Continuous Eigenfunctions of the Transfer Operator for the Dyson Potential

Anders Johansson   Anders Öberg   Mark Pollicott

"Tudo vale a pena se a alma não é pequena" - Fernando Pessoa

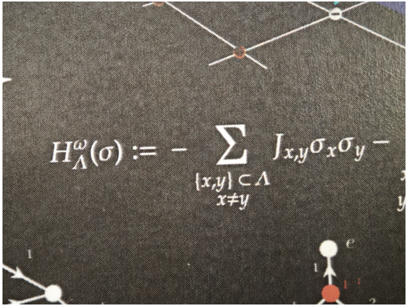
USP, August, 2026

Two gentlemen from Sweden

Overview

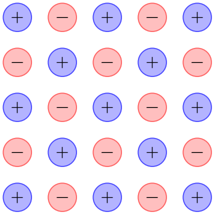
A physical viewpoint

Lattice gasses



It begins with ... The Ising Model

1D lattice gas configuration



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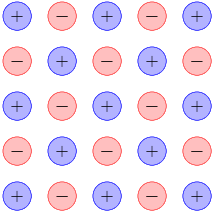

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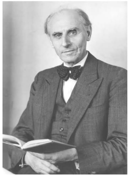

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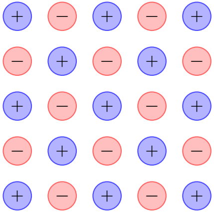

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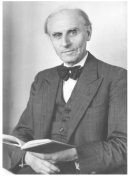

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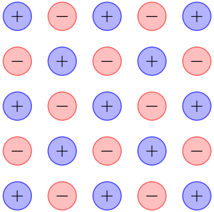

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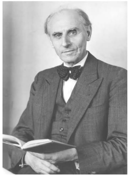

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- Freeman Dyson considered a 1D example *which does not have nearest-neighbour interactions*

The simpler case: nearest neighbour interactions (for 1D gases)

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Photo: Wikimedia Commons, CC BY-SA 2.0

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- The interaction between x_i and x_{i+k} takes the form:

$$J(k) = \beta k^{-\alpha}, \quad 1 < \alpha < 2,$$

and exhibits a phase transition at low temperature (i.e., large $\beta > 0$).

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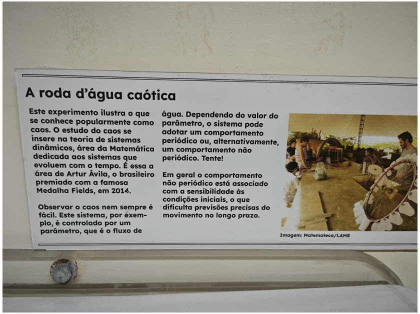
A dynamical viewpoint

Dynamical systems/Ergodic theory



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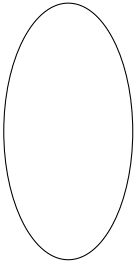

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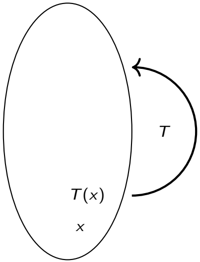

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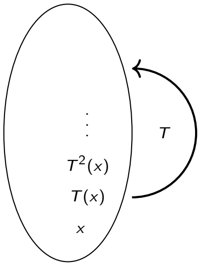

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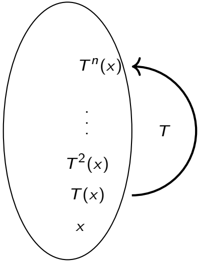

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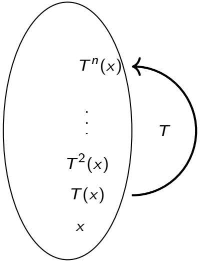

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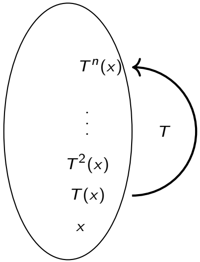

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The first problem is to find nice probability measures which actually have these properties.

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Transfer operator: Given a function $f : X \to \mathbb{R}$ we define a new function $(\mathscr{L}_\phi f) : X \to \mathbb{R}$ by

$$(\mathscr{L}_\phi f)(x) = \sum_{y \,:\, Ty = x} e^{\phi(y)} f(y).$$

(It helps to assume $\sum_{k=1}^{\infty} J(k) < +\infty$ so that the definition makes sense).

