Gibbs Measures in a Cold Climate

From lattice gases to thermodynamic formalism

Mark Pollicott (and Oliver Jenkinson)

Victoria, November 2026



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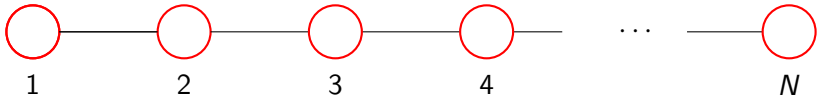
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1. Gibbs measures for lattice gases

A one-dimensional lattice gas and its energy

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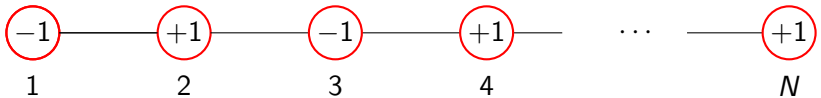
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2. To each configuration σ we can associate the *Hamiltonian (energy)* $H(\sigma)$ for a given configuration. For example,
$$H(\sigma) = \sum_{i=1}^{N} h(\sigma_i, \sigma_{i+1 \pmod N}) \quad \text{where } h(a, b) = \begin{cases} +1, & a = b, \\ -1, & a \neq b. \end{cases}$$

Hamiltonian Energy: Examples

Two illustrative configurations, assuming

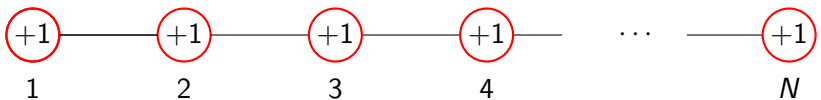
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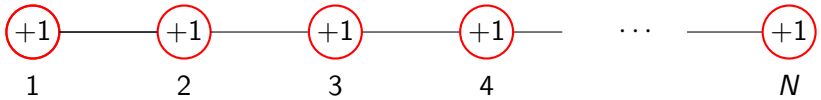
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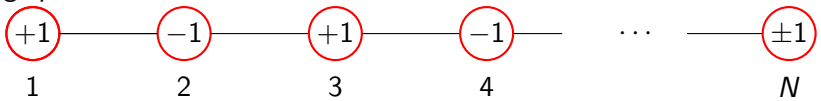
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(1) All spins the same:



Then $H(\sigma) = N$.

(2) Alternating spins:



Then $H(\sigma) = -N$ (for N even) or $H(\sigma) = 2 - N$ (for N odd).

Hamiltonian energy and the Gibbs distribution

We can multiply the Hamiltonian by the "inverse temperature" $\beta > 0$, and define the Gibbs probability on the configurations by:

$$\mathbb{P}_\beta(\sigma) = \frac{\mathrm{e}^{-\beta H(\sigma)}}{Z_N(\beta)},$$

where the normalizing constant is called the *partition function*:

$$Z_N(\beta) = \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)}.$$

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Cold-climate viewpoint

As $\beta \to +\infty$ (low temperature), the Gibbs distribution concentrates on configurations of smaller energy.

Aside: A useful tool is the transfer matrix

A convenient way to write the partition function is in terms of the *transfer matrix*

$$M = \left(\mathrm{e}^{-\beta h(i,j)}\right)_{i,j\in\{-1,+1\}} = \begin{pmatrix} \mathrm{e}^{-\beta} & \mathrm{e}^{\beta} \\ \mathrm{e}^{\beta} & \mathrm{e}^{-\beta} \end{pmatrix}.$$

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The partition function can then be written as

$$Z_N(\beta) = \sum_{\sigma\in\{-1,+1\}^N} \mathrm{e}^{-\beta\sigma_i\sigma_{i+1}} = \mathrm{tr}(M^N).$$

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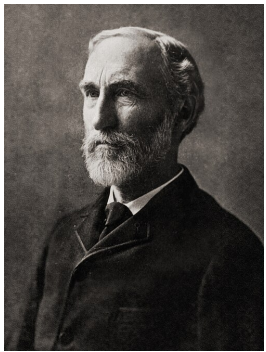
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If $\lambda_\beta > 0$ denotes the largest eigenvalue of M, then

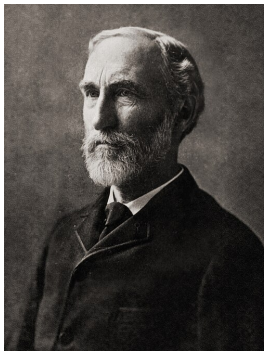
$$Z_N(\beta) \sim \lambda_\beta^N \qquad (\text{as } N \to \infty).$$

