the measurable Livsic theorem for infinite covers? i.e., $\exists$ continuous versions of measurable cocycles.
3. Is there some "restricted" result related to the "positive Livšic theorem" (or "revelation theorem") (i.e., where $\sum_{i=0}^{n-1} F(\sigma^i \underline{x}) \geq 0$ whenever $\sigma^n \underline{x} = \underline{x}$).
4. For quasi-hyperbolic toral automorphisms there is a Livšic theorem (due to Veech). Is there a restricted version of this theorem?
5. For partially hyperbolic systems there are versions of the Livšic theorem for partially hyperbolic diffeomorphisms where *us*-paths replace closed orbits. Is there a restricted version of these theorems?

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3. It has interesting applications (e.g., rigidity theorems).
4. It illustrates a simple "local-global principle"
(i.e., local information (on periodic orbits) gives global information (on a function)).

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Back

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$$F(x) = \frac{d}{dt} U(\phi_t x) + \omega(X)(x) \quad \textit{for any } x \in M$$

where X is the vector field for ϕ_t.

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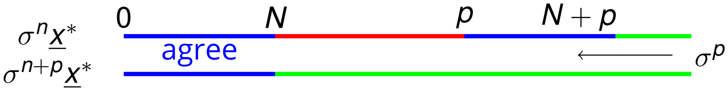
3. It will be more natural to find analogues of these results for Anosov flows (later).

Back

Hölder continuity and quantifying approximations



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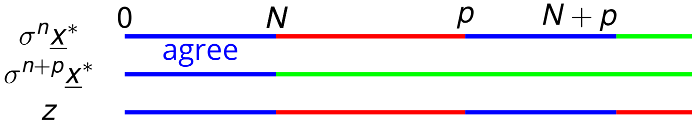

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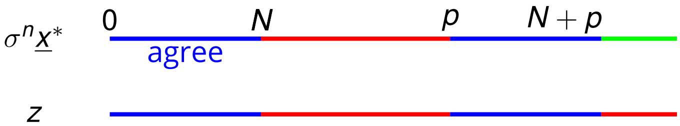

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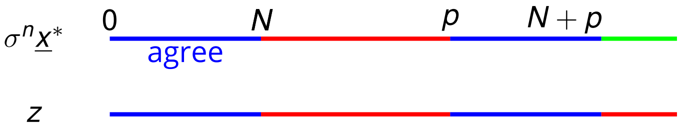

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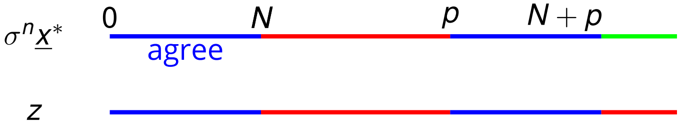

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