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We can define a measure μ on X by $\mu = h\nu$ for the purposes of ergodic theory.

Suitably motivated, we come to: When is h a continuous function?

Some classical results

What was known decades ago

An old result on continuous eigenfunctions

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Definition

We say that $\phi : X \to \mathbb{R}$ has *summable variation* if

$$\sum_{n=0}^{\infty} \operatorname{var}_n \phi < \infty \quad \text{where} \quad \operatorname{var}_n \phi := \sup_{x,y : x_k = y_k,\, k \leq n} |\phi(x) - \phi(y)|.$$

where $x = (x_1, x_2, \cdots) \in X$ and $y = (y_1, y_2, \cdots) \in X$.

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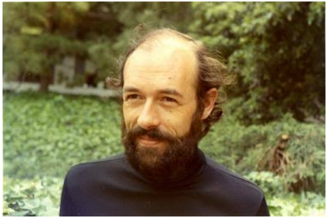

Theorem (Folklore, after Ruelle and Sinai)

There exists a positive continuous eigenfunction under the summable variation assumption.

Example: Dyson potential

Claim for Dyson potential

For $J(k) = \beta k^{-\alpha}$ denote the tail

$$r_n := \sum_{k=n+1}^{\infty} J(k) \text{ and then } \operatorname{var}_n \phi = 2r_n.$$

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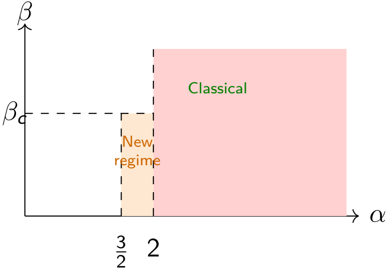

Aim: The claim is that we can reduce the value of α so that for $\beta > 0$ small enough we still have a continuous eigenfunction

Progress

Newer results

A first general result

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Theorem (Johansson–Öberg–Pollicott)

Let $r(x) := \sum_{n=0}^{\infty} r_n |x_n|$ where $r_n = \sum_{k>n} J(k)$ then if

$$\int_X e^{r(x)} \, d\nu(x) < \infty$$

then there exists a strictly positive continuous function $h : X \to \mathbb{R}^+$ such that $\mathscr{L}_\phi h = \lambda h$.

A first general result

Let $0 \leq J(k) \to 0$ as $k \to +\infty$ and consider general potentials of the form

$$\phi(x) = x_0 \sum_{k=1}^{\infty} J(k) x_k.$$

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A second (less) general result

Let $0 \leq J(k) \to 0$ as $k \to +\infty$ and set

$$\phi(x) = x_0 \sum_{k=n+1}^{\infty} J(k) x_k .$$

Theorem (Johansson–Öberg–Pollicott)

There exists a critical value $\beta_c > 0$ such that if

$$\sum_{n=0}^{\infty} (\mathrm{var}_n\, \phi)^2 < \infty \text{ and } \beta < \beta_c$$

then there exists a strictly positive continuous function $h : X \to \mathbb{R}^+$ such that $\mathscr{L}_\phi h = \lambda h$.

Example (even less general): Dyson potential

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For $J(k) = \beta k^{-\alpha}$ with $\alpha > \frac{3}{2}$ and $\beta < \beta_c$, then there exists a strictly positive continuous function $h : X \to \mathbb{R}^+$ such that $\mathscr{L}_\phi h = \lambda h$.

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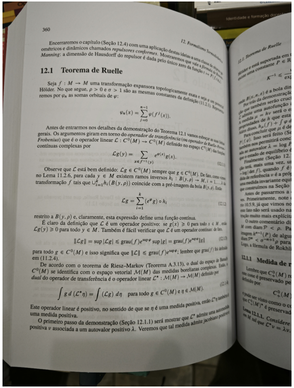
Strategy of the proof



360 12. Formalismo Termodinâmico

Encerraremos o capítulo (Seção 12.4) com uma aplicação destas ideias a uma classe de objetos geométricos e dinâmicos chamados *repulsores conformes*. Mostraremos que vale a *fórmula de Bowen–Manning*: a dimensão de Hausdorff do repulsor é dada pelo único zero da função $t \mapsto P(f, t\varphi_u)$.