A historical note



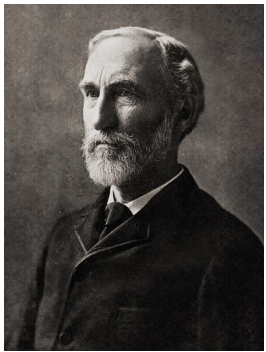
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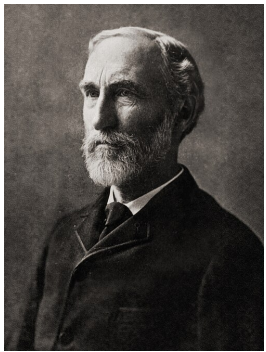
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2. Gibbs measures for dynamical systems

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The metric: A standard metric on Σ is

$$d(x,y) = \sum_{n=0}^{\infty} \frac{e(x_n,y_n)}{2^n} \text{ where } e(i,j) = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases}$$

for two sequences $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$, say, Then x and y are closer when they agree in more initial coordinates.

The shift map and partition functions

The dynamics. The transformation is the *shift map* $T : \Sigma \to \Sigma$ given by

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

on the space $\Sigma = \{1, 2, \ldots, k\}^{\mathbb{Z}_+}$.

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Hölder functions on Σ. In the dynamical setting, replace the interaction by a Hölder function

$$h(\cdot, \cdot) \quad \longrightarrow \quad F : \Sigma \to \mathbb{R}$$

In particular,

$$\exists C, \alpha > 0 \quad |F(x) - F(y)| \leq C\, d(x, y)^{\alpha}.$$

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Partition function. We can replace the partition function

$$Z_n(\beta) \quad \longrightarrow \quad Z_n(F) = \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x) \right)$$

where the summation over configurations is replaced by the summation over all fixed points of $T^n = \underbrace{T \circ \cdots \circ T}_{\times n}$ in Σ.

Gibbs measures in the dynamical setting : Short and sweet

Given a Hölder function $F : \Sigma \to \mathbb{R}$, for each $n \geq 1$ we can associate a probability measure

$$\mu_n^F = \frac{1}{Z_n} \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x) \right) \delta_x \text{ normalized by } Z_n(F) = \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x) \right),$$

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Finally we have:

Dynamical Gibbs measure. The (unique) Gibbs measure μ^F for F is (a T-invariant probability) obtained as the weak-star limit

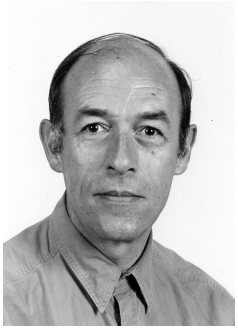
$$\mu_n^F \longrightarrow \mu^F \qquad (n \to +\infty).$$

i.e., for continuous $G : \Sigma \to \mathbb{R}$ we have $\int G \, d\mu_n \longrightarrow \int G \, d\mu^F$ (and $\int G \circ T d\mu^F = \int G d\mu^F$).

Sinai and Ruelle

Gibbs measures for dynamical systems were introduced and studied by Ya. G. Sinai (1935–) and D. Ruelle (1935–).





Gibbs measures in Dynamical setting: More traditional formulation

1. Assume that A is an aperiodic $k \times k$-matrix (i.e., $\exists N > 0$ such that $A^N > 0$) and let

$$\Sigma = \left\{ x = (x_n)_{n=0}^{\infty} \in \{1, \cdots, k\}^{\mathbb{Z}_+} \, : \, A(x_n, x_{n+1}) = 1 \text{ for } n \geq 0 \right\}$$

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3. There is a unique (T-invariant) Gibbs measure μ_F such that
$$\mu_F \left([i_0, \cdots, i_{n-1}] \right) \asymp \frac{\exp\left(\sum_{i=0}^{n-1} F(T^i x) \right)}{Z_n(F)}$$
for $x \in [i_0, \cdots, i_{n-1}] = \underbrace{\{ y = (y_n) \in \Sigma \,:\, y_j = i_j \text{ for } 0 \leq j \leq n-1 \}}_{\text{basis of open sets}}$
where $i_0 \cdots i_{n-1}$ is an "allowed word", i.e., $A(i_0, i_1) = \cdots = A(i_{n-2}, i_{n-1}) = 1$

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Question

What happens to the probability measures $\mu^{\beta F}$ as $\beta \to +\infty$? ("the zero temperature limit") Do they converge to some measure in the weak star topology?

3. Zero temperature limits

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- Moreover, they lie in the entropy-maximizing subset

$$\mathcal{M}_{\text{ext}} = \{\mu \in \mathcal{M}_{\max} : h(\mu) = h_*\},$$

where $h(\mu)$ is the entropy of the measure μ (which I won't define) and

$$h_* := \sup\{h(\nu) : \nu \in \mathcal{M}_{\max}\}.$$

Aside: The zero-temperature limit points

Let $F : \Sigma \to \mathbb{R}$ be Hölder continuous.
We can begin by identifying where we might find the limit points of $\{\mu^{\beta F}\}_{\beta>0}$ as $\beta \to +\infty$.

- In particular, one can show that the limit points lie in the special set of measures

$$\mathcal{M}_{\max} = \left\{ \nu \text{ is } T\text{-invariant probability, } : \int F \, d\nu = \sup_{\eta \ T\text{-invariant}} \int F \, d\eta \right\},$$

- Moreover, they lie in the entropy-maximizing subset

$$\mathcal{M}_{\text{ext}} = \{\mu \in \mathcal{M}_{\max} : h(\mu) = h_*\},$$

where $h(\mu)$ is the entropy of the measure μ (which I won't define) and

$$h_* := \sup\{h(\nu) : \nu \in \mathcal{M}_{\max}\}.$$

However, this doesn't address the question of whether $\{\mu^{\beta F}\}_{\beta>0}$ *converges as* $\beta \to +\infty$.

What is known: negative and positive results





Negative result: Chazottes–Hochman (2010)

There exists a Hölder potential $F : \Sigma \to \mathbb{R}$ such that $\{\mu^{\beta F}\}_{\beta>0}$ does not converge as $\beta \to +\infty$.

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There exists a Hölder potential $F : \Sigma \to \mathbb{R}$ such that $\{\mu^{\beta F}\}_{\beta>0}$ does not converge as $\beta \to +\infty$.

Positive result: Brémont (2003)

If $F(x) = F(x_0, x_1, \ldots, x_{N-1})$ is locally constant (i.e., $F(x)$ depends on only finitely many terms in x) then $\{\mu^{\beta F}\}_{\beta>0}$ converges as $\beta \to +\infty$.

"finite range interactions" and functions $F(x)$

We have already seen a simple example of a locally constant function.

Example 1 ($N = 1$)

Let $\Sigma = \{1, 2\}^{\mathbb{Z}_+}$. Let $F : \Sigma \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} +1 \text{ if } x_0 = x_1 \\ -1 \text{ if } x_0 \neq x_1 \end{cases} \qquad \text{where } x = (x_0, x_1, x_2, \ldots) \in \Sigma$$

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Then

$$\underbrace{Z_n(\beta) = \sum_{\sigma \in \{-1,+1\}^n} e^{-\beta H(\sigma)}}_{\text{Lattice gases}} = \underbrace{\sum_{T^n x = x} \exp\left(-\beta \sum_{i=0}^{n-1} F(T^i x)\right)}_{\text{Dynamical viewpoint}}$$

4. An elementary proof of Brémont's theorem

An alternative approach

We want to give an elementary proof of Brémont's result.

(a) We therefore need to show that if $F : \Sigma \to \mathbb{R}$ is locally constant, i.e.,

$$F(x) = F(x_0, x_1, \ldots, x_{N-1})$$

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Typically, we will assume $M \geq N$.

The main ingredient: a zeta function

Let $F : \Sigma \to \mathbb{R}$ be a locally constant function and associate a *zeta function* formally defined by

$$\zeta(z, F) := \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x) \right) \right).$$

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Simple Lemma

Let B_F be a square matrix with:

- Rows indexed by allowed words $i_0 i_1 \dots i_{M-1}$ (and columns by allowed words $j_0 j_1 \dots j_{M-1}$);
- Entries

$$B_F(i_0 i_1 \dots i_{M-1}, j_0 j_1 \dots j_{M-1}) = \begin{cases} \exp F(i_0, \dots, i_{M-1}) & \text{if } i_1 \dots i_{M-1} = j_0 \dots j_{M-2} \\ 0 & \text{if } i_1 \dots i_{M-1} \neq j_0 \dots j_{M-2}. \end{cases}$$

Then $\zeta(z, F) = \det(I - zB_F)$.