12.1 Teorema de Ruelle

Seja $f : M \to M$ uma transformação expansora topologicamente exata e seja φ um potencial Hölder. No que segue, $\rho > 0$ e $\sigma > 1$ são as mesmas constantes da definição (11.2.1). Representaremos por φ_n as somas orbitais de φ:

$$\varphi_n(x) = \sum_{j=0}^{n-1} \varphi(f^j(x)).$$

Antes de entrarmos nos detalhes da demonstração do Teorema 12.1 vamos esboçar as suas linhas gerais. Os argumentos giram em torno do *operador de transferência* (ou *operador de Ruelle–Perron–Frobenius*) que é o operador linear $\mathcal{L} : C^0(M) \to C^0(M)$ definido no espaço $C^0(M)$ das funções contínuas complexas por

$$\mathcal{L}g(y) = \sum_{x \in f^{-1}(y)} e^{\varphi(x)} g(x). \tag{12.1.1}$$

Observe que $\mathcal{L}$ está bem definido: $\mathcal{L}g \in C^0(M)$ sempre que $g \in C^0(M)$. De fato, como vimos no Lema 11.2.6, para cada $y \in M$ existem ramos inversos $h_i : B(y, \rho) \to M$, $i = 1, \dots, k$ da transformação f tais que $\cup_{i=1}^{k} h_i(B(y, \rho))$ coincide com a pré-imagem da bola $B(y, \rho)$. Então

$$\mathcal{L}g = \sum_{i=1}^{k} (e^{\varphi} g) \circ h_i \tag{12.1.2}$$

restrito a $B(y, \rho)$ e, claramente, esta expressão define uma função contínua.

É claro da definição que $\mathcal{L}$ é um operador positivo: se $g(x) \geq 0$ para todo $x \in M$, então $\mathcal{L}g(y) \geq 0$ para todo $y \in M$. Também é fácil verificar que $\mathcal{L}$ é um operador contínuo: de fato,

$$\|\mathcal{L}g\| = \sup |\mathcal{L}g| \leq \operatorname{grau}(f) e^{\sup \varphi} \sup |g| = \operatorname{grau}(f) e^{\sup \varphi} \|g\| \tag{12.1.3}$$

para todo $g \in C^0(M)$ e isso significa que $\|\mathcal{L}\| \leq \operatorname{grau}(f) e^{\sup \varphi}$; lembre que $\operatorname{grau}(f)$ foi definido em (11.2.4).

De acordo com o teorema de Riesz–Markov (Teorema A.3.13), o dual do espaço de Banach $C^0(M)$ se identifica com o espaço vetorial $\mathcal{M}(M)$ das medidas borelianas complexas. Então, o *dual* do operador de transferência é o operador linear $\mathcal{L}^* : \mathcal{M}(M) \to \mathcal{M}(M)$ definido por

$$\int g \, d(\mathcal{L}^* \eta) = \int (\mathcal{L}g) \, d\eta \quad \text{para todo } g \in C^0(M) \text{ e } \eta \in \mathcal{M}(M). \tag{12.1.4}$$

Este operador linear é positivo, no sentido de que se η é uma medida positiva, então $\mathcal{L}^*\eta$ também é uma medida positiva.

O primeiro passo da demonstração (Seção 12.1.1) será mostrar que $\mathcal{L}^*$ admite uma automedida positiva ν associada a um autovalor positivo λ. Veremos que tal medida admite jacobiano positivo e

From a measure on $\{-1,+1\}^{\mathbb{Z}}$ to One-Sided Eigenfunction

- Let μ be a suitable measure on $\{-1,+1\}^{\mathbb{Z}}$.
  Let $\nu =$ one-sided eigenmeasure (i.e., $\mathscr{L}_{\phi}^{*}\nu = \lambda\nu$) on $\{-1,+1\}^{\mathbb{N}}$ (which we get painlessly)

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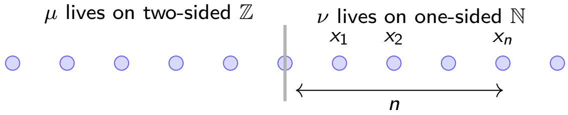

- For each $n > 0$ define a function $h_n : X \to \mathbb{R}$

$$h_n(x) = \frac{\mu([x]_n)}{\nu([x]_n)} \text{ for } x = (\ldots, x_{-1}, x_0, x_1, x_2, x_3, \ldots)$$

where $[x]_n$ = all infinite sequences which agree with x in the places $[1, n]$.

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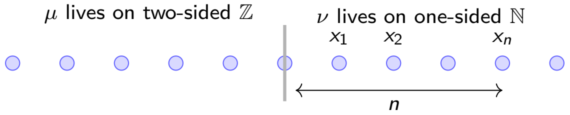

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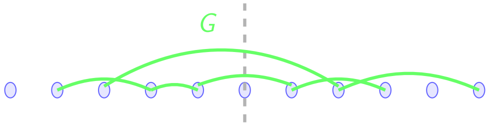
  where $[x]_n$ = all infinite sequences which agree with x in the places $[1, n]$.

Lemma. If we can choose μ such that $h_n \to h$ uniformly then h is continuous (and $\mathscr{L}_\phi h = \lambda h$)

To construct μ: use Random Graphs



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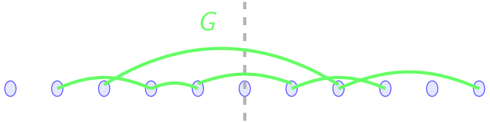

Let G denote a graph with vertex set $\mathbb{Z}$. The probability $\mathbb{P}$ on the space of such (random) graphs comes from edge weights $p(i,j) = 1 - e^{-J(|i-j|)}$ (for vertices $i, j \in \mathbb{Z}$).

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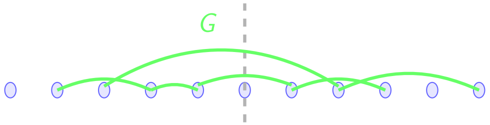

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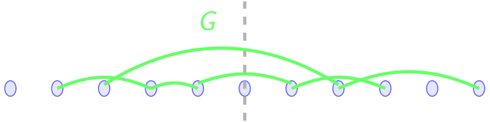

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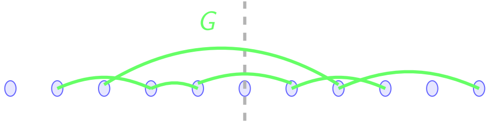

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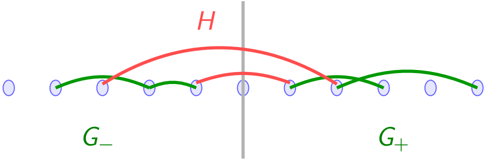
1. $\omega_n(G)$ = number of clusters (i.e., maximally connected subgraphs) of G intersecting $[1, n)$; and
2. $B_n(x, G) \in \{0, 1\}$ enforces that no path in G connects vertices of opposite signs in $[1, n)$.

Splitting the graphs

Cutting at the origin decomposes the random graph G into three parts:



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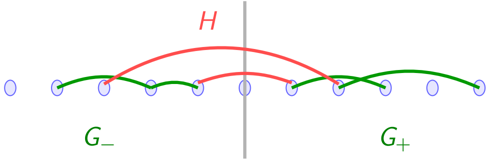

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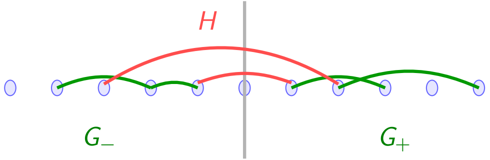

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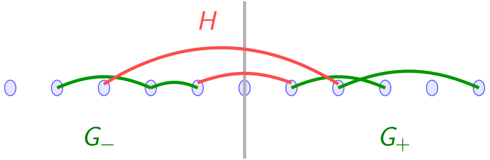

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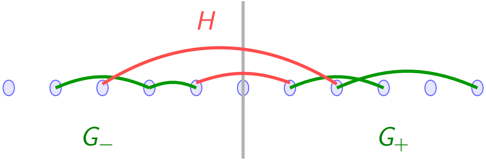

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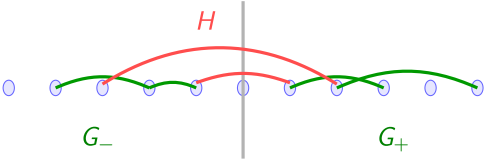

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The second main Theorem follows as corollary of the first main Theorem, i.e., If

$$\phi(x) = x_0 \sum_{k=n+1}^{\infty} J(k) x_k \implies \sum_{n=0}^{\infty} (\mathrm{var}(\phi))^2 < +\infty \text{ and } \beta < \beta_c$$

(Hypothesis of first theorem) then for $r(x) = \sum_{n=0}^{\infty} r_n |x_n|$ when $r_n = \sum_{k>n} J(k)$ then

$$\int_X e^{r(x)} d\nu(x) < +\infty. \text{ (Hypothesis of second theorem)}$$

The end

Obrigado pelo sua tempo