Back to the example

Example 2 ($N = 1$)

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Then for $n \geq 1$:

$$B_F = \begin{pmatrix} e & 1/e \\ 1/e & e \end{pmatrix} \implies \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x) \right) = \operatorname{trace}(B_F^n)$$

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Thus

$$\zeta(z, F) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{trace}(B_F^n) \right) = \det(I - zB_F)$$

The zeta function expanded

Let $F : \Sigma \to \mathbb{R}$ and $G : \Sigma \to \mathbb{R}$ be locally constant functions (depending on at most M terms, say).

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- Let $\beta, t \in \mathbb{R}$ and then first replace F by $\beta F + tG$ (still a locally constant function).
- $\zeta(z, \beta F + tG) = \det(I - zB_{\beta F + tG})$ can be expanded as a finite polynomial in z :

$$\zeta(z, \beta F + tG) = \sum_n z^n \left(\sum_j a_j^{(n)} \exp(\beta b_j^{(n)} + t c_j^{(n)}) \right), \text{say},$$

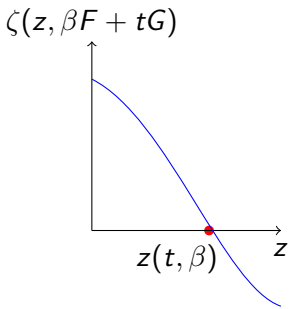
for some real numbers $\{a_j^{(n)}\}$, $\{b_j^{(n)}\}$, $\{c_j^{(n)}\}$ (and finite summations).

The smallest positive zero of the zeta function

There is a standard way to relate the zeta function to the integrals $\int G\, d\mu^{\beta F}$ for $\beta > 0$.

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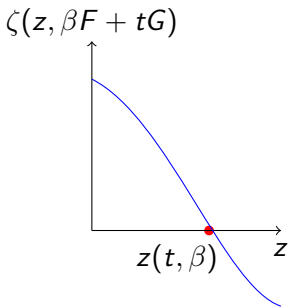


1. For fixed $t, \beta > 0$, the function $z \mapsto \zeta(z, \beta F + tG)$ has a smallest positive zero, denoted $z(t, \beta)$:

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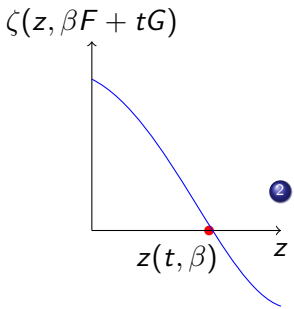
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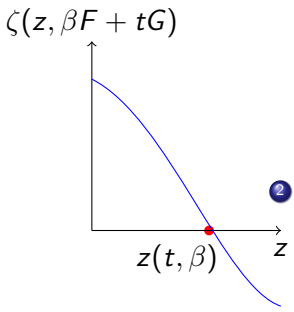
2. We can then write

$$\int G\, d\mu^{\beta F} = -\frac{1}{z(0, \beta)} \left. \frac{\partial z}{\partial t} \right|_{t=0,\ z=z(0,\beta)}. \qquad (0)$$

(after D. Ruelle: *Gibbs measures as a derivative of pressure*)

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To proceed we need to understand the right hand side of (0) as $\beta \to +\infty$.

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Let $F : \Sigma \to \mathbb{R}$ be a locally constant function and let $\beta > 0$.

- There is a unique Gibbs measure $\mu^{\beta F}$ (for $\beta F : \Sigma \to \mathbb{R}$).
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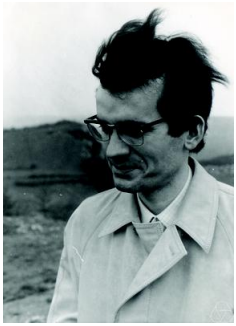
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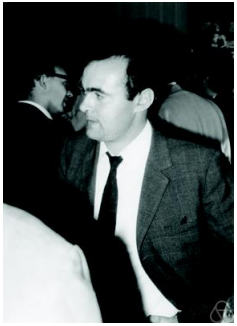
For the first part we have the following result.

Lemma 3 (Conze-Guivarc'h, 1993)

$\exists A > 0$ *and* $h^* > 0$ *such that*

$$\frac{z(0,\beta)}{\mathrm{e}^{-\beta A}} \longrightarrow h_*, \qquad \text{as } \beta \to +\infty \qquad (1)$$





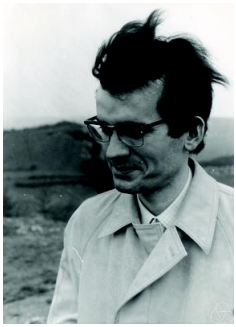
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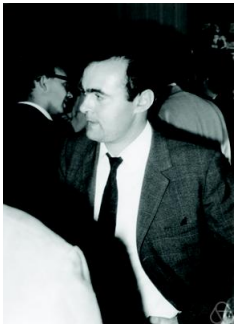
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This was a result in the grand tradition of these authors: It was contained in a preprint which was never published. However, it was reproved by others (e.g., Morris, Jenkinson, Coelho, Contreras-Lopes-Thieullen).

2. $\left.\frac{\partial z}{\partial t}\right|_{t=0,\, z=z(0,\beta)}$ as $\beta \to +\infty$

For the second part, recall that $z(t, \beta)$ is a zero for the zeta function $\zeta(z, \beta F + tG) = 0$.

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(where all the sums are finite).
This final expression is rather like a ratio of "finite Dirichlet series".

Putting the two parts together

We would *like* to combine (1) and (2) to deduce the following limit exists:

$$\begin{aligned}
\lim_{\beta\to+\infty}\int G\,d\mu^{\beta F} &= \lim_{\beta\to+\infty} -\frac{1}{z(0,\beta)}\left.\frac{\partial z}{\partial t}\right|_{t=0,\ z=z(0,\beta)} \\
&= \lim_{\beta\to+\infty}\left(\frac{e^{\beta A}}{h_*}\right)\left(-\frac{\sum_n z(0,\beta)^n\left(\sum_i a_i^{(n)}c_i^{(n)}\exp(\beta b_i^{(n)})\right)}{\sum_n nz(0,\beta)^{n-1}\left(\sum_i a_i^{(n)}\exp(\beta b_i^{(n)})\right)}\right) \\
&= \lim_{\beta\to+\infty}\left(\frac{e^{\beta A}}{h_*}\right)\left(-\frac{\sum_n\left(\frac{h_*}{e^{\beta A}}\right)^n\left(\sum_i a_i^{(n)}c_i^{(n)}\exp(\beta b_i^{(n)})\right)}{\sum_n n\left(\frac{h_*}{e^{\beta A}}\right)^{n-1}\left(\sum_i a_i^{(n)}\exp(\beta b_i^{(n)})\right)}\right) \\
&= \lim_{\beta\to+\infty} -\frac{\sum_n(h_*e^{-\beta A})^{n-1}\left(\sum_j a_i^{(n)}c_i^{(n)}\exp(\beta b_i^{(n)})\right)}{\sum_n n(h_*e^{-\beta A})^{n-1}\left(\sum_j a_i^{(n)}\exp(\beta b_i^{(n)})\right)}.
\end{aligned}$$

The final step



Oliver Jenkinson and I got stuck at this last step 19 years ago since we didn't know:

Problem (2007)

Why does the limit of the quotient of such finite sums exist?

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- **Solution I (2026):** follows immediately from an O-minimality result of A. Wilkie (1996).
- **Solution II (2026):** there is now an elementary algebraic proof (in less than a page) inspired by knowing the result was true.

The irony

Although I never met Alex Wilkie, I should have known more about his work.

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Although I never met Alex Wilkie, I should have known more about his work.

He succeeded me as a professor in Manchester University.

From Wikipedia:

Holders [edit]

Previous holders of the Fielden Chair (and lectureship) are:[2]

- A. T. Bentley (1876–1880) Lecturer in Pure Mathematics
- J. E. A. Steggall (1880–1883) Lecturer in Pure Mathematics
- R. F. Gwyther (1883–1907) Lecturer in Mathematics
- F. T. Swanwick (1907–1912) Lecturer in Mathematics
- H. R. Hasse (1912–1918) Lecturer in Mathematics
- George Henry Livens (1920–1922) Lecturer in Mathematics[3]
- Louis Mordell (1923–1945)
- Max Newman (1945–1964)
- Frank Adams (1964–1971)
- Ian G. Macdonald (1972–1976)
- Norman Blackburn (1978–1994)
- Mark Pollicott (1996–2004)
- Alex Wilkie (2007–)
- Radha Kessar (2024–)[4]

The end

Thank you for your time